05 Review of Linear Algebra and Transformations

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Part of the slides taken and modified based on materials from Prof. M. Vasilescu's Computer Vision Course taught in Stony Brook University (SUNY)

Overview

- n-dimensional vectors
- Dot (Scalar) product
- Bases and Frames
- Homogeneous Coordinates
- 2D and 3D Geometric Transformations

Vectors

Notation:

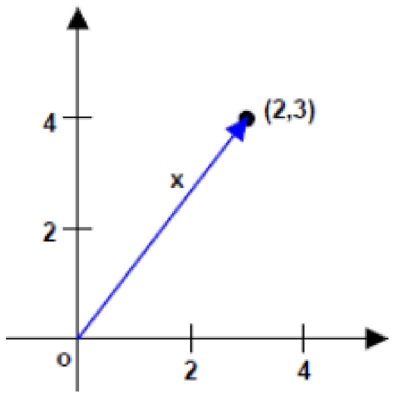
$$\mathbf{x} \in \mathbb{R}^n, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$$

Length:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Geometric Meanings:

- An origin + n pairwise perpendicular vectors → defines a frame or a coordinate system
- For a fixed frame, a point → a n-dimensional vector



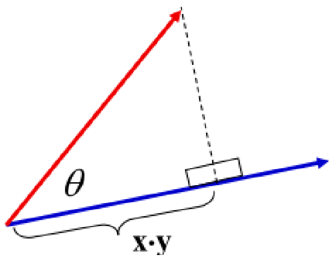
Dot Product

Dot Product of two vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle$$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Geometric Meanings:



$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$$

Properties

- Commutative: $$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$
- Distributive: $$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$$
- Linearity

$$(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y})$$
$$\mathbf{x} \cdot (c\mathbf{y}) = c(\mathbf{x} \cdot \mathbf{y})$$

$$(c_1\mathbf{x}) \cdot (c_2\mathbf{y}) = (c_1 c_2)(\mathbf{x} \cdot \mathbf{y})$$

- Non-negativity:

$$\forall \mathbf{x} \neq 0 : \ \langle \mathbf{x}, \mathbf{x} \rangle > 0 \qquad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = 0$$

- Orthogonality:

$$\forall \mathbf{x} \neq 0, \mathbf{y} \neq 0 \quad \mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$$

Norms

- Euclidean norm (sometimes called 2-norm):

$$\|\mathbf{x}\| = \|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- The length of a vector is defined to be its (Euclidean) norm.
- A unit vector is of length 1.
- Non-negativity properties also hold for the norm:

$$\forall \mathbf{x} \neq 0 : \ \|\mathbf{x}\|^2 > 0 \qquad \|\mathbf{x}\|^2 = 0 \Leftrightarrow \mathbf{x} = 0$$

Linear Dependence

- Linear combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

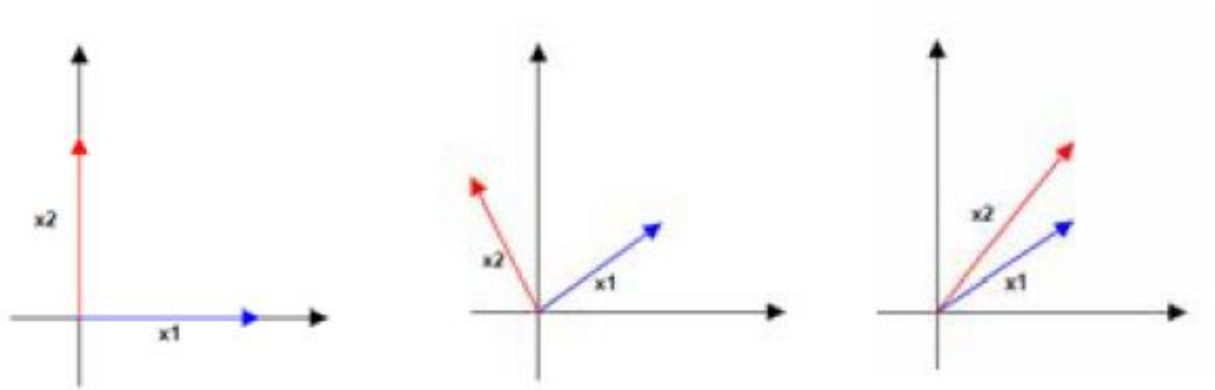
- A set of vectors $X=\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ are linearly dependent if there exists a vector $\mathbf{x}_i \in X$

  that is a linear combination of the rest of the vectors.

- In $\mathbb{R}^n$
  - sets of n+1vectors are always dependent
  - there can be at most n linearly independent vectors

Bases

- A basis is a linearly independent set of vectors that spans the "whole space". ie., we can write every vector in our space as linear combination of vectors in that set.
- Every set of n linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$
- A basis is called
  - ***orthogonal***, if every basis vector is orthogonal to all other basis vectors
  - ***orthonormal***, if additionally all basis vectors have length 1.



Bases

- Standard basis in $\mathbb{R}^n$ is made up of a set of unit vectors:

$$\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \hat{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \hat{\mathbf{e}}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

- We can write a vector in terms of its standard basis:

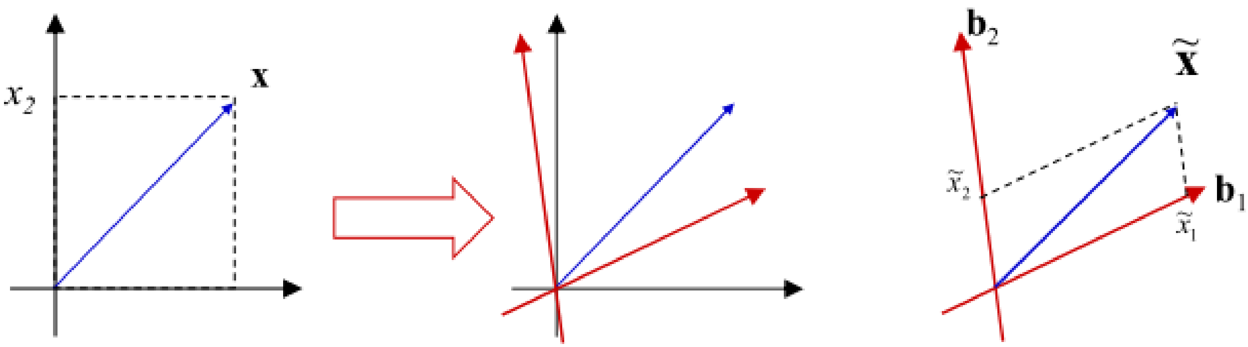
$$\begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix} = 4\,\hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3$$

- Observation: -- to find the coefficient for a particular basis vector, we project our vector onto it.

$$x_i = \hat{\mathbf{e}}_i \cdot \mathbf{x}$$

Change of Bases

- Suppose we have a new basis $\mathbf{B} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$, $\mathbf{b}_i \in \mathbb{R}^m$ and a vector $\mathbf{x} \in \mathbb{R}^m$ that we would like to represent in terms of **B**



- Compute the new components $\tilde{\mathbf{x}} = \mathbf{B}^{-1}\mathbf{x}$
- When B is orthonormal
  - $\tilde{\mathbf{x}}$ is a projection of **x** onto **b**$_i$
  - Note the use of a dot product

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{b}_1^T \mathbf{x} \\ \vdots \\ \mathbf{b}_n^T \mathbf{x} \end{bmatrix}$$

Note: B is an orthonormal matrix, whose inverse is its transpose. Therefore, we have

Rank of a Matrix

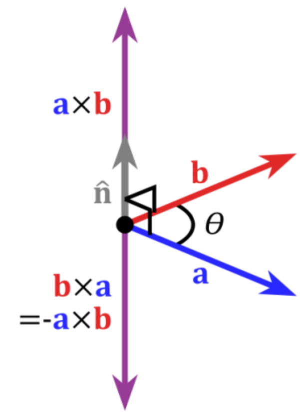
The rank of a matrix is the number of linearly independent rows or columns.

Examples: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has rank 2, but $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ only has rank 1.

Equivalent to the dimension of the range of the linear transformation.

A matrix with full rank is called *non-singular*, otherwise it is singular.

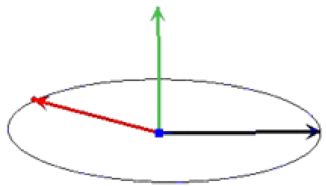
Cross Product



Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the 3D space

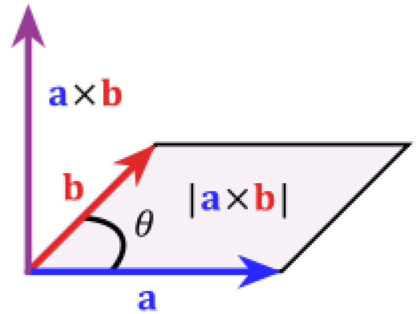
- ❑ the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and therefore normal to the plane containing them

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \; \mathbf{n}$$



- ❑ If the two given vectors are parallel or one vector has zero length, then the cross product is zero

- ❑ Anti-commutative : $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$
- ❑ Geometric meaning:
  - ❑ Magnitude: the area of a parallelogram
  - ❑ Direction: right-hand rule



- ❑ <u>Example 1</u>: Prove if $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ then $\boldsymbol{a} \cdot \boldsymbol{c} = 0$

Homogeneous Coordinates (2D)

Consider a point $\boldsymbol{x}$ in the 2D plane:

- Conventional Notation (inhomogeneous coordinates)
  - A 2D point can be represented by a pair of coordinates $(x, y) \in R^2$
  - A 2D point ← a 2D vector $\boldsymbol{x} = (x, y)^T$; the 2D plane ← R^2 vector space
- Homogeneous Notation (homogeneous coordinates)
  - $\tilde{\boldsymbol{x}} = (x, y, 1)^T$, referred to as an augmented vector
  - 3D vectors $(x, y, w)^T$ can be converted to its corresponding augmented vectors $w(x/w, y/w, 1)$ if $w \neq 0$;
    - In this case, we consider $(x, y, w)^T$ and $\left(\frac{x}{w}, \frac{y}{w}, 1\right)^T$ to be equivalent
  - The set of equivalence classes of vectors from 3D space $R^3 - (0,0,0)^T$ forms the **projective space** P^2
    - Homogeneous vector representation of a point $\tilde{\boldsymbol{x}} = (x_1, x_2, x_3)^T$ corresponds to the point with the 2D inhomogeneous coordinates $\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)^T$
- <u>Example 2</u>:
  - A 2D point $(3,4)$, what is its homogeneous coordinates?
    - → $(3,4,1)$
    - What about $(6,8,2)$?

Homogeneous Coordinates (cont.)

- Motivation of using Homogeneous Coordinates:
  - To consider and represent linear transformations using matrices
  - Simplify a lot of computations

- A line in the plane: $ax + by + c = 0$, can be put as $(a, b, c)^T$ in Homogeneous Notation
  - The correspondence is **NOT** one-to-one: $k(a, b, c)^T, k \neq 0$ represents the same line
  - The set of equivalence classes of vectors in $R^3 - (0,0,0)^T$ forms the projective space P^2

- A planar point $x = (x, y)^T$ lies on the line $l = (a, b, c)^T$ iff $ax + by + c = 0$
  - If we write the point as an augmented vector: $\tilde{\boldsymbol{x}} = (x, y, 1)^T$
  - → inner product of vectors: $(x, y, 1)(a, b, c)^T = 0$, or simply $\tilde{x} \cdot l = 0$

Homogeneous Coordinates for 3D Points

- Inhomogensou Coordinates $(x, y, z)^T$ → homogeneous coordinates $(x, y, z, 1)^T$

- A 3D plane can be represented as $\boldsymbol{m} = (a, b, c, d)^T$, because any point $\boldsymbol{x} = (x, y, z)^T$ on this plane satisfies:

$$ax + by + cz + d = \widetilde{\boldsymbol{x}} \cdot \boldsymbol{m} = 0$$

2D Transformations

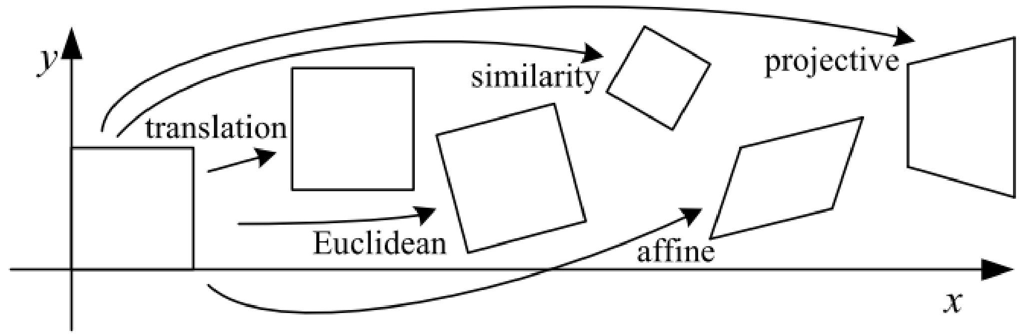


Figure 2.4 Basic set of 2D planar transformations.

- Basic Transformations: Translations, Rotations, Scaling, Affine, Projective
- Euclidean Transformation = Rigid Transformation = Translations + Rotations
- Similarity Transformation = Uniform scaling + Translations + Rotations + Reflection
- Affine Transformation = Similarity + non-uniform scaling + shear mapping
- Projective Transformation

Linear Transformations

- A transformation $T(\boldsymbol{x}) = \boldsymbol{y}, \boldsymbol{x} \in R^n, \boldsymbol{y} \in R^m$, is a linear transformation if it can be represented by a matrix: $\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x}, \boldsymbol{M} \in R^m \times R^N$, where M is called the transformation matrix
- With homogeneous coordinates, all basic transformations listed in the previous slide can be treated as linear transformations

Example: 2D Rotation

To rotate the left figure to the right figure (the left bottom corners of both objects remain at the origin)

= To find a transformation T such that $T(\boldsymbol{e}_1) = \boldsymbol{f}_1$, $T(\boldsymbol{e}_2) = \boldsymbol{f}_2$

→ We can solve a linear system using elementary trigonometry:

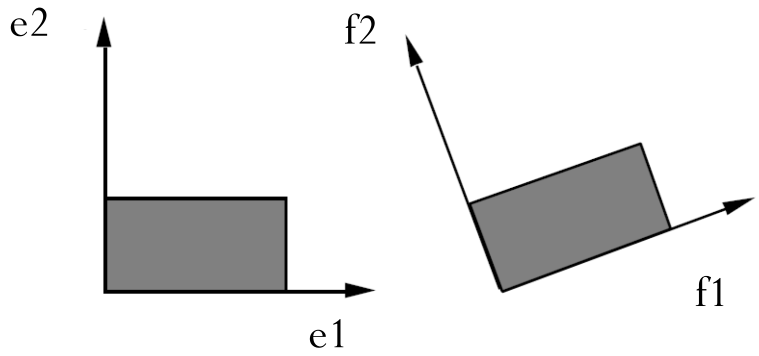
$$\boldsymbol{f}_1 = \boldsymbol{M}\boldsymbol{e}_1 \; ; \boldsymbol{f}_2 = \boldsymbol{M}\boldsymbol{e}_2$$

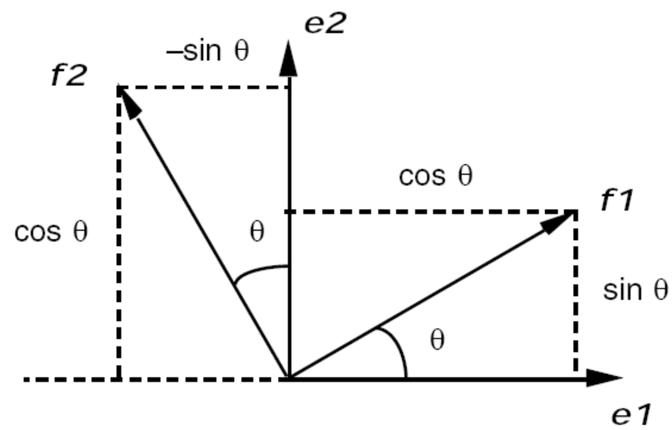
under the coordinate system of $(\boldsymbol{e}_1, \boldsymbol{e}_2)$,

$$\boldsymbol{f}_1 = (cos\theta, sin\theta)^T \; ; \boldsymbol{f}_2 = (-sin\theta, cos\theta)$$

⇨

$$\boldsymbol{M} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$





Example: Scaling

2D scaling transformation matrix:

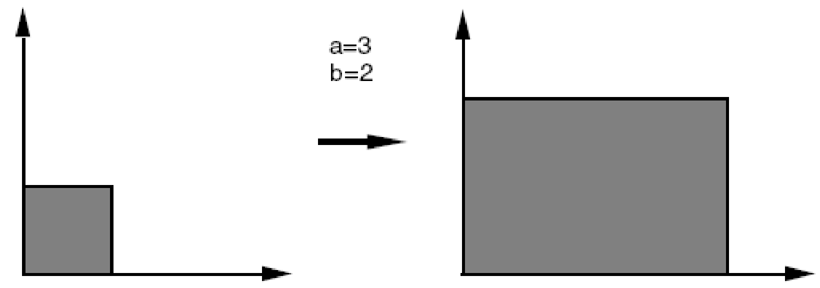
$$\boldsymbol{M} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

as

$$\boldsymbol{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

where a and b are called the scaling factors along x and y axes.

- If $a = b$: uniform (isotropic) scaling, shape preserved, only size changed
- If $a = b = 1$: identity transform



- Directly extendible to 3D

Example: Shear

Starting from the 2D scaling transformation matrix:

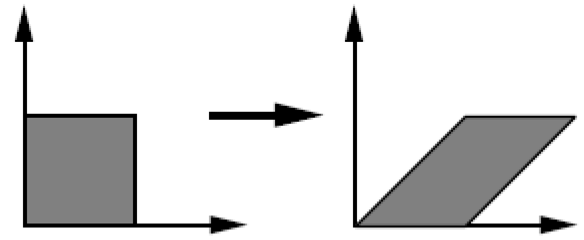
$$\boldsymbol{M} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Let one of the off-diagonal element be non-zero:

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \text{ or } \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

$$\boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + cy \\ by \end{bmatrix}$$

- A general definition: shearing = to shift the figure along a direction by an amount proportional to its signed distance from the line parallel to that direction
- Shearing changes the shape (angles changes under shearing) but preserves the area

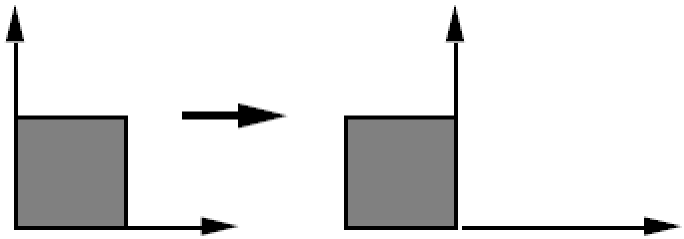


$a = b = 1, c = 1$

Examples: Reflection & Orthography

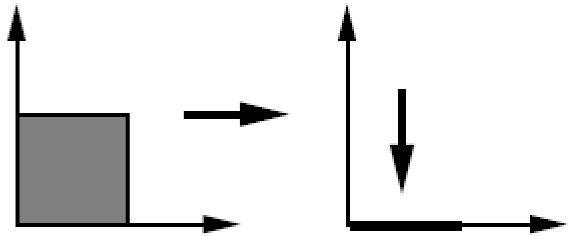
- Reflection = scaling with negative factors

$$\boldsymbol{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



- Orthographic Projection

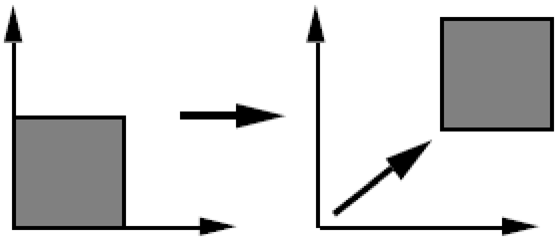
$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



  - Does not map a basis onto another basis
  - A singular transformation (can't be inverted, lost depth information along a direction)

Example: Translation

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} x + d_1 \\ y + d_2 \end{bmatrix}$$



Not a linear transformation (can't be represented as a matrix-vector multiplication)

But if the homogeneous coordinates are used:

$$\begin{bmatrix} 1 & & d_1 \\ & 1 & d_2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_1 \\ y + d_2 \\ 1 \end{bmatrix}$$

Rigid Transformation = translation + rotation

- If a transformation preserves distance → called an isometry
- Rigid transformations are definitely isometries, but isometries may not be rigid
  - e.g. bending a paper to a cylinder

Example: General Projection

A general projection, or perspective transformation, or also called homography, is defined on homogeneous coordinates,

$$\tilde{\boldsymbol{x}}' = \boldsymbol{H}\,\tilde{\boldsymbol{x}}$$

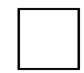
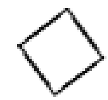
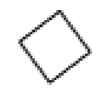
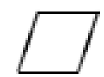
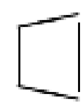
where $\boldsymbol{H}$ is an arbitrary 3×3 matrix, defined up to a scale.

- Converting back to inhomogeneous coordinates:

$$x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \text{ and } y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}}.$$

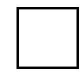
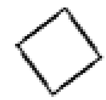
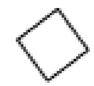
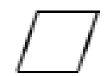
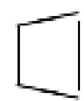
- Perspective transformations preserve straight lines such that they remain straight after the transformation

Summary of 2D Transformations*

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times3}$	8	straight lines	

#DOF : the Number of Degrees of Freedom = the number of independent variables (unknowns) that can change freely

Summary of 3D Transformations*

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} \mid \mathbf{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} \mid \mathbf{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} \mid \mathbf{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

#DOF : the Number of Degrees of Freedom = the number of independent variables that can change freely