EE 4610, Solution of Homework 1

Problem 2.2

4) The signal $\cos(t)$ is periodic with period $T_1 = 2\pi$ whereas $\cos(2.5t)$ is periodic with period $T_2 = 0.8\pi$. It follows then that $\cos(t) + \cos(2.5t)$ is periodic with period $T = 4\pi$. The trigonometric Fourier series of the even signal $\cos(t) + \cos(2.5t)$ is

$$\begin{aligned}\cos(t) + \cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi \frac{n}{T_0} t) \\ &= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2} t)\end{aligned}$$

By equating the coefficients of $\cos(\frac{n}{2}t)$ of both sides we observe that $a_n = 0$ for all n unless $n = 2, 5$ in which case $a_2 = a_5 = 1$. Hence $x_{4,2} = x_{4,5} = \frac{1}{2}$ and $x_{4,n} = 0$ for all other values of n.

Problem 2.3

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(2\pi \frac{n}{T_0} t) + b_n \sin(2\pi \frac{n}{T_0} t) \right]$$

The even part of $x(t)$ is

$$\begin{aligned}x_e(t) &= \frac{x(t) + x(-t)}{2} \\ &= \frac{1}{2} \Bigg(a_0 + \sum_{n=1}^{\infty} a_n (\cos(2\pi \frac{n}{T_0} t) + \cos(-2\pi \frac{n}{T_0} t)) \\ &\qquad + b_n (\sin(2\pi \frac{n}{T_0} t) + \sin(-2\pi \frac{n}{T_0} t)) \Bigg) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T_0} t)\end{aligned}$$

The last is true since $\cos(\theta)$ is even so that $\cos(\theta) + \cos(-\theta) = 2\cos\theta$ whereas the oddness of $\sin(\theta)$ provides $\sin(\theta) + \sin(-\theta) = \sin(\theta) - \sin(\theta) = 0$.

The odd part of $x(t)$ is

$$\begin{aligned}x_o(t) &= \frac{x(t) - x(-t)}{2} \\ &- \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T_0} t)\end{aligned}$$

Frequency shifting property:

We start with the inverse Fourier transform of $X(f - f_0)$,

$$\mathcal{F}^{-1}[X(f - f_0)] = \int_{-\infty}^{\infty} X(f - f_0) e^{j2\pi f t} df$$

With a change of variable of $u = f - f_0$, we obtain

$$\begin{aligned}
\mathcal{F}^{-1}[X(f-f_0)] &= \int_{-\infty}^{\infty} X(u) e^{j2\pi f_0 t} e^{j2\pi t u} du \\
&= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} X(u) e^{j2\pi t u} du \\
&= e^{j2\pi f_0 t} \mathcal{F}^{-1}[X(f)] \\
&= e^{j2\pi f_0 t} x(t)
\end{aligned}$$

Scaling property:

We start with

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt$$

and make the change in variable $u = at$, then,

$$\begin{aligned}
\mathcal{F}[x(at)] &= \frac{1}{|a|} \int_{-\infty}^{\infty} x(u) e^{-j2\pi f u/a} du \\
&= \frac{1}{|a|} X\left(\frac{f}{a}\right)
\end{aligned}$$

where we have treated the cases $a > 0$ and $a < 0$ separately.

Note that in the above expression if $a > 1$, then $x(at)$ is a contracted form of $x(t)$ whereas if $a < 1$, $x(at)$ is an expanded version of $x(t)$. This means that if we expand a signal in the time domain its frequency domain representation (Fourier transform) contracts and if we contract a signal in the time domain its frequency domain representation expands. This is exactly what one expects since contracting a signal in the time domain makes the changes in the signal more abrupt, thus, increasing its frequency content.

Problem 2.11

$$\begin{aligned}
\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2}) + \delta(t-\frac{1}{2}))] &= \int_{-\infty}^{\infty} \frac{1}{2}(\delta(t+\frac{1}{2}) + \delta(t-\frac{1}{2})) e^{-j2\pi f t} dt \\
&= \frac{1}{2}(e^{-j\pi f} + e^{-j\pi f}) = \cos(\pi f)
\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\begin{aligned}
\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2})) e^{j\pi f} \\
&= \frac{1}{2} e^{j\pi\frac{1}{2}} \delta(f+\frac{1}{2}) + \frac{1}{2} e^{-j\pi\frac{1}{2}} \delta(f-\frac{1}{2}) \\
&= \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2})
\end{aligned}$$

Problem 2.17

(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned}
\text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\
&= \text{sinc}(t)
\end{aligned}$$

Problem 2.18

$$\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right) y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} X(\theta) \left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right) d\theta \\
&= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f)
\end{aligned}$$

Problem 2.12

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f) - 4\text{sinc}^2(2f)$$

b)

$$x(t) = 2\Pi(\frac{t}{4}) - \Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\text{sinc}(4f) - \text{sinc}^2(f)$$

c)

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt \\
&= \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right) e^{-j2\pi ft}\bigg|_{-1}^{0} + \frac{j}{2\pi f}e^{-j2\pi ft}\bigg|_{-1}^{0} \\
&\quad + \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right) e^{-j2\pi ft}\bigg|_{0}^{1} - \frac{j}{2\pi f}e^{-j2\pi ft}\bigg|_{0}^{1} \\
&= \frac{j}{\pi f}(1 - \sin(\pi f))
\end{aligned}$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f)$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0(t - \frac{1}{4f_0})\right) - \Pi\left(2f_0(t - \frac{1}{4f_0})\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned} X(f) &= \left[\frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0})\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\ &\quad \star\frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\ &= \frac{1}{2f_0}\text{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\text{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right) \end{aligned}$$