Chapter 1  Signals

1.1  Signals are waveforms corresponding to a variety of physical quantities or measures, such as electric voltage, acoustic waves, the light intensity or electromagnetic waves.

Example:

* natural signals: speech signals, optical images, brain waves, MRI, EEG, etc.

* artificial signals: digital voltages in the computer data bus, electromagnetic fields for wireless communications, radar imaging signals, etc.
Classification of signals

1. One-dimensional signals: signal waveforms are one-dimensional, i.e., physical quantities versus time, space, or frequency, etc.

Example 1: Single-tone sinusoid at \( f_0 = 100 \text{Hz} \)

\[ s(t) = 7 \sin \left( \frac{2\pi f_0 t}{\Phi} \right) \]

Example 2: Speech signal

\[ s(t) \]
One-dimensional signal can be characterized as a function of one independent variable such as $s(t)$; thus, signals can be considered as single-value functions of time that carry information.

* Multidimensional signal: signal waveforms are multi-dimensional or in a vector space; i.e., a coordinate of physical quantities versus time, space, frequency, etc.

**Example 1: Optical Image**

```
Vertical axis  ↑
(spatial)          ↓

Horizontal axis  (spatial)
```
Example 2: MRI Image

Vertical axis (time, spatial)

Horizontal axis (time, spatial)

Continuous-time signals: Signals that are defined at every instant of time. It is also called analog signal.

Example: Single-tone continuous waveform of a sinusoid, \( f_0 = 250 \text{ Hz} \)

\[
S(t) = \sin (2\pi f_0 t) = \sin (500\pi t)
\]
Discrete-time signals are sampled from continuous-time signals at certain specific instants of time.

Example: discrete-time sampler of a single-tone sinusoid, \( f_0 = 250 \text{ Hz} \).

\[ s(t) = \sin (500 \pi t) \]

sampled at \( t = \frac{n}{f_s} \), \( f_s \): sampling frequency

\[ T_s = \frac{1}{f_s} \): sampling period

\( f_s \geq 2f_0 \), \( n \) is integer

\[ f_s = 1000 \text{ Hz} \]

\[ s(t) = \sin (500 \pi t) \] — original analog signal

\[ s(n) = \sin \left( 500 \pi \frac{n}{1000} \right) = \sin \left( \frac{\pi}{2} n \right), \quad n \in \mathbb{Z} \]

\( n = 0, 1, 2, 3, \ldots \)
1.2 Elementary continuous-time signals

A continuous-time signal will be denoted by \( f(t) \) with \( t \) ranging from minus infinity to positive infinity, written as \( f(t) \) for all \( t \) in \(( -\infty, \infty )\). Because \( t = 0 \) is mainly used for reference, we call it the reference time.

Typical Elementary continuous-time signals

\[ g(t) := \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \]

\[ g(t) \]

\[ g(t) \]

\[ t \]

\[ t g(t) \]

\[ r(t) := \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \]

\[ r(t) := t g(t) \]
Variation of \( r(t) \):

1. Acceleration function:

\[
f'(t) = \begin{cases} 
-t^2, & t \geq 0 \\
0, & t < 0 
\end{cases}
\]

\[
= t \cdot r(t) = t^3 g(t)
\]

2. Polynomial function:

\[
f''(t) = \begin{cases} 
b_0 + b_1 t + \ldots + b_n t^n, & \text{for } t \geq 0 \\
0, & \text{for } t < 0 
\end{cases}
\]

\[
= \sum_{k=0}^{n} b_k t^k g(t)
\]
Periodic function: A continuous-time signal \( f(t) \) is said to be periodic with period \( p \) if \( f(t+p) = f(t) \) for all \( t \).

If \( f(t) \) is periodic, then

\[
f(t) = f(t+p) = f(t+2p) = \ldots = f(t+np)
\]

\( n \) is any arbitrary positive integer.

Sinusoidal function:

\[
f(t) = A \sin (\omega t + \theta)
\]

- \( A \): amplitude
- \( \omega \): frequency (unit: rad/sec)
- \( \theta \): phase (unit: radians)

\( f(t) \) is periodic with fundamental period:

\[
p = \frac{2\pi}{\omega} \quad (\text{unit: seconds})
\]

\[
\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}
\]

Unit: cycle/second or Hz.
A real exponential function - time constant

\[ f(t) = e^{at}, \text{ where } a \text{ is real constant} \]

The rate of increase or decrease depends on the magnitude of \( a \). If \( a \) is negative, then

\[ \frac{f(t + \frac{1}{|a|})}{f(t)} = e^{\frac{a}{|a|}} = e^{\frac{a}{|a|}} = e^{-1} \]

\[ = \frac{1}{2.7} = 0.37 = 37\% \]

which means that the value of \( e^{at} \) decreases to 37% of its original value whenever the time increases by \( 1/|a| \). \( 1/|a| \) is called the time constant.
Complex exponential functions - positive and negative frequencies

\[ f(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t), \quad \text{where} \quad j = \sqrt{-1}. \]

\[ \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}. \]

Graphic representation of a complex exponential
1.2.1 Boundedness in magnitude and infinity time

A continuous-time signal $f(t)$ is said to be bounded in a time interval if there exists a constant $M$ such that $|f(t)| \leq M$, where $M$ is called the upper bound of $|f(t)|$.

Example:

What is the upper bound for $e^{-0.5t}$?

Answer:

$e^{-0.5t}$ is not bounded in $(-\infty, \infty)$ but it is bounded in $(0, \infty)$. The upper bound is 1.
For an exponential function $e^{ax}$, as $a < 0$, the function decreases to less than 1% of its original value after five time constants and may be considered to have reached 0, since $(0.37)^5 = 0.007$

We may consider the infinite time $\infty \approx 5 \times (\text{time constant})$ in many applications.

\[ \frac{1}{|a|} \]

Example: If we consider $e^{-bx}$ for $b > 0$, to have reached 0 in five time constants, what are infinite times for $b = 0.01$, 1 and 100?

Answer:

$\infty \approx \frac{1}{|a|} \times 5 = \frac{5}{|a|}$

- $b = 0.01 \Rightarrow \infty \approx \frac{5}{0.01} = 500 \text{ sec}$
- $b = 1 \Rightarrow \infty \approx \frac{5}{1} = 5 \text{ sec}$
- $b = 100 \Rightarrow \infty \approx \frac{5}{100} = 0.05 \text{ sec}$
13 Manipulation of Continuous-time signals.

Shifting: Let \( f(t) \) be a function and \( T \) be a positive number. Then \( f(t-T) \) shifts \( f(t) \) to the right or advances \( f(t) \) by \( T \) seconds.

Example: If \( g(t) \) denotes a unit step function, plot \( g(t-1) \) and \( g(t+2) \).

Answer: 

\[ g(t) \]

\[
\begin{array}{c}
0 \\
1 \\
\hline \\
0 \\
\end{array}
\]

\[ t \]

\[ g(t-1) \]

\[
\begin{array}{c}
0 \\
1 \\
\hline
0 \quad 1 \\
\end{array}
\]

\[ t \]

\[ g(t+2) \]

\[
\begin{array}{c}
-2 \\
1 \\
\hline \\
0 \\
\end{array}
\]

\[ t \]
Flipping: \( f(t) \) be a function. Then \( f(-t) \) is the flipping of \( f(t) \) with respect to \( t=0 \), to the negative time.

Example: \( f(t) = g(-t) \). Plot \( f(t) \).

\[ g(t) \]
\[ +1 \]
\[ 0 \rightarrow t \]

\[ f(t) = g(-t) \]

Multiplication: Consider two signals \( f(t) \) and \( h(t) \) defined for all \( t \) in \((-\infty, \infty)\). Then their product \( g(t) = f(t)h(t) \) forms a new signal.

Example: Plot \( f(t) = g(t)g(-t) \).
\[ f(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{elsewhere} \end{cases} \]

\[ g(t) = f(t) + h(t), \quad \text{forms a new signal} \]

**Example:** Plot \( f(t) = 2 \delta(t) - 2 \delta(t-1) - 2 \delta(t+1) \).
\[ f(x) = \begin{cases} 
2x, & x \leq 1 \\
2x - 2(x-1) = 2, & x \in [1, 3] \\
2x - 2(x-1) - 2 = 0, & x \geq 3 
\end{cases} \]
1.4 Discrete-time and digital signals

A continuous-time (analog) signal is sampled to generate a discrete-time signal through an Analog-to-digital converter (ADC). Then the amplitude of a discrete-time signal can be quantized to a finite set of values, so the waveforms of the quantized discrete-time signals are called digital signals.

![Diagram of discrete-time signal and quantization process]
On the contrary, a digital signal can be converted to an analog signal through a DAC (digital-to-analog converter). Basically DAC interpolates a continuous-time function in between any pair of successive discrete-time samples along the waveform.

\[ s(n) \quad \text{(quantized)} \]

\[ s(t) \]

DAC

\[ n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \]

\[ n, t \]
1.5 Elementary Discrete-time Signals and Their Manipulation

\[ f[k] := f(kT) = f\left(\frac{k}{f_s}\right), \quad \text{where } T = \frac{1}{f_s} \]

\[ \delta[k] = \begin{cases} 0 & \text{for } k \neq 0 \\ 1 & \text{for } k = 0 \end{cases} \]

Example: Plot \[ f[k] = 4\delta[k+2] - 3\delta[k] + 2\delta[k-1] \]
For any discrete-time sequence (signal) \( f[k] \), it can be written as

\[
f[k] = \sum_{i=-\infty}^{\infty} f[i] \delta[k-i]
\]

**Example:**

\[
\]

\[
= \delta[k] + \delta[k-1] - \delta[k-2] - 2 \delta[k-3]
\]

**Unit step sequence:**

\[
g[k] = \begin{cases} 1, & \text{for } k \geq 0 \\ 0, & \text{for } k < 0 \end{cases}
\]

\[
g[k-i] = \begin{cases} 1, & \text{for } k \geq i \\ 0, & \text{for } k < i \end{cases}
\]
\[ s[k] = g[k] - g[k-1] = g[k]g[-k] \]

\[ g[k] + g[-k] - s[k] = 1 \]

Example: Express a window function \( P_N[k] \) by a linear combination of unit-step functions, where

\[ P_N[k] = \begin{cases} 1 \text{, for } -N \leq k \leq N \\ 0 \text{, for } k < -N \text{ and } k > N \end{cases} \]

Answer:

\[ P_N[k] \]
Hence $p_n(k) = g(k+N) - g(k-N-1)$

Example: Plot $f[k] = (\frac{1}{2})^k$ and $g[k] = (\frac{1}{2})^k g[k]$
1.6 Sinusoidal sequence and its frequency

Consider an analog sinusoid \( f(t) = \sin (2\pi f_0 t) \) = \( \sin (\omega_0 t) \) with its fundamental period \( P = \frac{2\pi}{f_0} \) = \( \frac{1}{f_0} \) where \( \omega_0 = 2\pi f_0 \). We sample it with sampling period \( T > 0 \) yield

\[
f[k] = f(kT) = \sin (\omega_0 kT), \quad k = 0\pm 1, \pm 2, \ldots
\]

A discrete-time signal \( f[k] \) is said to be periodic with period \( N \), where \( N \) is a positive integer if \( f[k] = f[k+N] \). If \( f[k] \) is periodic, then \( f[k] = f[k+2N] = f[k+3N] = \cdots = f[k+nN] \) for any \( k \) and any integer \( n \).

\[
\sin (\omega_0 kT) = \sin (\omega_0 (k+N)T)
\]

\[
\omega_0 NT = 2\pi n \quad \text{or} \quad N = \frac{2\pi n}{\omega_0 T} \rightarrow \text{period}
\]

The smallest such \( N \) is called fundamental period.
Example:

Which of the following sequences are periodic? If any, find their periods and specify fundamental periods.

(a) $\sin(0.1k)$
(b) $\cos(\frac{3\pi}{7}k)$

Answer:

(a) $N = \frac{2k\pi}{0.1}$ cannot be integer no matter what kind of $n$ is chosen. Hence it is not periodic.

(b) $N = \frac{2k\pi}{\frac{3\pi}{7}} = \frac{14}{3}n$ can be integer whenever $n = 3l$, $l$ is any positive integer.

The fundamental period is $N = 14$. (the least integer $N$)
How can we visualize the continuous-time and discrete-time sinusoids.

If we sample a continuous-time sinusoid \( \sin(\overline{\omega}t) \) at \( t = kT \) to generate a discrete-time sinusoid \( \sin(w_0 \overline{k}T) \), \( N = \frac{2\pi \overline{k}}{w_0 T} \)

\[ \sin(0.5\pi \overline{k}) , \quad w_0T = 0.5\pi , \quad \overline{k} = w_0 \]

![Graph showing envelope of \( \sin(0.5\pi \overline{k}) \)]

However, if we sample another continuous-time sinusoid \( \sin((w_0 + 2\pi \overline{n})T) \) at \( t = kT \), we have \( \sin(w_0 \overline{k}T + 2\pi \overline{n}k) = \sin(w_0 \overline{k}T) \), where \( n \) is integer.

We still have the same sequence!

![Graph showing envelope of \( \sin(0.5\pi \overline{k}) \)]
Hence we cannot specify \( \omega \) according to \( \sin(\omega_0 k T) \). We need to specify a primary envelope of \( \sin(\omega_0 k T) \) with a unique frequency in the range of \(-\pi < \omega T \leq \pi\) or \(-\frac{\pi}{T} < \omega \leq \frac{\pi}{T}\) such that we can have a unique relationship \( \frac{\sin(\omega_0 k T)}{k} = \sin(\omega_0 k T) \).

**Example:** What is the frequency \( \tilde{\omega} \) of a primary envelope for \( \sin(1.1\pi k) \)? What are all possible frequencies \( \tilde{\omega} \) of envelopes for \( \sin(1.1\pi k) \)? Both sampled at \( T = 0.5 \).

**Answer:** Since \( \omega_0 = \frac{1.1\pi}{0.5} = 2.2\pi > \frac{\pi}{T} = 2\pi \),

we need to carry \( \omega_0 \) down one period \( \frac{2\pi}{T} \).

\[
\tilde{\omega} = \omega_0 - \frac{2\pi}{T} = 1.38\pi - 2\pi
\]

\[
\tilde{\omega} = \omega + \frac{2n\pi}{T} = -1.8\pi + 4n\pi, \ n \in \mathbb{Z}
\]
Definition: The frequency of a digital signal, \( \sin(w_0 t) \), is defined as the frequency of an analog primary envelope \( \sin(w t) \) with \( -\frac{\pi}{T} < W < \frac{\pi}{T} \).

Example: Find the frequencies of the following sequences for \( T = 1 \).

(a) \( \sin(4.2\pi t) \)  
(b) \( \sin(-2.1\pi t) \)

Answer: (a) \( w_0 = 4.2\pi > \pi \), so we need to carry \( w_0 \) down \( 4\pi \) such that \( W = w_0 - 4\pi = 0.2\pi < \pi \) 
\[ W = 0.2\pi \text{ rad/sec} \]

(b) \( w_0 = -2.1 \neq \pi \)
Hence \( W = w_0 = -2.1 \text{ rad/sec} \)

Sampling Theorem: If the sampling period \( T \) is less than half of the fundamental period \( w_0 \) of an analog signal such as \( \sin(w_0 t) \), then the frequency of \( \sin(w_0 t) \) can be determined from its sampled sequence.

\[ s(w_0 t) \] → \( \sin(w t) \) (primary envelope)
Complex exponential sequence:

\[ f[k] = e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t) \]

\[ f[k+N] = e^{j\omega_0 t} e^{j\omega_0 NT} \]

\[ f[k] \text{ is periodic if and only if }\omega_0 NT = 2\pi n \text{ or } \frac{\omega_0 T}{\pi} \text{ is a rational number.} \]

Example: Check if the following sequences are periodic. If so, figure out its fundamental period. (a) \( e^{j0.7\pi k} \)
(b) \( e^{-0.3k} \)

Answer:

(a) \( \frac{\omega_0 T}{\pi} = \frac{0.7\pi}{\pi} = 0.7 = \frac{7}{10} \) ... rational
\( e^{j0.7\pi k} \) is periodic,
\[ N = \frac{2\pi n}{\omega_0 T} = \frac{2\pi}{0.7} = \frac{20}{7} \]
\( N = 20 \) is the fundamental period.

(b) \( \frac{\omega_0 T}{\pi} = \frac{-0.3}{\pi} \) ... irrational
\( e^{-0.3k} \) is not periodic.