EE 7620 Solutions to Exam 2 Nov. 10, 2008

Problem 1

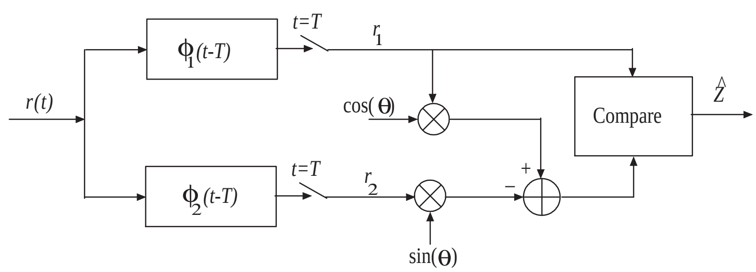
1. An O.N. spanning set for the signal set is given by

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t), \quad \text{and} \quad \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_0 t), \qquad 0 \le t \le T$$

Then $s_0(t) = \sqrt{E}\phi_1(t)$ and thus $\mathbf{s}_0 = (\sqrt{E}, 0)$. Also sincd

$$s_1(t) = \sqrt{E}\cos(\theta)\phi_1(t) - \sqrt{E}\sin(\theta)\phi_2(t)$$

we have $\mathbf{s}_1 = (\sqrt{E}\cos(\theta), \sqrt{E}\sin(\theta)$. A block diagram or the optimal receiver is shown below.



Note that we have ignored the term $\sqrt{E}$ *in the multipliers.*

2. We have $P(E) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$, where

$$d^2 = ||\mathbf{s}_1 - \mathbf{s}_1||^2 = [\sqrt{E} - \sqrt{E}\cos(\theta)]^2 + [\sqrt{E}\sin(\theta)]^2 = 2E[1 - \cos(\theta)]$$

To minimize $P(E)$ we need to maximize d. From the above we see that d is maximized for $\theta = \pi$ which results in $d^2 4E$. For this value of θ the signal set is BPSK. We need $\frac{E_b}{N_0} \ge 9.6$dB and $W \ge R_b$.

Problem 2

1. $\frac{E_b}{N_0} = \frac{P}{RN_0} = \frac{150{,}000}{5{,}000} = 30$. Thus $10\log\frac{E_b}{N_0} = 14.7$ dB. Thus the available SNR is 14.77 dB. The available $R/W = \frac{5{,}000}{2\alpha} = 5/8$. Our system must require $\frac{E_b}{N_0} \le 14.7$ dB and must have $R_b/W \ge 5/8$. The simplest system with these requirements is BPSK.

$$s_0(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_0 t), \quad s_1(t) = \sqrt{\frac{2E}{T}}\cos(2\pi f_0 t + \pi), \qquad 0 \le t \le T$$

where $E = E_b = 30N_0$ and $T = T_b = 1/R = .2$ msec.

2. $\frac{E_b}{N_0} = \frac{P}{RN_0} = \frac{1{,}250{,}000}{5{,}000} = 250$. Thus $10\log\frac{E_b}{N_0} \approx 24$ dB. Thus the available SNR is 24 dB. The available $R/W = \frac{5{,}000}{2\alpha} = 4.54$. Our system must require $\frac{E_b}{N_0} \leq 24$ dB and must have $R_b/W \geq 4.54$. The simplest system with these requirements is 32PSK.

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left(2\pi f_0 t + \frac{2\pi(i-1)}{32}\right),\;\; i = 1,2,\cdots,32, \quad 0 \leq t \leq T$$

where $E = E_b\log_2 32 = 5E_b = 5\times 250N_0 = 1250N_0$ and $T = T_b\log_2 32 = 5T_b = 5/R = 1$ msec.

3. $\frac{E_b}{N_0} = \frac{P}{RN_0} = \frac{320{,}000}{5{,}000} = 64$. Thus $10\log\frac{E_b}{N_0} \approx 18.06$ dB. Thus the available SNR is 18.06 dB. The available $R/W = \frac{5{,}000}{2\alpha} = 2.94$. Our system must require $\frac{E_b}{N_0} \leq 18.06$ dB and must have $R_b/W \geq 2.94$. The simplest system with these requirements is 8ASK.

$$s_i(t) = a_i\sqrt{\frac{2}{T}}\cos(2\pi f_0 t), \quad i = 1,2,\cdots,8, \quad 0 \leq t \leq T$$

where

$$a_i \in \left\{\pm\frac{7A}{2}, \pm\frac{5A}{2}, \pm\frac{3A}{2}, \pm\frac{A}{2}\right\}$$

With this selection

$$E_{\text{ave}} = \frac{1}{8}\times 2\times\left[\frac{49A^2}{4} + \frac{25A^2}{4} + \frac{9A^2}{4} + \frac{A^2}{4}\right] = \frac{21A^2}{4}$$

Now $E_{\text{ave}} = E_b\log_2 8 = 3E_b = 192N_0$. Thus

$$A\sqrt{\frac{4E_{\text{ave}}}{21}} = 16\sqrt{\frac{N_0}{7}}$$

Also $T = T_b\log_2 8 = 3T_b = 3/R = .6$ msec.

Problem 3

1. We have $W = 2.25 - 2 = .25$ MHz. Therefore, $R_b/W = 4$ bits/sec/Hz. The system we choose must have $R_b/W \geq 4$.

   (a) We can use q-ary PSK for $q \geq 16$. 16PSK has the smallest power requirement for which $\frac{E_b}{N_0} = 17.4$ dB or 55. Thus

   $$\frac{E_b}{N_0} = \frac{P}{R_b N_0} = 55, \quad \Longrightarrow P = 55\times R_b\times N_0 = 110\, W$$

   (b) No FSK system can provide $R_b/W \geq 4$.

   (c) We can use q-ary ASK for $q \geq 16$. 16ASK has the smallest power requirement for which $\frac{E_b}{N_0} = 23.2$ dB or 207. Thus $P = 207\times R_b\times N_0 = 414$ W.

(d) Again we can use 16QASK which requires $\frac{E_b}{N_0} = 13.2$ dB or 20.9. Thus $P = 20.9 \times R_b \times N_0 = 41.8$ W.

(e)

$$\frac{E_b}{N_0} = \frac{2^{R_b/W} - 1}{R_b/W} = \frac{2^4 - 1}{4} = 3.75$$

Thus

$$P = 3.75 \times R_b \times N_0 = 7.5W$$

2. $R_b = 125$ Kbits/s.Then $R_b/W = .5$ bits/sec/Hz. The system we choose must have $R_b/W \geq .5$.

(a) Any PSK system can be used. For BPSK or QPSK we get the smallest P. $\frac{E_b}{N_0} = 9.6$ dB or 9.12. Thus

$$\frac{E_b}{N_0} = \frac{P}{R_b N_0} = 9.12, \quad \Longrightarrow P = 9.12 \times R_b \times N_0 = 2.28\ W$$

(b) Any FSK system can be used. For 8FSK we get the smallest P. $\frac{E_b}{N_0} = 7$. Thus

$$\frac{E_b}{N_0} = \frac{P}{R_b N_0} = 7, \quad \Longrightarrow P = 7/4 = 1.75\ W$$

(c) Any ASK system can be used. For BASK we get the smallest P. $\frac{E_b}{N_0} = 9.12$. Thus

$$\frac{P}{R_b N_0} = 9.12, \quad \Longrightarrow P = 2.28\ W$$

(d)

$$\frac{E_b}{N_0} = \frac{2^{R_b/W} - 1}{R_b/W} = \frac{2^{.5} - 1}{.5} = .828$$

Thus

$$P = .207W$$

Problem 4

1. There are 8 signals in the signal set. THus $T = T_b \log_2 8 = 3/R_b = 3$msec.

2.

$$\int_0^T \{A\cos[2\pi f_0 t + \theta]\}^2 dt = \frac{A^2 T}{2}$$

Thus

$$E_{\text{ave}} = \frac{1}{8}\left\{4 \times \frac{a^2 T}{2} + 4 \times \frac{(3a)^2 T}{2}\right\} = 2.5a^2 T$$

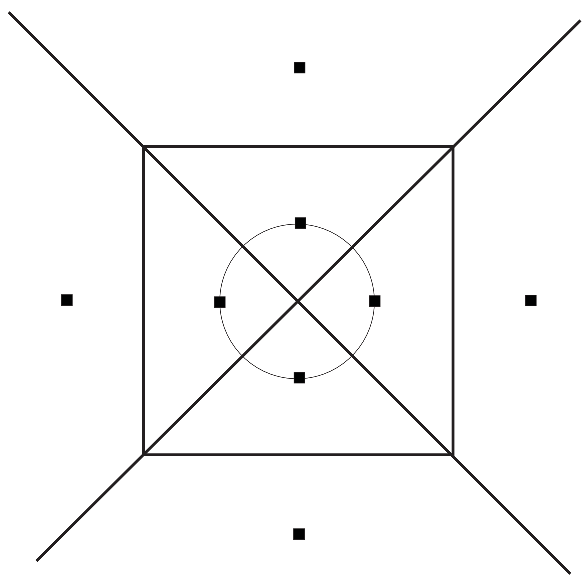
3. The following is a basis for the signal set.

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t),\ \ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_0 t),\quad 0 \le t \le T$$

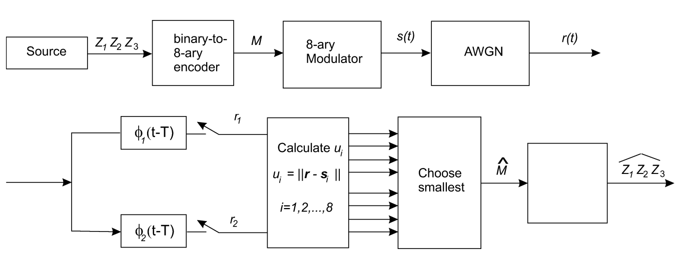
The signals are then given by

$$\begin{aligned} s(t) &= A\cos(\theta)\cos(2\pi f_0 t) - A\sin(\theta)\sin(2\pi f_0 t) \\ &= A\sqrt{\frac{T}{2}}\cos(\theta)\phi_1(t) - A\sqrt{\frac{T}{2}}\sin(\theta)\phi_2(t) \end{aligned}$$

for which $\mathbf{s} = (A\sqrt{\frac{T}{2}}\cos(\theta), A\sqrt{\frac{T}{2}}\sin(\theta))$. The vector representation and the decision regions are plotted below.



4. The transmitter and receiver block diagrams are shown below.



Source
$Z_1 Z_2 Z_3$
binary-to-8-ary encoder
M
8-ary Modulator
$s(t)$
AWGN
$r(t)$
ϕ_1(t-T)
r_1
ϕ_2(t-T)
r_2
Calculate u_i
$u_i = ||\mathbf{r} - \mathbf{s}_i||$
$i=1,2,...,8$
Choose smallest
$\hat{M}$
$\widehat{Z_1 Z_2 Z_3}$

5. Using the union bound we get

$$P(E) \leq \frac{1}{8}\sum_{i=1}^{8}\sum_{j\neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right) \leq 7Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 7Q\left(\sqrt{\frac{a^2}{N_0}}\right)$$

since $d_{\min} = a\sqrt{2}$.

Problem 5

1. Let $y(t)$ denote the output of the filter. Then $R = y(T)$ and

$$\begin{aligned} y(t) = r(t) * h(t) &= \int_0^t r(\tau)h(t-\tau)\, d\tau \\ &= \int_0^t s(\tau)e^{-a(t-\tau)}\, d\tau + \int_0^t N_W(\tau)e^{-a(t-\tau)}\, d\tau \end{aligned}$$

Then

$$R = y(T) = e^{-aT}\int_0^T s(\tau)e^{a\tau}\, d\tau + N = s + N$$

where

$$N = e^{-aT}\int_0^T N_W(\tau)e^{a\tau)}\, d\tau$$

For $s_0(t)$ we have $s = 0$ and for $s_1(t)$ we get $s = \frac{1}{a}\sqrt{\frac{E}{T}}(1-e^{-aT})$. N is Gaussian and we need to find its mean and variance. Now $E(N) = 0$ and

$$E[N^2] = e^{-2aT}\int_0^T\int_0^T E[N_W(\tau_1)N_W(\tau_2)]e^{a\tau_1)}e^{a\tau_2)}\, d\tau_1 d\tau_2$$

$$= e^{-2aT}\int_0^T\int_0^T N_0/2\delta(\tau_1-\tau_2)e^{a\tau_1}e^{a\tau_2)}\, d\tau_1 d\tau_2$$

$$= e^{-2aT}N_0/2\int_0^T e^{2a\tau_1}\, d\tau_1 = \frac{N_0}{4a}\left[1-e^{-2aT}\right]$$

Therefore $\sigma_N^2 = \frac{N_0}{4a}\left[1-e^{-2aT}\right]$. Now $P(E) = Q\left(\frac{d}{2\sigma_N}\right)$. Thus

$$P(E) = Q\left[\frac{\sqrt{E}(1-e^{-aT})}{\sqrt{aTN_0(1-e^{-2aT})}}\right]$$

2. We need to maximize the argument of the Q function above. Let

$$f(a) = \frac{\sqrt{E}(1-e^{-aT})}{\sqrt{aTN_0(1-e^{-2aT})}}$$

Taking the derivative of $f(a)$ with respect to a we see that $a = 0$ sets the derivative to zero. For $a = 0$

$$h(t) = \begin{cases} 1 & 0 \leq t \\ 0 & \text{otherwise.} \end{cases}$$

But since we sample at $t = T$, this is the same as

$$h(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

which is the matched filter. No surprize.

3. For the optimal matched filter receiver we have

$$P(E) = Q\left(\sqrt{\frac{ET}{2N_0}}\right)$$