Partition Information and Its Transmission Over Boolean Multi-Access Channels

Shuhang Wu, Shuangqing Wei, Yue Wang, Ramachandran Vaidyanathan, and Jian Yuan

***Abstract*— In this paper, we propose a novel reservation system to study partition information and its transmission over a noise-free Boolean multiaccess channel. The objective of transmission is not to restore the message, but to partition active users into distinct groups so that they can, subsequently, transmit their messages without collision. We first calculate (by mutual information) the amount of information needed for the partitioning without channel effects, and then propose two different coding schemes to obtain achievable transmission rates over the channel. The first one is the brute force method, where the codebook design is based on centralized source coding; the second method uses random coding, where the codebook is generated randomly and optimal Bayesian decoding is employed to reconstruct the partition. Both methods shed light on the internal structure of the partition problem. A novel formulation is proposed for the random coding scheme, in which a sequence of channel operations and interactions induces a hypergraph. The formulation intuitively describes the transmitted information in terms of a strong coloring of this hypergraph. An extended Fibonacci structure is constructed for the simple, but nontrivial, case with two active users. A comparison between these methods and group testing is conducted to demonstrate the potential of our approaches.**

***Index Terms*— Partitioning information, conflict resolution, Boolean algebra, Fibonacci numbers.**

I. Introduction

ONE primary objective of many coordination processes is to order a set of participants. For example, multiaccess can be viewed as (explicitly or implicitly) ordering a set of users for exclusive access to a resource. Information interaction plays a key role in establishing such an order. To formalize this interactive information and derive fundamental limits on its transmission, we propose, in this paper, a novel partition reservation model over a noise-free Boolean multi-access channel and use an information theoretic approach in its analysis.

For the simplest variant of the problem that we study, let $\mathcal{N} = \{1, \ldots, N\}$ be a set of N users and let $\mathcal{G}_{\mathbf{s}} = \{i_1, \ldots, i_K\} \subseteq \mathcal{N}$ be a set of K *active* users. The problem is to let all users obtain a common ordered *K-partition*[1] $\Pi = (\mathcal{B}_1, \ldots, \mathcal{B}_K)$ of $\mathcal{N}$, so that each group (or block) $\mathcal{B}_i$ has exactly one active user from $\mathcal{G}_{\mathbf{s}}$. Equivalently, we use a vector $\mathbf{z} = [z_1, \ldots, z_K]^\top$ to represent the ordered K-partition Π, where $z_i \in \mathcal{K} \triangleq \{1, 2, \ldots, K\}$ is the id of the group that user i belongs to; i.e., $i \in \mathcal{B}_k$ iff $z_i = k$. The desired partition is determined by a series of transmissions and observations over a shared slotted Boolean multi-access channel. Suppose that during slot t, each active user i transmits bit $x_{i,t} \in \{0, 1\}$ on the channel. A common feedback $y_t = \bigvee_{i \in \mathcal{G}_{\mathbf{s}}} x_{i,t}$ will be observed by all users; i.e., if no active user transmits a 1 during slot t, then $y_t = 0$, and if at least one active user transmits a 1, then $y_t = 1$. The idea is to schedule (off-line) T rounds of transmissions (represented by an accessing matrix $\mathbf{X} \triangleq [x_{i,t}]_{1 \le i \le N, 1 \le t \le T}$), and construct a decoding function $g(\cdot)$, so that after observing T rounds of channel feedback $\mathbf{y} \triangleq [y_1, \ldots, y_T]^\top$, an ordered *$K$-partition* of $\mathcal{N}$, denoted by $\mathbf{z} = g(\mathbf{y})$, can be obtained by all users. The objective is to find an achievable lower bound on the number of slots T, within which there exists a matrix $\mathbf{X}$ and function $g(\cdot)$ such that every possible active set $\mathcal{G}_{\mathbf{s}} \subseteq \mathcal{N}$ can be partitioned as outlined above.

In the problem we consider, we do not seek to restore the states of all users (that is, determine $\mathcal{G}_{\mathbf{s}}$ exactly), but to partition $\mathcal{G}_{\mathbf{s}}$ and to make the partition $\mathbf{z}$ known to all users. Thus, only information on a partition pertaining to the relationship between active users in $\mathcal{G}_{\mathbf{s}}$ is transmitted through the channel. We will formalize this partition information, and derive an achievable bound on its transmission rate over a Boolean multi-access channel. This problem plays a significant role in understanding the fundamental limits on establishing order in distributed systems.

The proposed problem is closely related to the well-known slotted conflict resolution problem [1], in which each active user must transmit without conflict at least once during T slots; i.e., if $x_{i,t} = 1$ denotes a trial of transmission for active user i at slot t, then there exists a slot $1 \le t_i \le T$ such that $x_{i,t_i} = 1$, and for all $j \in \mathcal{G}_{\mathbf{s}} - \{i\}$, we have $x_{j,t_i} = 0$. To achieve this

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[1] An ordered K-partition $\Pi = (\mathcal{B}_1, \ldots, \mathcal{B}_K)$ of $\mathcal{N}$ is a sequence of K non-empty subsets of $\mathcal{N}$ that satisfies the following conditions: (a) for all $1 \le i < j \le K$, $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ and (b) $\bigcup_{i=1}^{K} \mathcal{B}_i = \mathcal{N}$.

goal, primarily two types of systems have been studied: direct transmission systems and reservation systems with group testing [2]. Direct transmission focuses on directly designing an $N \times T_{dt}$ accessing matrix $\mathbf{X}_{dt}$ (subscript dt is for direct transmission), so that each node finds at least one slot for its exclusive access to the channel. Note that active users are implicitly partitioned during transmission to ensure the success of the transmission, (more specially, if the successful transmission slot t_i is known for each active user i, then the desired partition can be constructed by all users); however, the partition is not required to be known to all users. The second system, reservation with group testing, has two stages. In the first reservation stage, an accessing matrix $\mathbf{X}_g$ and decoding function $g(\cdot)$ are designed such that $\mathcal{G}_\mathbf{s}$ is determined exactly by $g(\mathbf{y})$, where $\mathbf{y}$ is the channel feedback. That is, (active or inactive) states of all users are restored and, subsequently, active users can transmit in a predetermined order without conflict in a second stage. The reservation stage is also called group testing [3] or compressed sensing [4] in different fields. The two stages can use slots of different time scales. Therefore, the separate reservation stage can use significantly smaller time slots than the subsequent payload transmission stage.

Compared to group testing and direct transmission systems, the proposed reservation system provides a new way to analyze the process of partitioning separately from data transmission. It can replace group testing as the reservation stage in conflict resolution problems, and holds the promise of requiring fewer resources, since it seeks only to partition $\mathcal{N}$, rather than restore $\mathcal{G}_\mathbf{s}$. (Notice that in the reverse direction once $\mathcal{G}_\mathbf{s}$ is restored, obtaining a partition is straightforward.) Compared with direct transmission, we observe that usually, the slot size for reservation can be much smaller in partition/reservation than that in payload transmission, thus it may need less time for conflict resolution in practical use.

The proposed partition reservation system has a number of applications in different areas. First, it can be directly applied to the reservation stage in conflict resolution instead of group testing. Second, since the partition reservation system conveys information about the partition to all users, more complex coordination problems can be addressed (than simply avoiding conflict in time domain); this additional possibility is not obvious in traditional conflict resolution schemes. For example, code-division multiple accessing (CDMA) codes could be assigned to users in different groups based on the partition obtained, so that active users can claim accessing code sequences from a common pool in a distributed way without coordination from a central scheduler. Other examples can be found in parallel and distributed computing [5]–[7], such as leader election [8] and broadcasting [9]. In this paper, the system is constrained to a case such that K active users non-adaptively access a noiseless Boolean channel. It is a fundamental case of the problem, but also has practical value. Consider a system with N users each of which stays active with a probability p. In this paper, we consider channel access by $K \approx pN$ active users. If p scales with N, then K can be assumed to be constant; this assumption is also reasonable for access to a single channel. We consider a non-adaptive model that is both simple and with low overheads. It also represents a fundamental structure for our new approach. It should be noted such non-adaptive channel model has also been considered in the MAC or group testing literature (for example, [3], [10]–[12], etc.). Our study will help us understand the fundamental limits on transmission resources to achieve a partitioned coordination among active users.

We first use source coding to quantify the partition information. Then we propose two coding schemes for the accessing matrix $\mathbf{X}$ and the decoding function $g(\cdot)$. The first is a brute force method to design $\mathbf{X}$ and $g(\cdot)$ directly based on results from source coding. Here the purpose of source coding is to compress the source information by determining a smallest set $\mathcal{C}$ of partitions, that contains a valid partition for nearly every possible active set $\mathcal{G}_\mathbf{s}$. The brute force method uses the channel to find a valid partition by checking every partition in $\mathcal{C}$. The second scheme, employing random coding, generates accessing matrix elements $x_{i,t}$ (i.i.d. by Bernoulli distribution); then, the partition is recovered by the optimal Bayesian decoding. The two methods can provide different views of the partition problem. In particular in the brute force method, if the number of slots used is $T_{BF} = \frac{K^{K+1}}{K!} f(N)$, where $f(N)$ is an arbitrary function satisfying $\lim_{N\to\infty} f(N) = \infty$, then we show that the average error probability $P_e^{(N)} \le e^{-f(N)}$ satisfies $\lim_{N\to\infty} P_e^{(N)} = 0$. For the simple, but non-trivial, $K = 2$ case we prove by random coding, for any $\xi > 0$, that if $\frac{\log N}{T} \le \max_{0\le p\le 1} C(p) - \xi$, (where $C(p) = -(1-(1-p)^2)\log\varphi(p) - (1-p)^2\log(1-p)$, and $\varphi(p) = \frac{p+\sqrt{4p-3p^2}}{2}$), then the average error probability $P_e^{(N)} \le \frac{1}{N^\Delta}$, for some $\Delta > 0$, such that $\lim_{N\to\infty} P_e^{(N)} = 0$ (i.e., with polynomial speed). The above two achievable bounds are shown to be better than that for group testing.

Moreover for the random coding approach, we introduce a hypergraph based framework to solve the problem. Here the problem is expressed in terms of a hypergraph in which interaction among a set of active nodes is represented as hyperedges. Channel effect is expressed as operations on hyperedges through strong hypergraph coloring. The joint work between the encoder and decoder is to ensure that the hypergraph becomes strongly colorable iff sufficient information is transmitted by active node interaction. In a simple, but nontrivial, case with $K = 2$ active users from a set of N users, the hypergraph is a simple graph and 2-colorability is used. A (suboptimal) odd-cycle based analysis is proposed, and a structure of extended Fibonacci numbers is found, which sheds lights on the inherent structure of the partition information and Boolean channel. This approach could be extended to $K > 2$.

In summary, the contributions of this paper are twofold. First, we formulate a novel partition reservation problem which captures the transmission and restoration of information about the relationship among active users. This problem is also represented in terms of a hypergraph. Second, we propose two types of coding approaches, and derive the corresponding achievable bounds on the communication period. This provides intuitive examples to study the transmission of relationship information over Boolean multi-access channels.

The rest of this paper is organized as follows: in Section II, we describe the related work. The problem formulation appears in Section III. In Section IV, the partition information is illustrated by centralized source coding, then a brute force method, directly inspired by source coding, is proposed in Section V. In Section VI, a random coding method is considered and the problem is reformulated in terms of a hypergraph. Based on this, the $K = 2$ case for random coding is analyzed in Section VII. In Section VIII, we compare our results with that of group testing. We summarize our results and make some concluding remarks in Section IX.

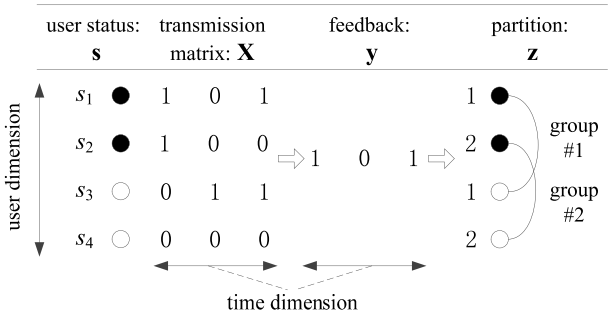

Fig. 1. Example of the formulation. ($N = 4$, $K = 2$, $\mathcal{G} = \{1, 2\}$ indicates that users 1 and 2 are active; the total number of time slots is $T = 3$.)

II. Related Work

Although the proposed partition model could be useful in many problem settings, typical applications are in conflict resolution problems. The work on conflict resolution is too extensive to be included in full here; we thus only refer to those most relevant to our setting.

To the best of our knowledge, Pippenger [14] first expressed conflict resolution as a two-stage problem: (a) partitioning active users into different groups; (b) payload transmission. Hajek [15] further studied this problem using a model in which K users are randomly distributed (uniform or Poisson) in the $[0, 1]$ real interval and a valid K-partition of $[0, 1]$ is sought, so as to separate active users into different groups. This model corresponds to the model we propose when $N \to \infty$. Hajek [15] derived an upper bound (achievability bound) on partition information on the above model. The tight lower bound discussed by Hajek, Körner, Simonyi and Marton [16]–[18] on this information still remains an open question.

This partition problem (without considering channel effect) is also closely related to perfect hashing, zero-error capacity, and list codes [19, Ch. V]. The problem is formulated in a combinatorial way: a subset $\mathcal{A}$ of $\mathcal{K}^L$ is called K-separated if every subset of $\mathcal{A}$ consisting of K sequences is separated, i.e., if for at least one coordinate i, the ith coordinates of the said sequences all differ. Let $A_L = A_L(K)$ denote the size of a maximal K-separated subset $\mathcal{A} \subseteq \mathcal{K}^L$. It can be seen that A_L corresponds to N users in our problem settings, and the set $\mathcal{A}$ can be viewed as a set of K partitions with size L, so that for any active set of A_L users, there exists a valid K partition.

The relationship between this combinatorial model and the probabilistic model is stated by Körner [17]. We note that these problems do not consider the channel effect, and are, therefore a form of source coding from the information theoretic perspective. For completeness, we will discuss the source coding problem further in Section IV of this paper. In contrast, the problem we are focusing on in this paper is a transmission problem; i.e., construction of a valid partition relationship among active users using the feedback from their explicit transmission over a collision Boolean multi-access channel. To the best of our knowledge, this problem has not been addressed previously.

In addition to the conflict resolution problems, there has been extensive work on direct transmission and group testing that considers channel effects from the *combinatoric* and *probabilistic* perspectives. Ding-Zhu and Hwang [3] provide an overview; more specific approaches can be found on superimposed codes for either disjunct or separable purposes [11], [20]–[23], on selective families [9], on the broadcasting problem [24], and for other methods [10], [22], [25]. It should be noted that recently, group testing has been reformulated using an information theoretic framework to study the limits of restoration of the IDs of all active nodes over Boolean multiple access channels [26]. We address in this paper the transmission of partition information (rather than identification information) over the channel, and it is thus, different from existing work.

III. System Model

A. Formulation

In this paper, lower-case (resp., upper-case) boldface letters are used for column vectors (resp., matrices). For instance, w_i is used for the i-th element of vector $\mathbf{w}$, and $w_{i,j}$ is used for the (i, j)-th element of matrix $\mathbf{W}$. Logarithms are always to base 2. The probability of a random variable A having value $\tilde{A}$ is denoted by $p_A(\tilde{A}) \triangleq \Pr(A = \tilde{A})$. Similarly, $p_{A|B}(\tilde{A}|\tilde{B}) \triangleq \Pr(A = \tilde{A}|B = \tilde{B})$. Where there is no danger of ambiguity, we will drop the subscripts and simply write $p(A)$ or $p(A|B)$ to denote the above quantities.

Assume the number of active users K is known to all users. The users are also given a common $N \times T$ accessing matrix (or codebook) $\mathbf{X}$, and a decoding function $g(\cdot)$. We use a Boolean vector $\mathbf{s} = [s_1, \ldots, s_N]^\top$ to represent the active or inactive states of users, where $s_i = 1$ iff user i is active (that is, $i \in \mathcal{G}_\mathbf{s}$). Active users will use T slots to transmit according to codebook $\mathbf{X}$ and observe the feedback $\mathbf{y} = [y_t : 1 \le t \le T]^\top$ over these T slots. Then users derive the partition $\mathbf{z} = g(\mathbf{y})$. There are two dimensions in this problem, the user dimension of size N and the time dimension of size T.

An Example: Our approach is illustrated by an example in Fig. 1 with four users from $\mathcal{N} = \{1, 2, 3, 4\}$, of which the users in set $\mathcal{G}_\mathbf{s} = \{1, 2\}$ are active. The $N \times T$ codebook is $\mathbf{X}$. In each slot $1 \le t \le 3 = T$, user i writes to the channel iff i is active and $x_{i,t} = 1$. For example, in slot 1, that has $x_{1,1} = x_{2,1} = 1$ and $x_{3,1} = x_{4,1} = 0$, both active users 1 and 2 write to the channel, resulting in a channel feedback of $y_1 = 1$. In slot 2, $x_{3,2} = 1$, however, since user 3 is not active, there is

no write and $y_2 = 0$. In slot 3, users 1 and 3 are called upon to write, but only user 1 writes as user 3 is not active. The channel feedback over the three slots is $\mathbf{y} = [y_1, y_2, y_3]^\top = [1, 0, 1]^\top$. From this feedback, the knowledge of $K = 2$ and the accessing matrix $\mathbf{X}$, the following conclusions can be drawn.

- Because $x_{3,2} = 1$ and $y_2 = 0$, it can be concluded that user 3 is not active.
- Because $x_{1,3} = x_{3,3} = 1$ and $y_3 = 1$, it can be concluded that user 1 is active (as user 3 is inactive), also $\mathcal{G}_\mathbf{s} \nsubseteq \{2, 4\}$.
- The interaction in slot 1 only says that $\mathcal{G}_\mathbf{s} \nsubseteq \{3, 4\}$.
- Since K is known to be 2, we conclude that exactly one of users 2 and 4 must be active and the other inactive.
- Thus, the partition $\{\{1, 3\}, \{2, 4\}\}$ of $\mathcal{N}$ separates active nodes into different groups, and $\mathbf{z} = [1\ 2\ 1\ 2]^\top$ can be one of the results of decoding $\mathbf{y}$.

Observe that (unlike the restoration of $\mathcal{G}_\mathbf{s}$), we do not (and need not) know which among users 2 and 4 is active. Likewise although we happen to know that user 1 is active and user 3 is not, this knowledge is coincidental; the partition approach does not invest resources to seek this knowledge.

To have a more general formulation, the problem can be treated as a coding problem in multi-access channels from the information theoretic view, as shown in Fig. 2. Consider N users whose active states are given in a vector $\mathbf{s} \in \mathbb{S}_{K;N} \triangleq \{\mathbf{s} \in \{0, 1\}^N : \sum s_i = K\}$. The i-th row of $\mathbf{X}$, denoted by $\mathbf{x}_i^\top$ can be viewed as a codeword of user i (note that $\mathbf{x}_i$ is a column vector, we would like to use a row vector $\mathbf{x}_i^\top$ to represent the codeword, as is customary). It is also easy to see that user i sends $s_i\mathbf{x}_i^\top$ on the channel. The channel feedback is $\mathbf{y} = \bigvee_{i=1}^N s_i\mathbf{x}_i \triangleq [\bigvee_{i=1}^N s_i x_{i,t}]_{t=1}^T$. Then the decoded output is an ordered partition $\mathbf{z} \in \mathbb{Z}_{K;N}$, where:

$$\mathbb{Z}_{K;N} = \left\{\mathbf{z} \in \mathcal{K}^N : \forall 1 \le k \le K, \exists z_i = k\right\}$$

is the set of all possible K-ordered partition. Recall that the definition of $\mathbf{z}$ is equivalent to that of an order partition as in footnote 1. A *distortion function* is defined for any status vector $\mathbf{s} \in \mathbb{S}_{K;N}$ and a partition vector $\mathbf{z} \in \mathbb{Z}_{K;N}$ as follows:

$$d(\mathbf{s}, \mathbf{z}) = \begin{cases} 0, & \text{if } \forall i, j \in \mathcal{G}_\mathbf{s}, \quad (i \neq j) \Longrightarrow (z_i \neq z_j) \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

The objective is to design a proper matrix $\mathbf{X}$ and a corresponding decoding function $\mathbf{z} = g(\mathbf{y})$, so that $d(\mathbf{s}, g(\mathbf{y})) = 0$ for nearly all $\mathbf{s} \in \mathbb{S}_{K;N}$.

To simplify the notation, we write $\mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s}$, where $\otimes$ denotes Boolean matrix multiplication in which the traditional arithmetic multiplication and addition operations are replaced by logical AND and OR, respectively. For any given $\mathbf{s}$, we denote the set of all possible $\mathbf{z}$ as $\mathbb{Z}_{K;N}(\mathbf{s}) = \left\{\mathbf{z} \in \mathbb{Z}_{K;N} : d(\mathbf{s}, \mathbf{z}) = 0\right\}$. The set of all possible vectors $\mathbf{s}$ that are compatible with a given $\mathbf{z}$ to produce 0 distortion is denoted by $\mathbb{S}_{K;N}(\mathbf{z}) = \{\mathbf{s} \in \mathbb{S}_{K;N} : d(\mathbf{s}, \mathbf{z}) = 0\}$. In some situations, we will need to know the number of users, n_k, in a given group $k \in \mathcal{K}$. The set of all possible $\mathbf{z}$ with group sizes $(n_1, \ldots, n_K)$, where $\sum_{k=1}^K n_k = N$, is denoted by:

$$\mathbb{Z}_{K;N}(n_1, \ldots, n_K) \triangleq \left\{\mathbf{z} \in \mathbb{Z}_{K;N} : \left(\sum_{i=1}^N \mathbb{1}(z_i = k)\right) = n_k, 1 \le k \le K\right\};$$

here the indicator function $\mathbb{1}(A)$, which accepts a Boolean value A as input, is 1 if A is true, and 0 if A is false.

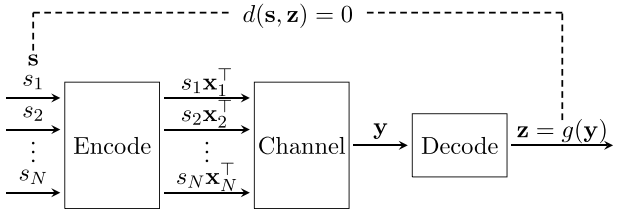

Fig. 2. Encoding-channel-decoding system with distortion criterion.

B. Performance Criteria

In this paper, we use a probabilistic model and consider average error. Assume each input $\mathbf{s} \in \mathbb{S}_{K;N}$ is with equal probability, i.e., $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$, where $\mathcal{U}(\mathbb{A})$ denotes the uniform distribution over a set $\mathbb{A}$. Thus $\forall \tilde{\mathbf{s}} \in \mathbb{S}_{K;N}$, $p_\mathbf{s}(\tilde{\mathbf{s}}) \triangleq \Pr(\mathbf{s} = \tilde{\mathbf{s}}) = 1/\binom{N}{K}$. For a given $\mathbf{X}$, the average error probability is defined as follows:

$$\begin{aligned} P_e^{(N)}(\mathbf{X}) &\triangleq \sum_{\mathbf{s} \in \mathbb{S}_{K;N}} p(\mathbf{s}) \Pr(d(\mathbf{s}, g(\mathbf{y})) \neq 0 | \mathbf{s}, \mathbf{X}) \\ &= \frac{1}{\binom{N}{K}} \sum_{\mathbf{s} \in \mathbb{S}_{K;N}} \sum_{\mathbf{y}} \mathbb{1}(d(\mathbf{s}, g(\mathbf{y})) \neq 0) \mathbb{1}(\mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s}) \end{aligned} \tag{2}$$

Note that we use $p(\mathbf{s})$ instead of $p_\mathbf{s}(\tilde{\mathbf{s}})$ for simplification. The first term $\mathbb{1}(d(\mathbf{s}, g(\mathbf{y})) \neq 0)$ reveals the effect of decoding, and the second term $\mathbb{1}(\mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s})$ the effect of the channel.

We define a number of slots $T_c^{(N)}$ to be achievable, if for any $T > T_c^{(N)}$, there exists a $N \times T$ matrix $\mathbf{X}^{(N)}$ and a decoding function $g^{(N)}(\cdot)$, such that $\lim_{N \to \infty} P_e^{(N)}(\mathbf{X}^{(N)}) = 0$. The aim is to find $T_c^{(N)}$, when $N \to \infty$.

Remark: In this problem, the objective is to restore a partition vector $\mathbf{z} \in \mathbb{Z}_{K,N}$, so that for given input $\mathbf{s}$, $d(\mathbf{s}, \mathbf{z}) = 0$. Whereas in group testing, the objective is to restore every user's state; i.e., the output should be a binary vector $\mathbf{z}_g \in \mathbb{S}_{K;N}$, and correct restoration means $\mathbf{z}_g = \mathbf{s}$. Thus, if we use the definition of distortion

$$d_g(\mathbf{s}, \mathbf{z}_g) \triangleq \mathbb{1}(\mathbf{z}_g = \mathbf{s}), \tag{3}$$

instead of the one defined in Equation (1), the problem above is exactly a noiseless group testing problem. Thus, the main difference between our partition problem and group testing problem lies in the different definitions of distortion functions or, more importantly, in the different forms of information to transmit. Furthermore, since knowing $\mathcal{G}_\mathbf{s}$ will always induce a correct partition of $\mathcal{N}$ by the distortion definition

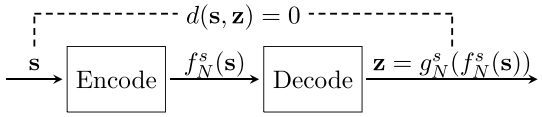

Fig. 3. Source coding part with distortion criterion.

in Equation (1), the partition problem requires no more information transfer than that in the case of group testing. In the next section, we rigorously analyze the amount of the information used to solve the partition problem.

A further remark can be made that the transmitted partition information will remain the same, if we slightly extend the definition of the output vector to be $\tilde{\mathbf{z}} \in \tilde{\mathbb{Z}}_{K;N}$, where:

$$\tilde{\mathbb{Z}}_{K;N} = \left\{\tilde{\mathbf{z}} \in (\{0\} \cup \mathcal{K})^N : \forall 1 \le k \le K, \exists z_i = k\right\}, \quad (4)$$

and the distortion function to be

$$\tilde{d}(\mathbf{s},\tilde{\mathbf{z}}) = \begin{cases} 0, & \forall i, j \in \mathcal{G}_{\mathbf{s}}, \\ & (i \neq j) \Longrightarrow (\tilde{z}_i, \tilde{z}_j \neq 0 \text{ and } \tilde{z}_i \neq \tilde{z}_j); \\ 1, & \text{otherwise} \end{cases} \quad (5)$$

this is shown in the next section and in the proof of Lemma 1 in Appendix A. The difference between this extended $\tilde{\mathbf{z}}$ and the original $\mathbf{z}$ is that we could have $\tilde{z}_i = 0$, indicating user i is inactive. In the main body of this paper, we will always use the original definition that $\mathbf{z} \in \mathbb{Z}_{K;N}$ and $d(\mathbf{s},\mathbf{z})$ is defined as in Equation (1).

IV. Source Coding

In this section, we first focus on the inputs and outputs of the system without considering channel effects (i.e., a centralized source coding scheme as illustrated in Fig. 3), to find the amount of information needed for describing the source with the purpose of partition. In other words, the purpose is to find a set of partitions $\mathcal{C}$ with minimum size, so that for nearly every possible $\mathbf{s} \in \mathbb{S}_{K;N}$, there is a partition $\mathbf{z} \in \mathcal{C}$ and $d(\mathbf{s},\mathbf{z}) = 0$. With the help of source codebook $\mathcal{C}$, for any unknown input $\mathbf{s}$, we can utilize the channel to check every partition in $\mathcal{C}$ to find the valid partition; details appear in the next section.

For group testing, the objective is to restore all states of users, if we use a *source codebook* $\mathcal{C}_g \triangleq \{\mathbf{s}_1, \ldots, \mathbf{s}_{L^{(N)}}\}$ to represent all $\mathbf{s} \in \mathbb{S}_{K;N}$, the codebook size $L^{(N)}$ should be $|\mathbb{S}_{K;N}| = \binom{N}{K}$. However, in the partition reservation system, for a given $\mathbf{z} \in \mathbb{Z}_{K;N}$, there can be more than one $\mathbf{s}$ so that $d(\mathbf{s},\mathbf{z}) = 0$. Actually when $\mathbf{z} \in \mathbb{Z}_{K;N}(n_1, \ldots, n_K)$, the number of possible active vectors so that $d(\mathbf{s},\mathbf{z}) = 0$ (for $\mathbf{s} \in \mathbb{S}_{K;N}(\mathbf{z})$) is $|\mathbb{S}_{K;N}(\mathbf{z})| = \Pi_{k=1}^K n_k$. Thus, we can use codebook with size smaller than $\mathbb{S}_{K;N}$ to represent the inputs. Strictly speaking, for $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$, we are seeking a source encoding function:

$$f_N^s : \mathbb{S}_{K;N} \to \{1, 2, \ldots, L^{(N)}\},$$

(we use superscript "s" to represent it is "source coding", and it is the same for the definition belows), and a source decoding function:

$$g_N^s : \{1, 2, \ldots, L^{(N)}\} \to \mathbb{Z}_{K;N},$$

so that we can map $\mathbf{s}$ to a decoding output $\mathbf{z} = g_N^s(f_N^s(\mathbf{s}))$, with the average source decoding error

$$P_e^{s,(N)} \triangleq \sum_{\mathbf{s} \in \mathbb{S}_{K;N}} p(\mathbf{s}) \mathbb{1}(d(\mathbf{s}, g_N^s(f_N^s(\mathbf{s}))) \neq 0) \quad (6)$$

such that $P_e^{s,(N)}$ approaches 0 as $N \to \infty$. Thus, we will call $(L^{(N)}, f_N^s, g_N^s)$ an achievable source code sequence for the uniform source $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$. The range of $g_N^s(\cdot)$ is defined as the source codebook. The minimum of $\log L^{(N)}$ for all achievable source code sequences will be called the partition information for $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$.

In this section, we first compute the minimum constrained mutual information, W_N^I, between $\mathbf{s}$ and valid partition $\mathbf{z}$ (see Lemma 1), and then prove the existence of an achievable source code sequence $(L^{(N)}, f_N^s, g_N^s)$ for a set of values $L^{(N)} > 2^{W_N^I}$ specified in Theorem 1.

Constrained mutual information is always related to the rate distortion problem [28], [29]. Thus, we first calculate the constrained minimum mutual information for $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$ and valid $\mathbf{z}$, i.e.,

$$W_N^I \triangleq \min_{p(\mathbf{z}|\mathbf{s}) \in \mathcal{P}_{z|s}} I(\mathbf{s},\mathbf{z}) \quad (7)$$

where the constraint is:

$$\mathcal{P}_{z|s} \triangleq \{p(\mathbf{z}|\mathbf{s}) : p(\mathbf{z}|\mathbf{s}) = 0, \text{if } d(\mathbf{s},\mathbf{z}) = 0\}, \quad (8)$$

which means only a valid partition $\mathbf{z}$ can be chosen for a given $\mathbf{s}$. The result corresponds to that of Hajek [15], when $N \to \infty$.

Lemma 1:

$$W_N^I \triangleq \min_{p(\mathbf{z}|\mathbf{s}) \in \mathcal{P}_{z|s}} I(\mathbf{s},\mathbf{z}) = \log \frac{\binom{N}{K}}{\prod_{k=1}^K n_k^*} \quad (9)$$

where

$$(n_1^*, \ldots, n_K^*) = \arg\left(\max_{n_k} \prod_{k=1}^K n_k\right), \quad (10)$$

$$\text{subject to } \sum_{k=1}^K n_k = N, \text{ and } \forall k \in \mathcal{K}, n_k \ge 1.$$

W_N^I can be achieved by choosing

$$\mathbf{z}|\mathbf{s} \sim \mathcal{U}\left(\mathbb{Z}_{K;N}(n_1^*, \ldots, n_K^*) \bigcap \mathbb{Z}_{K;N}(\mathbf{s})\right). \quad (11)$$

Equation (11) means that for any given $\mathbf{s}$, the partition $\mathbf{z}$ should be chosen from the "correct" set $\mathbb{Z}_{K;N}(\mathbf{s})$ under the constraint of $\mathcal{P}_{z|s}$, and to minimize the mutual information we require that $\mathbf{z} \in \mathbb{Z}_{K;N}(n_1^*, \ldots, n_K^*)$. This means that there are n_k^* users assigned to group k. The partition $\mathbf{z}$ can be chosen uniformly from the set satisfying these two conditions. In the proof, which appears in Appendix A, we first partition $\mathbb{Z}_{K;N}$ as $\bigcup_{(n_1,\ldots,n_K)} \mathbb{Z}_{K;N}(n_1, \ldots, n_K)$, and then for each set of partitions, log sum inequality is used to obtain the lower bound. For the achievability, we directly construct the optimal $p(\mathbf{z}|\mathbf{s})$ using Equation (11).

Theorem 1 (Source Coding): There exists a codebook $\{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$ of size $L^{(N)}$, and a source coding sequence $(L^{(N)}, f_N^s, g_N^s)$, so that for all N, the average source decoding

error probability is bounded by $P_e^{s,(N)} \leq e^{-2^{\left(\log L^{(N)} - W_N^I\right)}}$. *When* $\log L^{(N)} > W_N^I$ *and* $\left(\log L^{(N)} - W_N^I\right) \xrightarrow{N\to\infty} \infty$, *sequence* $(L^{(N)}, f_N^s, g_N^s)$ *is achievable.*

Proof Outline: The complete proof is in Appendix B. The core of the proof is to use a random coding method to construct the codebook $\{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$; in particular, we choose $\mathbf{z}_\ell$ i.i.d. from $\mathcal{U}\left(\mathbb{Z}_{K;N}(n_1^*, \ldots, n_K^*)\right)$, and show the average of $P_e^{s,(N)}$ over all possible codebooks satisfies the bound in Theorem 1, thus there must exist at least one codebook satisfying this bound. Then by assigning the source encoding function $f_N^s(\mathbf{s}) = \arg\min_{1\leq\ell\leq L^{(N)}} d(\mathbf{s}, \mathbf{z}_\ell)$, and the source decoding function $g_N^s(\ell) = \mathbf{z}_\ell$, we obtain the source coding sequence $(L^{(N)}, f_N^s, g_N^s)$ with the error probability bounded by Theorem 1.

From Theorem 1, we see that W_N^I can be used to measure the amount of asymptotic partition information of the source. It explicitly shows the partition information, as well as its difference from the required information to restore all states in further remarks.

Remark 1: For group testing, if we define $W_{G,N}^I$ as that in Equation (7), we have:

$$W_{G,N}^I = \log \binom{N}{K}$$

Thus $W_N^I = \log \binom{N}{K} - \log\left(\prod_{k=1}^K n_k^*\right)$ of partition problem is smaller by a $\log\left(\prod_{k=1}^K n_k^*\right)$ term than that of group testing. We next remark on the effect of the order of K (compared to N) on the achieved mutual information, as well as the error probability.

Remark 2: First, we derive the explicit expression for W_N^I. From the restriction on $[n_k]_{k=1}^K$ in Equation (10), it is easy to see without requiring n_k to be an integer that the optimal values of n_k are

$$n_1^* = n_2^* = \ldots = n_K^* = \frac{N}{K}.$$

Thus

$$W_N^I \geq \log \binom{N}{K} - \log \left(\frac{N}{K}\right)^K \tag{12}$$

The equality is achieved when K divides N, and it has a good approximation when $N \gg K$. Also, we have the inequalities:

$$\frac{\binom{N}{K}}{\left(\frac{N}{K}\right)^K} \overset{(a)}{\leq} \frac{K^K}{K!} \overset{(b)}{\leq} e^K, \tag{13}$$

Equality in Equation (13)(a) will be approximately achieved when $K \ll N$, and the equality of Equation (13)(b) requires $K \gg 1$.

Remark 3: When $K = O(N)$, e.g. $K = \eta N$ for a constant $0 < \eta < 1$, we have:

$$\lim_{N\to\infty} \frac{1}{N} W_N^I = -(1-\eta)\log(1-\eta) \tag{14}$$

$$\lim_{N\to\infty} \frac{1}{N} W_{G,N}^I = -(1-\eta)\log(1-\eta) - \eta\log\eta \triangleq H(\eta) \tag{15}$$

They are obtained by a tight bound of $\binom{N}{K}$ derived by Wozencraft and Reiffen [28], (see Section 17.5). Thus we can define an achievable source information rate R_s for the partition problem (note the unit of the rate defined here is bits/user), so that for any $R \geq R_s + \xi$, where $\xi > 0$ is any constant, there exists an achievable coding sequence $(L^{(N)} = 2^{NR}, f_N^s, g_N^s)$, and

$$P_e^{s,(N)} \to 0, \text{ when } N \to \infty$$

By Theorem 1 and Equation (14), we can see that when $K = \eta N$, we have $R_s = -(1-\eta)\log(1-\eta)$, since we can always construct the achievable coding sequence of $L^{(N)} = 2^{NR}$ that, for all $\xi > 0$, and $\forall R \geq R_s + \xi$, satisfies

$$P_e^{s,(N)} \leq e^{-2^{N(R-R_s)}} \text{ and } \lim_{N\to\infty} P_e^{s,(N)} = 0$$

Note that the error is doubly exponential. While for group testing, if we define R_s^g similarly to R_s, we can see by Equations (14) and (15) that $R_s^g = R_s + (-\eta\log\eta) > R_s$. Thus, we need higher rate to represent the states of users than to partition them.

Remark 4: When $K = o(N)$, $\lim_{N\to\infty} W_N^I = \log\frac{K^K}{K!}$. A special example is that K is a constant, then $\lim_{N\to\infty} W_N^I$ is also a constant. We can see the proposed achievable rate $R_s = 0$, since $\frac{1}{N} W_N^I \leq \frac{K}{N}\log e \to 0$. By Theorem 1, for any $L^{(N)} = f(N)$ (where $f(N)$ is a function satisfying $f(N) \xrightarrow{N\to\infty} \infty$), we can always construct a source coding sequence with codebook size $L^{(N)} = f(N)$, and

$$P_e^{s,(N)} \leq e^{-2^{\left(\log f(N) - \log\frac{K^K}{K!}\right)}} \to 0, \text{ when } N \to \infty$$

We can choose $L^{(N)}$ to be of any order of N to guarantee the convergence of $P_e^{s,(N)}$; for example, $L^{(N)} = \log\log N$. On the other hand, group testing requires that $L^{(N)} = \binom{N}{K}$ to represent the source, which can be much larger than that of partition problem. However, different choices of $f(N)$ will influence the speed of convergence, e.g., if an exponential convergence speed is required, i.e., $P_e^{s,(N)} \leq e^{-\Delta N}$ for some $\Delta > 0$, then $L^{(N)} = O(N)$.

V. The Brute Force Method

Given a randomly generated source codebook for the source coding problem, here we propose a corresponding channel coding scheme. In this scheme, the channel codebook $\mathbf{X}$ is created by first collecting all partitions (or codewords) in the source codebook; the decoder then checks each partition (or source codeword) exhaustedly with the help of the Boolean channel. More specifically, if the partition set $\mathcal{C}$ is given as a source codebook, and T_0 slots are needed to check if a partition $\mathbf{z} \in \mathcal{C}$ is a valid partition, then at most $T = T_0 \cdot |\mathcal{C}|$ slots are needed to check all partitions in $\mathcal{C}$. This is the brute force method.

For a given $L^{(N)}$, we can find a source codebook $\{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$ to represent the source under error probability $P_e^{s,(N)}$ (using Theorem 1). Thus if a matrix $\mathbf{X}$ is designed to check, one by one, whether $\mathbf{z}_\ell$ is the correct output, the average error probability $P_e^{(N)}$ will behave the same as $P_e^{s,(N)}$, and thus

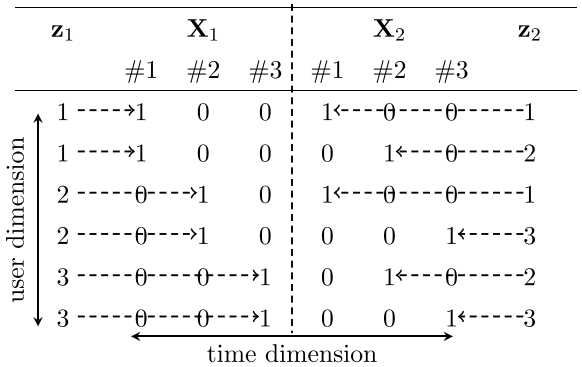

Fig. 4. Example of the generation of $\mathbf{X}$ in brute force method, where $N = 6$, $K = 3$, and source codebook $\{\mathbf{z}_1, \mathbf{z}_2\}$ of size $L^{(N)} = 2$ is chosen.

approaches zero when $\log L^{(N)} - W_N^I \to \infty$ as $N \to \infty$. The following holds for the brute force method:

1) *Source Coding:* For $L^{(N)}$, choose the codebook $\{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$, and the source coding sequence $(L^{(N)}, f_N^s, g_N^s)$ based on Theorem 1.
2) *Joint Coding:* Generate $\mathbf{X}$ by $L^{(N)}$ submatrices of dimension $N \times K$,

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_{L^{(N)}}].$$

Thus, the dimension of $\mathbf{X}$ is $N \times T$, where $T = KL^{(N)}$ ($T_0 = K$ to check each possible partition). Each $\mathbf{X}_\ell$ is an $N \times K$ matrix, so that $\forall 1 \le i \le N,\ 1 \le k \le K$, the (i,k)-th element of $\mathbf{X}_\ell$, denoted by $x_{\ell;i,k}$, satisfies:

$$x_{\ell;i,k} = \begin{cases} 1, & \text{if } z_{\ell;i} = k; \\ 0, & \text{otherwise.} \end{cases}$$

See Fig. 4 for an example.
3) *Decoding:* Now the outputs are separated into $L^{(N)}$ blocks:

$$\mathbf{y} = [\mathbf{y}_1; \ldots; \mathbf{y}_{L^{(N)}}],$$

and

$$\mathbf{y}_\ell = \mathbf{X}_\ell^\top \otimes \mathbf{s}$$

is a $K \times 1$ column vector. If there exists $\mathbf{y}_\ell = \mathbf{1}_{K\times 1}$, where $\mathbf{1}_{K\times 1}$ is a $K \times 1$ column vector with all components equal to 1, then the joint decoder is $g(\mathbf{y}) = \mathbf{z}_\ell$; if there exist more than one, we can select one of them, e.g., the first one; otherwise there is decoding error.

Note that if $\mathbf{y}_\ell = \mathbf{1}_{K\times 1}$, then there exists at least one active user in each of k groups assigned by $\mathbf{z}_\ell$. And since we know that there are exactly K active users, only one active user is assigned to each group. Thus definitely $d(\mathbf{s}, \mathbf{z}_\ell) = 0$ (based on the following fact):

$$\forall i \ne j \in \mathcal{G}_\mathbf{s}, \quad z_i \ne z_j \iff \bigcup_{i\in\mathcal{G}_\mathbf{s}} \{z_i\} = \mathcal{K}.$$

Clearly, in the brute force method the number of channel uses is $T_{BF} = KL^{(N)}$. In addition, for this method there exists $\mathbf{z}_\ell$ in codebook $\{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$ so that $d(\mathbf{s}, \mathbf{z}_\ell) = 0$, if and only if $d(\mathbf{s}, g(\mathbf{y})) = 0$. Consequently, the average error of the brute force method is the same as centralized source coding. Based on the analysis of centralized source coding as in Theorem 1, we have

Theorem 2 (Brute Force Method): For the brute force method, if the size of centralized source codebook is $L^{(N)}$, then $T_{BF} = KL^{(N)}$, and the average error probability is $P_e^{(N)} \le e^{-\left(\frac{T_{BF}}{K}|dle/2^{W_N^I}\right)} = e^{-2^{\left(\log L^{(N)} - W_N^I\right)}}$.

Although the brute force method is very simple and clearly not optimal, it highlights some features of the partition problem. First, if K is a fixed number, then as stated in Remark 4 in the last section, only $T_{BF} = \frac{K^{K+1}}{K!} f(N)$ is needed for the convergence of $P_e^{(N)}$(since $P_e^{(N)} \le e^{-f(N)}$), where $\frac{K^{K+1}}{K!}$ is a constant and $f(N)$ is any function satisfying $\lim_{N\to\infty} f(N) = \infty$. In this case, the threshold effect of the convergence doesn't exist as in group testing or compressive sensing [4], and the choice of $f(N)$ is related to the speed of convergence of $P_e^{(N)}$. However, when K is large, (e.g. when $1 \ll K \ll N$, $2^{W_N^I} \to e^K$), T_{BF} should be larger than Ke^K to guarantee the convergence of $P_e^{(N)}$; here T_{BF} may be even larger than the time $T_G = O(K \log N)$ needed for group testing. This is to be expected as the brute force method is not optimal. In particular, when K increases, the size of the centralized source codebook increases exponentially, and it becomes exceedingly inefficient to check each element one by one.

VI. Random Coding and Reformulation as Hypergraph

The brute force method was inspired by a centralized source coding and it works well only for small K. To find the achievable bound of T for general case, we design the code from another approach, by randomly generating $\mathbf{X}$ and then employing the optimal Bayesian decoding. However, for a more insightful approach to derive an achievable rate, a new angle from graph theory is proposed in this section, which transforms the effect of the channel to a series of operations on hypergraphs. It is shown that seeking an acceptable partition is equivalent to obtaining a common strong colorable hypergraph by all users, and then coloring this hypergraph. Because we are only concerned about an achievable rate, the computational cost associated with the coloring is not relevant for our framework.

A. Random Coding and the Optimal Bayesian Decoding

Random coding is frequently used in the proof of achievability in information theory, including for group testing [26]. The binary matrix $\mathbf{X}$ is generated randomly, where each element $x_{i,t} \sim \mathcal{B}(p)$ follows the i.i.d Bernoulli distribution with p parameter (other distributions of $\mathbf{X}$ can also be considered, but that is beyond the scope of this paper). Let the probability of $\mathbf{X}$ be $Q(\mathbf{X})$. Then the average probability of error over the realization of $\mathbf{X}$ is given by:

$$\begin{aligned} P_e^{(N)} &= \sum_{\mathbf{X}} Q(\mathbf{X}) P_e^{(N)}(\mathbf{X}) \\ &= \sum_{\mathbf{X}} Q(\mathbf{X}) \sum_{\mathbf{s}\in\mathbb{S}_{K;N}} \sum_{\mathbf{y}} p(\mathbf{s}) p_{y|s;X}(\mathbf{y}|\mathbf{s}) \mathbb{1}(d(\mathbf{s}, g(\mathbf{y})) \ne 0) \\ &\overset{(a)}{=} \sum_{\mathbf{X}} Q(\mathbf{X}) \sum_{\mathbf{y}} p_{y|s;X}(\mathbf{y}|\mathbf{s}_0) \mathbb{1}(d(\mathbf{s}_0, g(\mathbf{y})) \ne 0) \end{aligned} \tag{16}$$

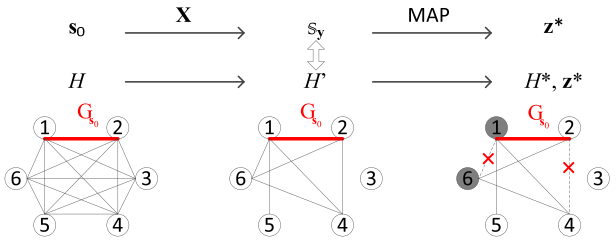

Fig. 5. Reformulation from hypergraph.

Since we do not consider observation noise in this paper,

$$p_{y|s;X}(\mathbf{y}|\mathbf{s}) = \mathbb{1}(\mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s}).$$

The equality of Equation (16)(a) above follows from the symmetry of the generation of $\mathbf{X}$, so we can choose any given $\mathbf{s}_0$ as input to analyze. We will choose $\mathcal{G}_{\mathbf{s}_0} = \{1, \ldots, K\}$ in the rest of the paper. Since the derived achievable $T_c^{(N)}$ for random coding is $O(\log N)$, we define an achievable rate S_c, so that for any T satisfying $\frac{\log(N)}{T} \leq S_c - \xi$ (where $\xi > 0$ is an arbitrary constant), we have $P_e^{(N)} \xrightarrow{N\to\infty} 0$. Which also implies there exists an $\mathbf{X}^*$ such that $P_e^{(N)}(\mathbf{X}^*) \xrightarrow{N\to\infty} 0$. In this section, we will derive such an S_c.

We employ the optimal Bayesian decoding, for which given feedback $\mathbf{y}$, we choose $\mathbf{z}^* = g(\mathbf{y})$ so that $\forall \mathbf{z} \neq \mathbf{z}^* \in \mathbb{Z}_{K;N}$, the following holds

$$\sum_{\mathbf{s}\in\mathbb{S}_{K;N}} \mathbb{1}\left(d(\mathbf{s}, \mathbf{z}^*) = 0\right) p_{y|s;X}(\mathbf{y}|\mathbf{s}) \geq \sum_{\mathbf{s}\in\mathbb{S}_{K;N}} \mathbb{1}\left(d(\mathbf{s}, \mathbf{z}) = 0\right) p_{y|s;X}(\mathbf{y}|\mathbf{s}) \tag{17}$$

If there is more than one $\mathbf{z}^*$ with the maximum value, then choose any one. Note that here we search all possible $\mathbf{z}^*$ in all possible $\mathbf{z} \in \mathbb{Z}_{K;N}$; however, considering the source coding results, we can just search $\mathbf{z}^* \in \mathbb{Z}_{K;N}(n_1^*, \ldots, n_K^*)$ without loss of generality.

As seen in the definition of the optimal Bayesian decoding, to find the output $\mathbf{z}$, we should count all $\mathbf{s} \in \mathbb{S}(\mathbf{z})$ satisfying $\mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s}$. Many $\mathbf{s} \in \mathbb{S}(\mathbf{z})$ support the same set of active users. Thus, it is extremely difficult to compare the posterior probability for different $\mathbf{z}$. This obstacle arises because in the Bayesian decoding, few inherent structures of the problem are found and utilized. To further reveal the inherent structure of the problem, a novel formulation from the perspective of hypergraph is proposed in the next section, which proves to be helpful in reducing the complexity of performance analysis.

B. Reformulation as Hypergraph

The process of random coding is illustrated in the upper part of Fig. 5. For an input $\mathbf{s}_0$, the channel output $\mathbf{y} = \bigvee_{i\in\mathcal{G}_{\mathbf{s}_0}} \mathbf{x}_i$ is observed, and then a candidate subset of $\mathbb{S}_{K;N}$ that is capable of generating $\mathbf{y}$ can be inferred as follows:

$$\mathbb{S}_\mathbf{y} = \left\{\mathbf{s} \in \mathbb{S}_{K;N} : \mathbf{y} = \mathbf{X}^\top \otimes \mathbf{s}\right\}$$

the optimal Bayesian decoder tries to find $\mathbf{z}^*$ with the largest number of $\mathbf{s} \in \mathbb{S}_\mathbf{y}$ satisfying $d(\mathbf{z}^*, \mathbf{s}) = 0$.

This process can be viewed in terms of hypergraphs, as shown in Fig. 5.

1) *Source:* Since all real sources $\mathbf{s}_0 \in \mathbb{S}_{K;N}$ are equiprobable, a complete K-uniform hypergraph $H(V(H), E(H))$ can be used to express the knowledge of the source before observation, where the set of nodes $V(H) = \mathcal{N}$ represents N users, and the set of hyperedges $E(H) = \{e \subseteq V(H) : |e| = K\}$ represents all possible inputs [30], [31]. It means every hyperedge in H could be $\mathbf{s}_0$. Actually the real input is just an edge $\mathcal{G}_{\mathbf{s}_0} \in E(H)$, the objective of group testing is to find exactly this edge to obtain every user's state; while for partition reservation system, the objective is to separate each vertex of $\mathcal{G}_{\mathbf{s}_0}$.

2) *Transmission and Observation:* The transmission and corresponding observation can be seen as a series of edge deleting operations on the hypergraphs. Because after observing each feedback y_t, $1 \leq t \leq T$, some $\mathbf{s}$ could be determined to be not possible, and the candidate set $\mathbb{S}_\mathbf{y}$ could shrink. A sub-hypergraph $H'(V(H'), E(H')) \subseteq H(V(H), E(H))$ is used to denote the candidate set $\mathbb{S}_\mathbf{y}$ after observing the feedback $\mathbf{y}$. Note that we consider the node set $V(H') = V(H)$ to be invariant, but actually there will be many isolated nodes in $V(H')$ with zero degree. The details of the operations will be shown in the next subsection. Note that for the noiseless case that we consider, we always have $\mathbf{s}_0 \in \mathbb{S}_\mathbf{y}$, so $\mathcal{G}_{\mathbf{s}_0} \in H'$.

3) *Partition:* Finally, the partition $\mathbf{z}^*$ should be decided by observing H'. First, we introduce the concept of strong coloring. A strong coloring of a hypergraph H is a map $\Psi : V(H) \to \mathbb{N}^+$, such that for any vertices $u, v \in e$ for some $e \in E(H)$, $\Psi(u) \neq \Psi(v)$. The value of $\Psi(u)$ is called the color of node u. In other words, all vertices of any hyperedge should have different colors. The corresponding strong chromatic number $\chi_s(H)$ is the least number of colors so that H has a proper strong coloring [32]. Obviously for a K-uniform hypergraph, $\chi_s(H) \geq K$. We call a strong coloring with K colors to be K-strong coloring. If z_i^* is viewed as a color of node i, then $\mathbf{z}^* \in \mathbb{Z}_{K;N}$ gives a coloring of $V(H)$ with K colors.

For the Bayesian decoding in (17), the method of finding $\mathbf{z}^*$ from $\mathbb{S}_\mathbf{y}$ is equivalent to finding a hypergraph $H^*(V(H^*), E(H^*)) \subseteq H'(V(H), E(H'))$, such that $\chi_s(H^*) = K$, i.e., H^* is K-strong colorable, and the number of deleted edges $|E(H')\backslash E(H^*)|$ is minimum. Then the output $\mathbf{z}^*$ can be any strong coloring of H^*.

From the perspective of hypergraphs, the process generates $H \to H' \to (H^*, \mathbf{z}^*)$, corresponding to the expression from vectors $\mathbf{s}_0 \to \mathbb{S}_\mathbf{y} \to \mathbf{z}^*$. This process is shown in Fig. 5 through an example of $N = 6$, $K = 2$. Note that the hypergraph becomes a graph when $K = 2$. Compared with group testing, whose objective is to obtain $H^* = H'$ with only one edge $\mathcal{G}_{\mathbf{s}_0}$ by deleting edges through transmissions and observations, our partition problem allows H' and H^* to have more edges, so less effort is needed to delete edges, which translates to higher achievable rate than that for group testing. We observe that $\mathbf{z}^*$ is correct iff $\mathcal{G}_{\mathbf{s}_0} \in E(H^*)$ and H^* is K-strong colorable, we will use this equivalence in our analysis to determine if the decoding is correct.

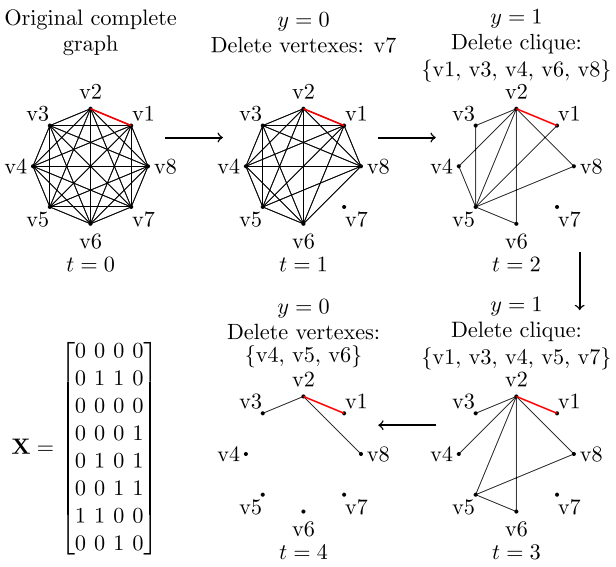

Fig. 6. Example of $\mathbf{X}$ effects on the operations of a graph. (Here $N = 8$, $K = 2$, $T = 4$, and $\mathcal{G}_{\mathbf{s}_0} = \{1, 2\}$.)

From the view point of the algorithm, first all users obtain a common "good" H' which will lead to a correct partition. Second, we obtain H^* and choose a common consistent $\mathbf{z}^*$. The second step, to obtain H^* by deleting the minimum number of edges from H' and finding the K-strong coloring, does not influence the transmission time needed to arrive at a "good" H'. This is because once all users have the same copy of H', the remaining computation, including removal of edges and coloring, can be done locally without further expending communication resources. A further explanation of operations of the deleting edges is introduced in the next subsection.

C. Reduction Step: Obtaining H' From H

The effect of transmissions on the basis of matrix $\mathbf{X}$ and observation of the channel feedback result in two hypergraph operations: deleting vertices and deleting a clique. Assume at time t, that the set of users transmitting 1 is $\mathcal{G}_{\mathbf{X}}(t) = \{i \in \mathcal{N} : x_{i,t} = 1\}$. The operation at time t can be classified based on the feedback y_t as follows.

1) If $y_t = 0$, then none of the users in $\mathcal{G}_{\mathbf{X}}(t)$ is active, so the corresponding vertices (and hyperedges) can be deleted; i.e., all hyperedges containing these vertices can be deleted.
2) If $y_t = 1$, it implies that at least one active user is transmitting 1 at time t. Here every hyperedge consisting only of vertices from $\mathcal{N} \setminus \mathcal{G}_{\mathbf{X}}(t)$ can be deleted.

The vertices and hyperedges removal are illustrated in an example in Fig. 6 where $K = 2$ means the hypergraph is a graph. There are 8 users and 4 slots are used for transmission. We can see the edges removing process starting from a complete graph at $t = 0$, to a graph of only 3 edges at time $t = 4$. At $t = 1, 4$, when $y_t = 0$, the corresponding vertices are removed, while at time $t = 2, 3$ we have $y_t = 1$ and hypercliques are removed.

Now it is clear that our problem can be viewed as a K-strong hypergraph coloring problem, and the objective is to schedule a series of edge removing operations efficiently to construct such a hypergraph so that all K active users could be assigned a unique color (or partition group). In next section, a special case of $K = 2$ is solved; even in this simple case, the problem is nontrivial.

VII. Random Coding for $K = 2$

For $K = 2$, two sub-optimal decoding methods inspired by the Bayesian decoding are proposed to further simplify the calculation.

A. Two Simplified Decoding Methods

In the optimal Bayesian decoding, the decoder will find a K-strong colorable graph H^* from H' by deleting the minimum number of edges, and the decoding result is correct if $\mathcal{G}_{\mathbf{s}_0} \in H^*$. For $K = 2$, hypergraph H^* is a graph and 2-strong colorability is equivalent to H^* being bipartite, or equivalently, having *no odd cycles*. Without loss of generality, assume $\mathcal{G}_{\mathbf{s}_0} = \{1, 2\}$. Odd cycles can be of three kinds:

Type 1) The cycle contains vertices 1 and 2; the cycle may or may not contain edge $\{1, 2\}$;

Type 2) The cycle contains only one of the vertices 1 and 2;

Type 3) The cycle contains neither vertex 1 nor 2.

For simplicity call any odd cycle that contains edge $\{1, 2\}$ (a kind of Type-1 odd cycle) an "odd1 cycle". Since $\{1, 2\}$ always exists in H' for our channel model, it is easy to see H' contains no Type-1 cycles iff H' contains no odd1 cycles. Thus in the rest of paper, we just consider the existence of odd1 cycles and Type-2 and 3 odd cycles. We can assert that if there is no odd1 cycle in H', then the decoding result is correct. The reason is that the optimal Bayesian decoding breaks all odd cycles in H' to get H^* by deleting least edges. If there is no odd1 cycle in H', set $\mathcal{G}_{\mathbf{s}_0}$ will not be deleted during this process. Thus, $\mathcal{G}_{\mathbf{s}_0} \in H^*$, which implies correct decoding. Thus, we have

$$P_e^{(N)} \le P_e^{odd1} \triangleq \sum_{\mathbf{X}} Q(\mathbf{X}) \Pr(H' \text{ contains odd1 cycles} | \mathbf{X}, \mathbf{s}_0)$$
$$\le P_e^{odd} \triangleq \sum_{\mathbf{X}} Q(\mathbf{X}) \Pr(H' \text{ contains odd cycles} | \mathbf{X}, \mathbf{s}_0)$$

In the following, P_e^{odd} and P_e^{odd1} are both upper bounded by their respective union bounds, and it is shown their upper bounds are nearly the same when $N \to \infty$, which points to the possibility of using a suboptimal decoding method to advantage: when H' is 2-colorable, find any $\mathbf{z}$ consistent with it; otherwise announce an error. The reason is if the optimal Bayesian decoding is used, it is necessary to obtain H^* by deleting the minimum number of edges of H', which is a NP Hard problem [33]; however, it is easy to judge whether H' is a bipartite graph in a linear number of steps in N. So while the suboptimal decoding method needs more channel use, it is easier to compute.

B. Main Result: Achievable Bound of T for $K = 2$ Case

To upper bound $P_e^{(N)}$ by P_e^{odd1}, let

$$C(p) = -(1 - (1 - p)^2) \log \varphi(p) - (1 - p)^2 \log(1 - p)$$

where

$$\varphi(p) = \frac{p + \sqrt{4p - 3p^2}}{2}. \tag{18}$$

We have the following lemma:

Lemma 2: For $K = 2$ and for any constant $\xi > 0$ such that $\frac{\log N}{T} \le S_c - \xi$ and $S_c \triangleq \max_{0 \le p \le 1} C(p)$, we have $P_e^{(N)} \le P_e^{odd1}$, and $\lim_{N\to\infty} P_e^{odd1} = 0$.

And similarly, by bounding $P_e^{(N)}$ by P_e^{odd}, we have the following theorem:

Theorem 3: For $K = 2$ and for any constant $\xi > 0$ such that $\frac{\log N}{T} \le S_c - \xi$, we will have $P_e^{(N)} \le P_e^{odd}$, and $\lim_{N\to\infty} P_e^{odd} = 0$.

The proofs of Lemma 2 and Theorem 3 are given in Appendix C. In fact, if the elements of $\mathbf{X}$ are generated i.i.d. by a Bernoulli distribution of parameter p, then P_e^{odd1} and P_e^{odd} approach 0, if $\frac{\log N}{T} \le C(p) - \xi$; this implies $S_c = \max_p C(p)$. We can see the achievable bound that makes $P_e^{(N)} \to 0$ is S_c for both methods. The main idea in the proof is to calculate the probability of existence of a particular odd cycle in H'; this calculation is similar for the three types of odd cycles. Observe that $\varphi(p)$ in Equation (18) is related to the solution of the extended Fibonacci numbers; this observation could provide additional insight.

A sketch of the proof of Lemma 2 is given below. The proof of Theorem 3 follows the same structures:

1) Consider the problem conditioning on $[\mathbf{x}_1, \mathbf{x}_2]$ in a strong typical set $\mathcal{A}_\epsilon^{(T)}$; this will simplify the algebra. Assume the probability of existence of a particular odd1 cycle of M vertices in H' to be $P_{e;M}$; there are $\binom{N-2}{M-2}(M-2)! \le N^{M-2}$ such odd cycles and all of them are equiprobable. Thus,

$$P_e^{odd1} \le \sum_{M \ge 3, M \text{ is odd}} 2^{(M-2)\log N} P_{e;M} + \Pr([\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) \tag{19}$$

Since $\Pr([\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) \le 2^{-q(p,\epsilon)T}$, and $2^{-q(p,\epsilon)T} \xrightarrow{T\to\infty} 0$, where $q(p, \epsilon)$ is some constant, according to the properties of strong typical set [34]. We will show that $P_{e;M} \le 2^{-(M-2)C(p)T}$. Thus when $\log N < (C(p) - \xi)T$, we have $2^{(M-2)\log N} \times P_{e;M} \le 2^{-(M-2)\xi}$; this implies the $P_e^{odd1} \to 0$. Note that specifically, P_e^{odd1} goes to 0 exponentially with T, and polynomially with N; specifically, $P_e^{odd1} \le 2^{-\Delta_1 T} = \frac{1}{N^{\Delta_2}}$, where Δ_1 and Δ_2 are constants.

3) Divide the T slots into four parts $\mathcal{T}_{u,v} = \{t : (x_{1,t}, x_{2,t}) = (u, v)\}$, for the four different $(u, v) \in \{0, 1\}^2$, according to the codewords of the real input $[\mathbf{x}_1, \mathbf{x}_2]$. In the strong typical set, we just need to consider when $|\mathcal{T}_{u,v}| \approx p_x(u)p_x(v)T$, where $p_x(u) \triangleq p \cdot \mathbb{1}(u = 1) + (1 - p) \cdot \mathbb{1}(u = 0)$ is the probability distribution of the Bernoulli variable. And due to symmetry and independence of the generation of $\mathbf{X}$, for any (u, v), we just need to consider an arbitrary slot $t \in \mathcal{T}_{u,v}$. 2) For $t \in \mathcal{T}_{u,v}$, denote by $\mu_{u,v;M}$ the probability that an odd1 cycle of length M will not be deleted by the operations. Then

$$P_{e;M} = \Pi_{u,v} \left(\mu_{u,v;M}\right)^{|\mathcal{T}_{u,v}|} \approx \Pi_{u,v} \left(\mu_{u,v;M}\right)^{p_x(u)p_x(v)T} \tag{20}$$

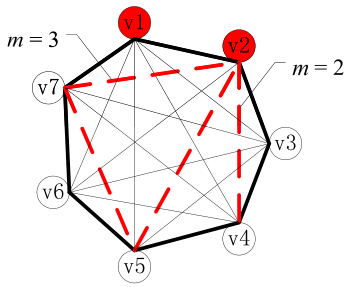

Fig. 7. Random clique deletion while keeping particular odd cycle at a particular t such that $\mathbf{y}(t) = 1$. (Here the size of the odd cycle is $M = 7$, $K = 2$, only the cliques of size 2 (edge) or 3 (triangle) consisting of non consecutive vertices can be deleted. ((2, 4) and (2, 5, 7) for example.))

4) We have shown that for all t, where $y_t = 0$, (i.e., $t \in \bigcup_{(u,v)\neq(0,0)} \mathcal{T}_{u,v}$, $\mu_{u,v;M} = (1 - p)^{M-2}$), and exponent $p_x(0)p_x(0) = (1 - p)^2$, thus

$$\left(\mu_{0,0;M}\right)^{p_x(0)p_x(0)T} = (1 - p)^{(M-2)(1-p)^2 T} = 2^{(M-2)T(1-p)^2 \log(1-p)} \tag{21}$$

For $y_t = 1$, (i.e., $t \in \mathcal{T}_{u,v}$, $(u, v) \neq (0, 0)$), we have shown that $\mu_{u,v;M} = \varphi^{M-2}(p)$, and $\sum_{(u,v)\neq(0,0)} p_x(u)p_x(v) = 1 - (1 - p)^2$. Thus,

$$\Pi_{(u,v)\neq 0} \left(\mu_{u,v;M}\right)^{p_x(u)p_x(v)T} = 2^{(M-2)T(1-(1-p)^2)\log\varphi(p)} \tag{22}$$

Then, combining Equations (21) and (22), we obtain $P_{e;M} \le 2^{-(M-2)C(p)T}$, which completes the proof.

We now provide an intuitive explanation. The result in Lemma 2 can be expressed as

$$\forall p,\ T > \frac{\log N^{M-2}}{(M-2)C(p)}$$

Intuitively, we have at most N^{M-2} odd1 cycles of length M. After ideally eliminating all of them, the error probability becomes 0. Thus, $\log N^{M-2}$ can be viewed as an upper bound on source information, that describes the uncertainty of odd1 cycles; $(M - 2)C(p)$ can be viewed as the information transmitting rate of the channel, which represents the speed of eliminating the uncertainty of odd cycles with M vertices.

To further explain the meaning of $(M-2)C(p)$, we should use the effect of $\mathbf{X}$ on hypergraphs (see in Section VI-C). If a given odd1 cycle $H_{e;M}$ with M vertices exists in H', for $1 \le t \le T$, then none of the M vertices, or the cliques containing the edges of $H_{e;M}$ can be deleted. See an example in Fig. 7, where $H_{e;M}$ is the outer boundary. It will not be removed if none of the edges are deleted; here the clique to be deleted should not contain consecutive vertices on the outer boundary.

At any slot t, vertices are deleted if $y_t = 0$; the probability of this happening is $(1 - p)^2$. For a particular t with $y_t = 0$, an inactive vertex i is deleted, only if $x_{i,t} = 1$, so the probability that all M vertices are maintained at time t is $\mu_{0,0;M} = (1 - p)^{M-2}$.

On the other hand, all the edges of the odd cycle $H_{e;M}$ cannot be deleted by the clique deleting operation. At any slot t with $y_t = 1$, whose probability is $1-(1-p)^2$, there are 3 different cases to consider, $(x_{1,t}, x_{2,t}) \in \{(0,0),(1,0),(0,1)\}$, whose analysis is similar. Consider $(x_{1,t}, x_{2,t}) = (1,1)$. Assume that odd cycle $H_{e;M} = (1, 2, i_1, \ldots, i_{M-2})$, so at any slot t, the probability that $H_{e;M}$ is not removed by clique deletion can be derived to be:

$$\begin{aligned}\mu_{1,1;M} &= 1 - \Pr(H_{e;M} \text{ is removed at slot } t | t \in \mathcal{T}_{1,1}) \\ &= 1 - \Pr(\exists w \in \{1,\ldots,M-3\}, \\ &\qquad (x_{i_w}(t), x_{i_{w+1}}(t)) = (0,0)) \\ &\overset{(a)}{=} \frac{1}{p} F(M,p) \leq \varphi(p)^{M-2}\end{aligned}$$

The derivation of Equation (a) is shown in Appendix C, with

$$\begin{aligned}F(k,p) &= \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-j}{j} p^{k-1-j}(1-p)^j \\ &= \frac{\varphi(p)^k - \psi(p)^k}{\varphi(p) - \psi(p)},\end{aligned}$$

and

$$\varphi(p) = \frac{p + \sqrt{4p-3p^2}}{2}; \quad \psi(p) = \frac{p - \sqrt{4p-3p^2}}{2}$$

as the solution to a generalized Fibonacci sequence [35]. In fact, $\frac{1}{p}F(k+2,p)$ is the probability that there are no two consecutive 0s in a p-Bernoulli sequence of length k. This feature of Fibonacci sequences has also been used in generating codes without consecutive 1s, known as Fibonacci coding. The other probabilities $\mu_{1,0;M}$ and $\mu_{0,1;M}$ can be derived similarly. Thus, we can see $(M-2)C(p)$ can be explained as the rate of deleting vertices or cliques for an odd1 cycle with M vertices from above.

Lemma 1 and Theorem 2 above reveal the internal structure of the partition problem. Partitionability is related to the absence of odd cycles, and $\mathbf{X}$ is constructed to remove odd cycles by deleting vertices or cliques. The Fibonacci structure is related to consecutive 0s in Bernoulli sequences; thus the Fibonacci structures may be a key factor in the partition problem and could be extended to more general cases with $K > 2$. In the next section, the efficiency is compared with random coding based group testing approach.

VIII. Comparison

As in introduction, our partition reservation has close relation to direct transmission and group testing. Since the average error considered in direct transmission system is not the same as that used in this paper (see Equation (2)), we compare our result only with group testing.

Atia and Saligrama [26] have proved the achievable rate for group testing with random coding, which shows that if for any $\xi > 0$ and $\frac{\log N}{T} \leq S_{cg} - \xi$, $S_{cg} \triangleq \max_p C_g(p)$, where

$$C_g(p) = \min\left\{(1-p)H(p), \frac{1}{2}H((1-p)^2)\right\},$$

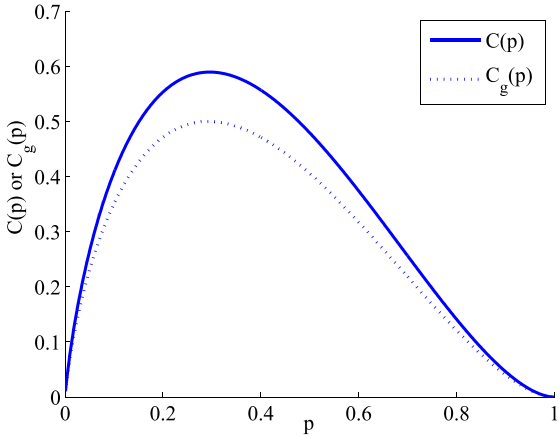

Fig. 8. Comparison of $C(p)$ and $C_g(p)$.

the average error probability $P_e^{(N)} \to 0$ (Also, they show S_{cg} to be a capacity). From Fig. 8, we can see $C_g(p) < C(p)$ for any $0 < p < 1$. In particular, $S_c = \max_p C(p) = 0.5 < 0.5896 = \max_p C_g(p) = S_{cg}$; i.e., our achievable rate is always larger than the capacity of group testing with random coding in the noiseless case. Further, expressing the upper bound of the error probability as only a function of N, the diminishing speeds $P_e^{(N)} \to 0$ of group testing and partition reservation are both polynomial in N; i.e., $P_e^{(N)} \leq \frac{1}{N^\Delta}$ for some constant $\Delta > 0$.

Compared with the brute force method, (see Section V), when $K = 2$, if $T_{BF} = \frac{K^{K+1}}{K!} f(N)$, where $f(N)$ is an arbitrary function satisfying $f(N) \xrightarrow{N\to\infty} \infty$, we have $P_e^{(N)} \leq e^{-f(N)}$ and since $\lim_{N\to\infty} e^{-f(N)} = 0$, we can say that $P_e^{(N)} \to 0$. This means that the threshold effect of the convergence doesn't exist as in group testing or compressive sensing [4]; i.e., $T = O(\log N)$. However, the choice of $f(N)$ will influence the convergence speed. For the convergence speed of the brute force method to be polynomial, $f(N) = O(\log N)$ and thus, $T_{BF} = O(\log N)$, which is of the same order as for partition and group testing (using random coding).

The random coding method is not as efficient as the brute force method when $K = 2$ on two counts: first, the result derived by the random coding method has a threshold effect for the convergence of $P_e^{(N)}$, namely, $P_e^{(N)} \to 0$ only when $T = c \log N$; while the result derived by brute force method does not have such constraint. Second, even when T is constrained to be $T = c \log N$, and $c \geq 1/C(p)$, the upper bound for error probability with the random coding approach satisfies $P_{e;1}^{(N)} \leq N^{-\Delta_1}$, and $P_{e;2}^{(N)} \leq N^{-\Delta_2}$ for the brute-force method, where $\Delta_1 < \Delta_2$. This means that the error probability under the brute-force approach decays faster than that using the random coding method. The reason is because intuitively in the brute force approach, we actually encode the coloring information in the codebook, which is not the case for random coding. The main point is if we use the source codebook to construct channel codebook as done in the brute force approach, we have to deliver the associated coloring information. While for the random coding approach, we actually only care about sending information enough for the nodes to form a 2-colorable graph. However, the random

coding approach shows the internal structure of the problem, and the possibility to attain consistent partition for generalized $K > 2$.

IX. Conclusion

In this paper, we study a new partition reservation problem that focuses on the coordination overhead in the multi-access conflict resolution. The partition information is related to the relationship between active users. Two codebook design methods, source coding based and random coding based are proposed and analyzed to estimate the achievable bound of partitioning overhead in non-adaptive (0, 1)-channel. It should be observed that the adopted hypergraph coloring approach and the Fibonacci structure found in the attained bound are important. They could provide much insight into understanding partition reservation and constructing distributed algorithms for the problem.

In this paper, we attain an achievable rate for the simple $K = 2$ case. Work on the converse bound and more general $K > 2$ case is ongoing. In addition, the effect of noise present in multiple access channels requires different machinery in the corresponding achievability analysis for the random coding approach [36].

Appendix A
Proof of Lemma 1

Proof: The following derivation is subject to $p(\mathbf{z}|\mathbf{s}) \in \mathcal{P}_{z|s}$. Since $I(\mathbf{s};\mathbf{z}) = H(\mathbf{s}) - H(\mathbf{s}|\mathbf{z})$ and $H(\mathbf{s}) = \log\binom{N}{K}$, $H(\mathbf{s}|\mathbf{z})$ is derived as follows. To simplify the notation, we use $\mathbf{n} \triangleq (n_1, \ldots, n_K)$ to represent the number of users in each group.

$$
\begin{aligned}
&-H(\mathbf{s}|\mathbf{z}) \\
&= \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}} \sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s})p(\mathbf{s}) \log \frac{p(\mathbf{z}|\mathbf{s})p_s(\mathbf{s})}{\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})p_s(\tilde{\mathbf{s}})} \\
&\overset{(a)}{=} p(\mathbf{s}) \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}} \sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s}) \log \frac{p(\mathbf{z}|\mathbf{s})}{\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})} \\
&\overset{(b)}{=} p(\mathbf{s}) \sum_{\mathbf{n}} \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}(\mathbf{n})} \sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s}) \log \frac{p(\mathbf{z}|\mathbf{s})}{\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})} \\
&\overset{(c)}{\geq} p(\mathbf{s}) \sum_{\mathbf{n}} \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}(\mathbf{n})} \left(\sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s}) \right) \\
&\qquad \times \log \left(\frac{\left[\sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s})\right]}{\sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} \left[\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})\right]} \right) \\
&= p(\mathbf{s}) \sum_{\mathbf{n}} \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}(\mathbf{n})} \left(\sum_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\mathbf{s}) \right) \log |\mathbb{S}_{K;N}(\mathbf{z})| \\
&\overset{(d)}{=} \sum_{\mathbf{n}} \Pr(\mathbb{Z}_{K;N}(\mathbf{n})) \log \prod_{k=1}^{K} n_k \\
&\overset{(e)}{\geq} \max_{\mathbf{n}} \log \prod_{k=1}^{K} n_k \qquad (23) \\
&\overset{(f)}{\geq} K \log\left(\frac{N}{K}\right), \qquad (24)
\end{aligned}
$$

where $\Pr(\mathbb{Z}_{K;N}(\mathbf{n})) = \sum_{\mathbf{z}\in\mathbb{Z}_{K;N}(\mathbf{n})} p_z(\mathbf{z})$ and $p_z(\mathbf{z})$ is the marginal distribution function with $p(\mathbf{z}|\mathbf{s}) \in \mathcal{P}_{z|s}$. In the derivation, line (a) is because of $\mathbf{s} \sim \mathcal{U}(\mathbb{S}_{K;N})$; line (b) is because set $\mathbb{Z}_{K;N}$ includes all partitions $\mathbf{z}$ of $\mathbf{n}$; i.e., $\mathbb{Z}_{K;N} = \bigcup_{\mathbf{n}} \mathbb{Z}_{K;N}(\mathbf{n})$; line (c) is derived from the log sum inequality; i.e., for non-negative sequence $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right) \log \frac{\sum_{j=1}^{n} a_j}{\sum_{\ell=1}^{n} b_\ell} \qquad (25)$$

with equality if and only if $\frac{a_i}{b_i}$ is a constant for all i. And here, sequence $[a_i] = [p(\mathbf{z}|\mathbf{s})]_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})}$, $[b_i] = \left[\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})\right]_{\mathbf{s}\in\mathbb{S}_{K;N}(\mathbf{z})}$ is a constant sequence. Line (d) holds for any $\mathbf{z} \in \mathbb{Z}_{K;N}(\mathbf{n})$, $|\mathbb{S}_{K;N}(\mathbf{z})| = \prod_{k=1}^{K} n_k$; line (f) is just an application of the inequality of arithmetic and geometric means. For the equality in (23), line (c), (e) should be equalities, which means by (b), $\forall \mathbf{s} \in \mathbb{S}_{K;N}(\mathbf{z})$,

$$\frac{p(\mathbf{z}|\mathbf{s})}{\sum_{\tilde{\mathbf{s}}\in\mathbb{S}_{K;N}(\mathbf{z})} p(\mathbf{z}|\tilde{\mathbf{s}})} = p(\mathbf{s}|\mathbf{z}) = \text{const.}, \qquad (26)$$

and by (e),

$$\Pr(\mathbb{Z}_{K;N}(\mathbf{n}^*)) = \begin{cases} \frac{1}{A}, & \mathbf{n}^* = \arg\max \prod_{k=1}^{K} n_k \\ 0, & \text{otherwise} \end{cases} \qquad (27)$$

where $A = \sum \Pr(\mathbb{Z}_{K;N}(\mathbf{n}^*))$ is a normalized factor. We can choose $\mathbf{z}|\mathbf{s} \sim \mathcal{U}(\mathbb{Z}_{K;N}(\mathbf{n}^*) \bigcap \mathbb{Z}_{K;N}(\mathbf{s}))$, and it is easy to see under this condition, both (b) and (d) will be equality, then so is (e). Thus the lower bound of Equation (23) is proved to be achieved, and so is the lower bound of $I(\mathbf{s};\mathbf{z}) \geq \log\binom{N}{K} - \max_{\mathbf{n}} \log \prod_{k=1}^{K} n_k$.

It is worth noting the result will not change with a generalized $\tilde{\mathbf{z}}$. Define $\tilde{\mathbf{z}} \in \{0, 1, \ldots, N\}^N$, such that $\tilde{z}_i = 0$ indicates an inactive i-th user, and $\tilde{z}_i = k$ indicates that the i-th user is assigned to the k-th group. The definition of distortion can be generalized as follows:

$$\tilde{d}(\mathbf{s}, \tilde{\mathbf{z}}) = \begin{cases} 0, & \forall i, j \in \mathcal{G}_{\mathbf{s}}, \\ & (i \neq j) \implies (\tilde{z}_i, \tilde{z}_j \neq 0 \text{ and } \tilde{z}_i \neq \tilde{z}_j); \\ 1, & \text{otherwise} \end{cases}$$

i.e., active users are assigned different groups, and they cannot be announced as inactive. The definition is consistent with our earlier definition where $\mathbf{z}$ is restricted to $\mathbb{Z}_{K;N}$. From the proof above, it is easy to see with this generalization, the lower bound is the same as in Equation (23). The equality in line (e) can be achieved by choosing $\tilde{\mathbf{z}}|\mathbf{s}$ uniformly in the same way. ∎

Appendix B
Proof of Theorem 1

Proof: First, for $\mathbf{n} \triangleq (n_1, \ldots, n_K)$ satisfying $\sum n_k = N$ and $n_k \geq 0$, observe that

$$\binom{N}{n_1, \ldots, n_K} = \frac{N!}{\Pi_k n_k!}$$

is the number of possible partitions in $\mathbb{Z}_{K;N}(\mathbf{n})$.

The proof is based on random coding; i.e., randomly generate from $p_z(\mathbf{z})$ the source codebook $\mathcal{C} = \{\mathbf{z}_\ell\}_{\ell=1}^{L^{(N)}}$, which is the marginal distribution function based on $p(\mathbf{z}|\mathbf{s})$ in Equation (11) in Lemma 1; $p_s(\mathbf{s}) = 1/\binom{N}{K}$. Therefore,

$$p(\mathbf{z}|\mathbf{s}) = \frac{1}{K!\binom{N-K}{n_1^*-1,\ldots,n_K^*-1}},\ \mathbf{z} \in \mathbb{Z}_{K;N}(\mathbf{n}^*) \bigcap \mathbb{Z}_{K;N}(\mathbf{s})$$
$$p(\mathbf{z}) = \frac{1}{\binom{N}{n_1^*,\ldots,n_K^*}},\ \mathbf{z} \in \mathbb{Z}_{K;N}(\mathbf{n}^*)$$

Reveal this codebook to the source encoder and decoder. For any $\mathbf{s} \in \mathbb{S}_{K;N}$, define the source encoding function $f_N^s(\mathbf{s}) = \ell$, such that $\ell = \arg \min_{1\le\ell\le L^{(N)}} d(\mathbf{s}, \mathbf{z}_\ell)$. If there is more than one such ℓ, choose the least. Then define the source decoding function $g_N^s(\ell) = \mathbf{z}_\ell$. Thus, any source $\mathbf{s}$ will be correctly reconstructed if and only if there exists ℓ such that $d(\mathbf{s}, \mathbf{z}_\ell) = 0$. The average error probability over the codebook $\mathcal{C}$ is

$$\begin{aligned}
\tilde{P}_e^{s,(N)} &= \sum_{\mathcal{C}} p(\mathcal{C}) \sum_{\mathbf{s}:\forall \mathbf{z}_\ell \in \mathcal{C}, d(\mathbf{s},\mathbf{z}_\ell)\neq 0} p(\mathbf{s}) \\
&= \sum_{\mathbf{s}} p(\mathbf{s}) \sum_{\mathcal{C}:\forall \mathbf{z}_\ell \in \mathcal{C}, d(\mathbf{z}^\ell,\mathbf{s})\neq 0} p(\mathcal{C}) \\
&\overset{(a)}{=} \sum_{\mathcal{C}:\forall \mathbf{z}_\ell \in \mathcal{C}, d(\mathbf{z}^\ell,\tilde{\mathbf{s}})\neq 0} p(\mathcal{C}) \\
&= \prod_{\ell=1}^{L^{(N)}} \sum_{\mathbf{z}_\ell : d(\mathbf{z}_\ell,\tilde{\mathbf{s}})\neq 0} p(\mathbf{z}_\ell) \\
&= \prod_{\ell=1}^{L^{(N)}} \left(1 - \sum_{\mathbf{z}_\ell : d(\mathbf{z}_\ell,\tilde{\mathbf{s}})=0} p(\mathbf{z}_\ell)\right) \\
&\overset{(b)}{=} \left(1 - \frac{K!\binom{N-K}{n_1^*-1,\ldots,n_K^*-1}}{\binom{N}{n_1^*,\ldots,n_K^*}}\right)^{L^{(N)}} \\
&= \left(1 - 2^{-W_N^I}\right)^{L^{(N)}}
\end{aligned}$$

The meaning of line (a) above is the probability of no correct codewords in a randomly chosen $\mathcal{C}$, for any given $\tilde{\mathbf{s}}$, that is derived by the symmetry of the random codebook. Line (b) is derived using the fact that $|\{\mathbf{z}_\ell : d(\mathbf{z}_\ell, \tilde{\mathbf{s}}) = 0\}| = |\mathbb{Z}_{K;N}(n_1^*, \ldots, n_K^*) \bigcap \mathbb{Z}(\tilde{\mathbf{s}})| = K!\binom{N-K}{n_1^*-1,\ldots,n_K^*-1}$. Due to the inequality $(1 - xy)^n \le 1 - x + e^{-yn}$, for $0 \le x, y \le 1$ and $n > 0$, we have

$$\tilde{P}_e^{s,(N)} \le e^{-2^{\left(\log L^{(N)} - W_N^I\right)}}$$

Since this is the average error probability over all possible codebooks $\mathcal{C}$, there must exist a codebook to achieve the error bound above. This completes the proof. ∎

APPENDIX C
PROOF OF LEMMA 2 AND THEOREM 3

Proof: The proofs of Lemma 2 and Theorem 3 are similar, so we put them together. In the proof, we will use the method of strong typical set; a definition of strong typical set can be found in Csiszar and Körner [34]. Recall that the $T \times 1$ vectors $\mathbf{x}_1$, $\mathbf{x}_2$ represent the codewords of users 1 and 2, let $[\mathbf{x}_1, \mathbf{x}_2]$ denote the $T \times 2$ matrix with $\mathbf{x}_1$ and $\mathbf{x}_2$ as its two columns. A strong typical set $\mathcal{A}_\epsilon^{(T)}$ as defined in Equation (28) is proposed at first, as in the following subsection.

A. Strong Typical Set

Note that the input is $\mathcal{G}_{\mathbf{s}_0} = \{1, 2\}$. Since the codewords $x_{1,t}$ and $x_{2,t}$ are generated from $\mathcal{B}(p)$, it is very likely that in the set $\{(x_{1,t}, x_{2,t})\}_{t=1}^T$, there are $p_{12}(u, v)T$ pairs (u, v), $\forall u, v \in \{0, 1\}$, where $p_{12}(u, v) \triangleq p_x(u)p_x(v)$, and $p_x(\tilde{x}) \triangleq p\mathbb{1}(\tilde{x} = 1) + (1 - p)\mathbb{1}(\tilde{x} = 0)$ is the pdf of $\mathcal{B}(p)$. Define $N((u, v)|[\mathbf{x}_1, \mathbf{x}_2])$ as the number of (u, v) in $\{(x_{1,t}, x_{2,t})\}_{t=1}^T$, for any given $\epsilon > 0$, define the strong typical set as follows:

$$\begin{aligned}
\mathcal{A}_\epsilon^{(T)} = \Bigg\{ &[\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2] \in \{0, 1\}^{2T} : \forall u, v \in \{0, 1\}, \\
&\left|\frac{1}{T} N((u, v)|[\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2]) - p_{12}(u, v)\right| < \frac{\epsilon}{4} \Bigg\} \qquad (28)
\end{aligned}$$

The parameter ϵ will be chosen at beginning to guarantee some good features of the set $\mathcal{A}_\epsilon^{(T)}$, we will describe the such requirements on ϵ during the proof. The first requirement is that

$$\epsilon/4 < \max_{u,v} (\max\{p_{12}(u, v), 1 - p_{12}(u, v)\}), \qquad (29)$$

so that $1 > p_{12}(u, v) \pm \epsilon/4 > 0$.

For a strong typical set, $\forall \epsilon > 0$, $\Pr(\mathbf{X} \notin \mathcal{A}_\epsilon^{(T)}) \to 0$, as $T \to \infty$. It means that the strong typical set contains almost all $[\mathbf{x}_1, \mathbf{x}_2]$ and $\Pr\left([\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}\right) \xrightarrow{T\to\infty} 1$. In this typical set, the probability of existence of any possible odd cycle (odd1 cycle) is calculated and P_e is bounded by using union bound.

To simplify notation, denote by $E^{(w)}$ with $w = 1, 2$ as the event that H' contains the odd1 cycles or odd cycles respectively, and $P_e^{(1)} \triangleq P_e^{odd1} \triangleq \Pr(E^{(1)})$, $P_e^{(2)} \triangleq P_e^{odd} \triangleq \Pr(E^{(2)})$. Since we have:

$$\begin{aligned}
P_e^{(w)} &= \Pr(E^{(w)}[\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) + \Pr(E^{(w)}, [\mathbf{x}_1, \mathbf{x}_2] \notin \mathcal{A}_\epsilon^{(T)}) \\
&\le \Pr(E^{(w)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) + \Pr([\mathbf{x}_1, \mathbf{x}_2] \notin \mathcal{A}_\epsilon^{(T)}), \qquad (30)
\end{aligned}$$

it suffices to show that for any $0 < p < 1$, when $\frac{\log N}{T} < C(p)$, both $\Pr([\mathbf{x}_1, \mathbf{x}_2] \notin \mathcal{A}_\epsilon^{(T)})$ and $\Pr(E^{(w)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)})$ approach 0 as $N \to \infty$. While the first one is directly from the property of strong typical set, the key point is to estimate the probability of $E^{(w)}$ in the typical set $\mathcal{A}_\epsilon^{(T)}$. Thus let us show that $\Pr(E^{(w)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) \to 0$ in the following parts.

B. Odd Cycles for Given $\mathcal{A}_\epsilon^{(T)}$

Consider $\Pr(E^{(w)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)})$. For simplicity, assume N is an odd number; this is without loss of generality. Denote by $A_M^{(g)}$, where $g \in \{1, 2, 3\}$, the event of existence of the

type-g odd cycles with size M, thus:

$$E^{(1)} = \bigcup_{m=1}^{(N-1)/2} A_{2m+1}^{(1)}$$
$$E^{(2)} = \bigcup_{m=1}^{(N-1)/2} \bigcup_{g=1,2,3} A_{2m+1}^{(g)}$$

For any given particular $[\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}$, let

$$P_{2m+1|[\mathbf{x}_1,\mathbf{x}_2]}^{(g)} \triangleq \Pr\left(A_{2m+1}^{(g)} \big| [\mathbf{x}_1, \mathbf{x}_2]\right),$$

as the probability that the type-g odd cycle of length $2m+1$ exists in H' for given $[\mathbf{x}_1, \mathbf{x}_2]$. Then by union bound, we have:

$$\begin{aligned}\Pr(E^{(1)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) &\le \sum_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} \sum_{M=3,5,\ldots,N} P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(1)} Q_{1,2}(\mathbf{x}_1, \mathbf{x}_2) \\ &\le \sum_{M=3,5,\ldots,N} \max_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(1)}, \end{aligned} \tag{31}$$

where $Q_{1,2}(\mathbf{x}_1, \mathbf{x}_2)$ is the probability of the first two codewords taking values $\mathbf{x}_1, \mathbf{x}_2$. Similarly,

$$\begin{aligned}\Pr(E^{(2)}, [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) &\le \sum_{M=3,5,\ldots,N} \sum_{g=1,2,3} \max_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(g)} \end{aligned} \tag{32}$$

Thus, the key point is to determine $P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(g)}$ for any given $[\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}$ and upper bound it.

C. The Probability of Existence of Odd1 Cycles of Length M: $P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(1)}$

Consider any particular odd1 cycle of length M, denoted by $H_{e;M}^{(1)} = (1, 2, i_1, \ldots, i_{M-2})$; there are at most $\binom{N-2}{M-2}(M-2)!$ such odd cycles out of N nodes, and because of symmetry, the existence of any of them is equiprobable. Let us now see for a given $[\mathbf{x}_1, \mathbf{x}_2]$, what values should the codewords $\{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_{M-2}}\}$ of the remaining $M-2$ vertices be to guarantee that $H_{e;M}^{(1)} \subseteq H'$.

Note that H' is obtained by a series of graph operations, whose order is not important. Let us determine the probability that $H_{e;M}^{(1)}$ is not deleted during any slot $1 \le t \le T$. By the symmetry of the generation of codewords, this probability only depends on the values of $(x_{1,t}, x_{2,t})$. Given $[\mathbf{x}_1, \mathbf{x}_2]$, let $\mathcal{T}_{u,v} = \{t : (x_{1,t}, x_{2,t}) = (u, v)\}$, and $T_{u,v} \triangleq |\mathcal{T}_{u,v}| = N((u, v)|[\mathbf{x}_1, \mathbf{x}_2])$. Thus, we just need to consider four situations for $t \in \mathcal{T}_{u,v}$. Denote by $\mu_{u,v;M}^{(1)}$ the probability that $H_{e;M}^{(1)}$ is not deleted at $t \in \mathcal{T}_{u,v}$, so by union bound of all possible $H_{e;M}^{(1)}$, we have for all $[\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}$:

$$\begin{aligned} P_{M|[\mathbf{x}_1,\mathbf{x}_2]}^{(1)} &\le \binom{N-2}{M-2}(M-2)! \prod_{u,v} \left(\mu_{u,v;M}^{(1)}\right)^{T_{u,v}} \\ &\le N^{M-2} \prod_{u,v} \left(\mu_{u,v;M}^{(1)}\right)^{(p_{12}(u,v)-\epsilon/4)T} \end{aligned} \tag{33}$$

Now $\mu_{u,v;M}^{(1)}$ is determined separately for cases of different (u, v) as follows:

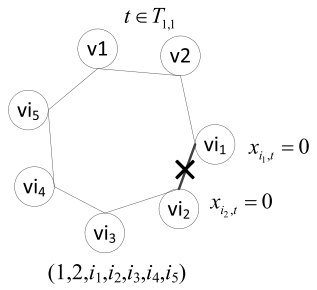

Fig. 9. An example illustrating the removal of an odd cycle by deletion of edge (i_1, i_2).

1) *Case $t \in \mathcal{T}_{0,0}$:* For $t \in \mathcal{T}_{0,0}$ and $y_t = 0$, the operation is to delete vertices. Then all of the codewords of the other $M-2$ vertices are 0; otherwise these vertices would be deleted. Thus

$$\begin{aligned}\mu_{0,0;M}^{(1)} &= \Pr(x_{i_w,t} = 0, \forall w \in \{1, \ldots, M-2\}) \\ &= (1-p)^{M-2}\end{aligned}$$

2) *Case $t \in \mathcal{T}_{1,1}$:* In these slots $y_t = 1$, and a clique deletion operation is performed on H'. At any slot t, $H_{e;M}^{(1)}$ will be broken up if its edges are deleted; this is equivalent to the existence of codewords of two consecutive vertices from $(1, 2, i_1, \ldots, i_{M-2}, 1)$ that are both 0 at slots $t \in \mathcal{T}_{1,1}$, as shown in Fig. 9. So we have:

$$\begin{aligned}\mu_{1,1;M}^{(1)} &\triangleq 1 - \Pr(\exists (i, j) \in \{(1, 2), (2, i_1), \ldots, (i_{M-3}, i_{M-2}), \\ &\qquad (i_{M-2}, 1)\}, x_{i,t} = x_{j,t} = 0) \\ &\overset{(a)}{=} 1 - \Pr\left(\exists w \in \{1, \ldots, M-3\}, x_{i_w,t} = x_{i_{w+1},t} = 0\right) \\ &\overset{(b)}{=} 1 - \sum_{M_1=\frac{M-3}{2}}^{M-2} \Pr\Bigg(\sum_{w=1}^{M-2} x_{i_w,t} = M_1, \text{ and} \\ &\qquad \exists w \in \{1, \ldots, M-3\}, \\ &\qquad x_{i_w,t} = x_{i_{w+1},t} = 0\Bigg) \\ &\overset{(c)}{=} \sum_{M_1=\frac{M-3}{2}}^{M-2} \binom{M_1+1}{M-2-M_1} p^{M_1}(1-p)^{M-2-M_1}. \end{aligned} \tag{34}$$

Line (a) is because now $x_{1,t} = x_{2,t} = 1$. For line (b), we change the sum by grouping items with different M_1, where M_1 is the number of values of $1 \le w \le M-2$ for which $x_{i_w,t} = 1$. It is easy to see there must be $M_1 \ge \lfloor \frac{M-2}{2} \rfloor = \frac{M-3}{2}$, otherwise there must exist a w such that $x_{i_w,t} = x_{i_{w+1},t} = 0$. In line (c), the sum of probabilities of the items with M_1 is determined, using the fact that the probability of each item is $p^{M_1}(1-p)^{M-2-M_1}$; the key point is to count the number of sequences $(x_{i_1,t}, \ldots, x_{i_{M-2},t})$ with M_1 ones and $M-2-M_1$ zeros. The counting method is to fix M_1 ones, then count the number of combinations for $M-2-M_1$ zeros in M_1+1 spots, as shown in Fig. 10.

For additional clarity, define a function $J_M(p)$ which determines the probability of an M length random Bernoulli sequence $(x_1, \ldots, x_M)$, where $x_w \sim \mathcal{B}(p)$ without consecutive

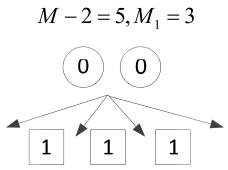

Fig. 10. Example of counting the number of sequence $(x_{i_1,t},\ldots,x_{i_{M-2},t})$ with M_1 ones; here $M-2=5$, $M_1=3$.

zeros; i.e.,

$$J_M(p) = 1-\Pr\left(\exists w\in\{1,\ldots,M-1\}, x_w = x_{w+1}=0\right) = \sum_{M_1=\lfloor\frac{M}{2}\rfloor}^{M}\binom{M_1+1}{M-M_1}p^{M_1}(1-p)^{M-M_1}. \quad (35)$$

Thus, we can see $\mu^{(1)}_{1,1;M} = J_{M-2}(p)$.

3) *Case $t\in\mathcal{T}_{1,0}$ or $t\in\mathcal{T}_{0,1}$:* The two cases are symmetric. Let us consider $t\in\mathcal{T}_{1,0}$ first. It is similar to the case where $t\in\mathcal{T}_{1,1}$, but since now $x_{2,t}=0$, the codeword of the vertices that is connected to vertex 2 (i.e., vertex i_1), is $x_{i_1,t}=1$. For the other $M-3$ vertices, it is required that codewords of any two consecutive vertices from $(i_2,\ldots,i_{M-2})$ at those $t\in\mathcal{T}_{1,0}$ not be both 0. Thus,

$$\mu^{(1)}_{1,0;M} = \Pr(x_{i_1t}=1)\times\left(1-\Pr\left(\exists w\in\{1,\ldots,M-1\}, x_w=x_{w+1}=0\right)\right) = pJ_{M-3}(p)$$

Similarly, for $t\in\mathcal{T}_{0,1}$, we have $\mu^{(1)}_{1,0;M} = pJ_{M-3}(p)$.

Then $P^{(1)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$ can be bounded by Equation (33).

D. The Probability of Existence of Type-2 and Type-3 Cycles of Length M: $P^{(2)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$ and $P^{(3)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$

For the Type-2 odd cycles, either 1 or 2 nodes are included. Denote $H^{(2),h}$ to be a Type-2 odd cycle containing vertex $h\in\{1,2\}$. We can choose a particular pair of odd cycles $H^{(2),1}_{e;M}=(1,i_1,\ldots,i_{M-1})$, $H^{(2),2}_{e;M}=(2,i_1,\ldots,i_{M-1})$, and a particular Type-3 odd cycle $H^{(3)}_{e;M}=(i_1,\ldots,i_M)$. There are $\binom{N-2}{M-1}\frac{(M-1)!}{2}$ such $H^{(2),h}_{e;M}$, and $\binom{N-2}{M}\frac{(M-1)!}{2}$ such $H^{(3)}_{e;M}$. Then following the same analysis as for $P^{(1)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$, we have for all $[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}^{(T)}_\epsilon$,

$$P^{(2)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \le \sum_{h=1,2}\binom{N-2}{M-1}\frac{(M-1)!}{2}\prod_{u,v}\left(\mu^{(2),h}_{u,v;M}\right)^{T_{u,v}} \le \frac{1}{2}N^{M-1}\sum_{h=1,2}\prod_{u,v}\left(\mu^{(2),h}_{u,v;M}\right)^{(p_{12}(u,v)-\epsilon/4)T}, \quad (36)$$

$$P^{(3)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \le \binom{N-2}{M}\frac{(M-1)!}{2}\Pi_{u,v}\left(\mu^{(3)}_{u,v;M}\right)^{T_{u,v}} \le N^M\Pi_{u,v}\left(\mu^{(3)}_{u,v;M}\right)^{(p_{12}(u,v)-\epsilon/4)T}, \quad (37)$$

where $\mu^{(g),h}_{u,v;M}$ is the probability that $H^{(g),h}_{e;M}$ will not be deleted at $t\in\mathcal{T}_{u,v}$. Then similarly, we have:

1) For $t\in\mathcal{T}_{0,0}$, $y_t=0$, every vertex cannot be deleted, thus:

$$\mu^{(2),h}_{0,0;M} = (1-p)^{M-1};\quad \mu^{(3)}_{0,0;M} = (1-p)^M$$

2) For $g=2$, let us consider $H^{(2),1}_{e;M}$ first, when $t\in\mathcal{T}_{1,1}$ or $t\in\mathcal{T}_{1,0}$, we have $x_{1,t}=1$. Thus,

$$\mu^{(2),1}_{1,0;M} = \mu^{(2),1}_{1,1;M} = 1-\Pr\left(\exists w\in\{1,\ldots,M-2\}, x_{i_w,t}=x_{i_{w+1},t}=0\right) = J_{M-1}(p)$$

When $t\in\mathcal{T}_{0,1}$, we have $x_{1,t}=0$. Here the codewords of vertices i_1 and i_{M-1} (which are connected to Node 1) would be 1, i.e.,

$$\mu^{(2),1}_{0,1;M} = \Pr(x_{i_1,t}=x_{i_{M-1},t}=1)\times(1-\Pr(\exists w\in\{2,\ldots,M-1\}, x_{i_w,t}=x_{i_{w+1},t}=0)) = p^2J_{M-3}(p)$$

For $H^{(2),2}_{e;M}$, due to symmetry, the result is easy to derive:

$$\mu^{(2),2}_{1,1;M} = \mu^{(2),2}_{0,1;M} = J_{M-1}(p);\quad \mu^{(2),2}_{1,0;M} = p^2J_{M-3}(p)$$

3) For $g=3$, for $t\notin\mathcal{T}_{0,0}$, we have $y_t=1$. Since neither vertices 1 nor 2 are in $H^{(3)}_{e;M}$, probabilities $\mu^{(3)}_{u,v;M}$ are the same for any $(u,v)\neq(0,0)$. Now we have:

$$\begin{aligned}\mu^{(3)}_{u,v;M} &= 1-\Pr(\exists(i,j)\in\{(i_1,i_2),\ldots,(i_{M-1},i_M),(i_M,i_1)\}, x_{i,t}=x_{j,t}=0)\\ &= 1-\Pr\left(\exists w\in\{1,\ldots,M-1\}, x_{i_w,t}=x_{i_{w+1},t}=0\right)\\ &\quad -\Pr((x_{i_1,t},x_{i_M,t})=(0,0);\ \text{and}\\ &\qquad \nexists w\in\{1,\ldots,M-1\}, x_{i_w,t}=x_{i_{w+1},t}=0)\\ &= J_M(p)-p^2(1-p)^2J_{M-4}(p)\end{aligned}$$

Now we can bound $P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$. In the next subsection, we will obtain explicit expressions for $J_M(p)$ and $\mu^{(g)}_{u,v;M}$. We can see that they have a close relationship to extended Fibonacci numbers.

E. Explicit Expressions for $J_M(p)$ and Extended Fibonacci Numbers

We will show that expression for $J_M(p)$ has a close relationship to a certain class of extended Fibonacci numbers. It is not surprising since Fibonacci numbers can be used for determining the number of consecutive 0s in a Bernoulli sequence [37].

Define extended Fibonacci numbers as:

$$F(k,p) = \sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}F(k,j)p^{k-1-j}(1-p)^j,$$

where

$$F(k,j) = \begin{cases}\binom{k-1-j}{j}, & 0\le j\le\lfloor\frac{k-1}{2}\rfloor\\ 0, & \text{otherwise}\end{cases}$$

The meaning of $F(k,j)$ can be seen directly from the Pascal triangle, as shown in Fig. 11. The quantity $F(k,p)$

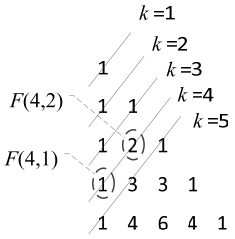

Fig. 11. $F(k, j)$ in the Pascal triangle.

is a weighted sum of $F(k, j)$ with weight $p^{k-1-j}(1-p)^j$. From Fig. 11 it can be shown that:

$$F(k, j) = F(k-1, j) + F(k-1, j-1),$$

so that

$$\begin{aligned} &F(k, p) \\ &= \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \left(F(k-1, j) + F(k-1, j-1)\right) p^{k-1-j}(1-p)^j \\ &= pF(k-1, p) + p(1-p)F(k-2, p). \end{aligned}$$

Then we can get the general terms of $F(k, p)$ by solving the corresponding difference equation, which gives us:

$$F(k, p) = \frac{\varphi(p)^k - \psi(p)^k}{\varphi(p) - \psi(p)}, \tag{38}$$

where

$$\varphi(p) = \frac{p + \sqrt{4p - 3p^2}}{2}; \quad \psi(p) = \frac{p - \sqrt{4p - 3p^2}}{2}$$

It is not difficult to see that

$$1 \geq \varphi(p) \geq 0 \geq \psi(p) \geq -1, \quad |\varphi(p)| \geq |\psi(p)|$$

Given $F(k, p)$ defined in Equation (38), it is straightforward to see that:

$$J(M, p) = \frac{1}{p}F(M+2, p) = \frac{1}{p}\frac{(\varphi(p))^{M+2} - (\psi(p))^{M+2}}{\varphi(p) - \psi(p)}$$

Which further enables us to determine $\mu^{(g)}_{u,v;M}$ and upper bound $P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$.

F. Bounds on $P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]}$

By Equations (33), (36) and (37), we now have for any $[\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}$ and $g \in \{1, 2, 3\}$,

$$P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \leq 2^{-(M-3+g)T\left(\left(h^{(g)} - ((1-p)^2 - \frac{\epsilon}{4})\log(1-p)\right) - \frac{\log N}{T}\right)} \tag{39}$$

where

$$\begin{aligned} h^{(1)} &\triangleq -\frac{1}{M-2}\Big((p^2 - \frac{\epsilon}{4})\log J_{M-2}(p) \\ &\qquad + (2p(1-p) - \frac{\epsilon}{2})\log pJ_{M-3}(p)\Big) \\ h^{(2)} &\triangleq -\frac{1}{M-1}\Big((p - \frac{\epsilon}{2})\log J_{M-1}(p) \\ &\qquad + (p(1-p) - \frac{\epsilon}{4})\log p^2 J_{M-3}(p)\Big) \\ h^{(3)} &\triangleq -\frac{1}{M}\left(1 - (1-p)^2 - \frac{3\epsilon}{4}\right) \\ &\qquad \times \log\Big(J_M(p) - p^2(1-p)^2 J_{M-4}(p)\Big) \end{aligned}$$

Next we will give a concise lower bound for $h^{(g)}$, which can be obtained by the monotonicity and concavity of $\log(\cdot)$.

1) *Bound on* $h^{(1)}$*:* Define a normalizing factor

$$W = (p^2 - \frac{\epsilon}{4}) + (2p(1-p) - \frac{\epsilon}{2}) = 1 - (1-p)^2 - \frac{3\epsilon}{4}.$$

If we choose ϵ so that

$$\begin{aligned} &W - \varphi(p)^2 \\ &= \frac{p}{2}\left((2-p) - \sqrt{(2-p)^2 - 4(1-p)^2}\right) - \frac{3\epsilon}{4} > 0. \end{aligned} \tag{40}$$

then when M is an odd number,

$$\begin{aligned} -h^{(1)} &\overset{(a)}{\leq} W\log\Big(\frac{p^2 - \epsilon/4}{W}J_{M-2}(p) \\ &\qquad + \frac{2p(1-p) - \epsilon/2}{W}pJ_{M-3}(p)\Big) \\ &\overset{(b)}{\leq} W\log\left(\frac{p^2 J_{M-2}(p) + 2p^2(1-p)J_{M-3}(p)}{W}\right) \quad (41) \\ &= W\log\Bigg(\frac{p\sqrt{4-3p}}{W(\varphi(p) - \psi(p))} \\ &\qquad \times \Bigg(\frac{\sqrt{4-3p} + \sqrt{p}}{2}\varphi(p)^{M-1} \\ &\qquad - \frac{\sqrt{4-3p} - \sqrt{p}}{2}\psi(p)^{M-1}\Bigg)\Bigg) \\ &\overset{(c)}{\leq} W\log\left(\frac{p\sqrt{4-3p}(\sqrt{4-3p} + \sqrt{p})}{2W(\varphi(p) - \psi(p))}\varphi(p)^{M-1}\right) \\ &= W\log\left(\frac{\varphi(p)^M}{W}\right) \quad (42) \\ &= W\log\left(\varphi(p)^{M-2}\right) + W\log\left(\frac{\varphi(p)^2}{W}\right) \\ &\leq W\log\left(\varphi(p)^{M-2}\right) \quad (43) \end{aligned}$$

Where line (a) is because of the concavity of $\log(\cdot)$; inequalities (b) and (c) are because of $\log(\cdot)$ increasing monotonically. Then

$$h^{(1)} - ((1-p)^2 - \frac{\epsilon}{4})\log(1-p) \geq C(p) - g_1(p)\epsilon,$$

where $g_1(p) \triangleq -\frac{1}{4}(3\log(\varphi(p)) + \log(1-p)) > 0$, and

$$C(p) \triangleq -(1-(1-p)^2)\log\varphi(p) - (1-p)^2\log(1-p).$$

2) *Bound of $h^{(2)}$:* Similarly, if ϵ satisfies Equations (29) and (40), we have

$$\begin{aligned}
&-h_2 - (p(1-p) - \epsilon/4)\log p \\
&\overset{(a)}{\leq} W\log\left(\frac{p-\epsilon/2}{W}J_{M-1}(p) + \frac{p(1-p)-\epsilon/4}{W}pJ_{M-3}(p)\right) \\
&\quad \overset{(b)}{\leq} W\log\left(\frac{p}{W}J_{M-1}(p) + \frac{p(1-p)}{W}pJ_{M-3}(p)\right) \\
&\quad \overset{(c)}{\leq} W\log\left(\frac{\varphi(p)^M}{W}\right) \\
&= W\log\left(\varphi(p)^{M-1}\right) + W\log\left(\frac{\varphi(p)}{W}\right)
\end{aligned}$$

Thus,

$$-h_2 \leq W\log\left(\varphi(p)^{M-1}\right) - \frac{\epsilon}{4}\log p. \tag{44}$$

Inequalities (a) and (b) are because of the concavity and monotonicity of $\log(\cdot)$; line (c) is because $\frac{p}{W}J_{M-1}(p) + \frac{p(1-p)}{W}pJ_{M-3}(p) = \frac{1}{W}\left(p^2J_{M-2}(p) + 2p^2(1-p)J_{M-3}(p)\right)$, and then is identical to the expressions in Equations (41) and (42); Equation (44) is derived from Equation (40). Therefore,

$$\begin{aligned}
&W\log\left(\frac{\varphi(p)}{W}\right) + (p(1-p)-\epsilon/4)\log p \\
&= \frac{W}{2}\log(\varphi(p)^2/W) - (2p-p^2)\log\sqrt{2p-p^2} \\
&\quad + p(1-p)\log p + \frac{3\epsilon}{8}\log(1-(1-p)^2) \\
&\quad + \frac{W}{2}\log\left(1 - \frac{3\epsilon}{4(2p-p^2)}\right) - \frac{\epsilon}{4}\log p \\
&\leq \left(-(2p-p^2)\log\sqrt{2p-p^2} + p(1-p)\log p\right) - \frac{\epsilon}{4}\log p \\
&= -p(1-H(p/2)) - \frac{\epsilon}{4}\log p \\
&\leq -\frac{\epsilon}{4}\log p,
\end{aligned}$$

and thus,

$$\begin{aligned}
&h_2 - ((1-p)^2 - \frac{\epsilon}{4}) \\
&\geq -W\log\varphi(p) - ((1-p)^2 - \frac{\epsilon}{4})\log(1-p) + \frac{\epsilon\log p}{4(M-1)} \\
&= C(p) + (3\log(\varphi(p)) + \log(1-p))\frac{\epsilon}{4} + \frac{\epsilon\log p}{4(M-1)} \\
&\geq C(p) - g_2(p)\epsilon,
\end{aligned}$$

where $g_2(p) = g_1(p) - \log p/8 > 0$.

3) *Bound of $h^{(3)}$:* Similarly, we can bound h_3, if ϵ satisfies (29) and (40) and M is an odd number. Here,

$$\begin{aligned}
h_3 &\geq -\frac{1}{M}\left(1-(1-p)^2 - \frac{3\epsilon}{4}\right)\log(\varphi(p)^M + \psi(p)^M) \\
&\overset{(a)}{\geq} \frac{1}{M}\left(1-(1-p)^2 - \frac{3\epsilon}{4}\right)\log(\varphi(p)^M),
\end{aligned}$$

where Line (a) is because $\psi(p) < 0$ and M is odd. Thus,

$$h_3 - ((1-p)^2 - \frac{\epsilon}{4})\log(1-p) \geq C(p) - g_1(p)\epsilon$$

Since $g_2(p) > g_1(p)$, as shown above, we can say that when ϵ satisfies Equations (29) and (40), we have

$$h^{(g)} - ((1-p)^2 - \frac{\epsilon}{4})\log(1-p) \geq C(p) - g_2(p)\epsilon$$

Thus from Equation (39), we have

$$\max_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \leq 2^{-(M-3+g)T\left(C(p)-g_2(p)\epsilon-\frac{\log N}{T}\right)}. \tag{45}$$

G. Completing the Proof

If $\frac{\log N}{T} \leq C(p) - \xi$ for any constant $\xi > 0$, we can always choose ϵ satisfying Equations (29) and (40), and

$$\epsilon < (C(p) - \delta)/g_2(p)$$

so that $C(p) - \frac{\log N}{T} - g_2(p)\epsilon \geq C(p) - \delta - g_2(p)\epsilon \triangleq \Delta > 0$, where Δ is a predetermined constant. Then by Equations (31), (32) and (45), we have:

$$\begin{aligned}
\Pr(E^{(1)}, [\mathbf{x}_1,\mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) &\leq \sum_{M=3,5,\ldots,N} \max_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} P^{(1)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \\
&\leq \sum_{M=3,5,\ldots,N} 2^{-(M-2)\Delta T} \\
&\leq \frac{2^{-\Delta T}}{1-2^{-2\Delta T}}
\end{aligned}$$

and

$$\begin{aligned}
&\Pr(E^{(2)}, [\mathbf{x}_1,\mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) \\
&\quad \leq \sum_{g=1,2,3}\sum_{M=3,5,\ldots,N} \max_{[\mathbf{x}_1,\mathbf{x}_2]\in\mathcal{A}_\epsilon^{(T)}} P^{(g)}_{M|[\mathbf{x}_1,\mathbf{x}_2]} \\
&\quad \leq \sum_{g=1,2,3}\sum_{M=3,5,\ldots,N} 2^{-(M-3+g)\Delta T} \\
&\quad \leq 3\times\frac{2^{-\Delta T}}{1-2^{-2\Delta T}}
\end{aligned}$$

Thus, when $N\to\infty$, which also means $T\to\infty$, we can see $\Pr(E^{(w)}, [\mathbf{x}_1,\mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)})$ approaches 0, $\forall w = 1, 2$. Since $\Pr(E^{(w)}, [\mathbf{x}_1,\mathbf{x}_2] \in \mathcal{A}_\epsilon^{(T)}) \to 0$ as well, we have $P_e^{(w)} \to 0$ when $\frac{\log N}{T} < C(p) - \xi$. This completes the proof. ■

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