EE 7620 Solutions to Exam I Oct. 21, 2008

1:00-3:00 pm.

Problem 1

1. (a) The MAP rule is given by

$$g(r) = 2 \text{ iff } \; p_M(2)p_{R|M}(r|2) \geq p_M(-2)p_{R|M}(r|-2)$$

We know that $p_{R|M}(r|i) = P_N(r-i)$. Thus

$$\begin{aligned} g(r) &= 2 \text{ iff } \; .75P_N(r-2) \geq .25p_N(r+2) \\ & \quad 3p_N(r-2) \geq p_N(r+2) \end{aligned}$$

Thus

$$g(r) = \begin{cases} 2 & -2 \leq r \leq 6 \\ -2 & -6 \leq r \leq 2 \end{cases}$$

(b)

$$P(E) = p_M(2)P(E|M=2) + p_M(-2)P(E|M=-2)$$

Now

$$P(E|M=2) \int_{-6}^{-2} p_{R|M}(r|2)\, dr = 0$$

and

$$P(E|M=-2) = \int_{-2}^{6} p_{R|M}(r|-2)\, dr = \int_{-2}^{2} \frac{1}{8}\, dr = 1/2$$

This gives $P(E) = .125$.

2. (a) Again

$$g(r) = 2 \text{ iff } \; p_M(2)p_{R|M}(r|2) \geq p_M(-2)p_{R|M}(r|-2)$$

Examining the PDF's we see that

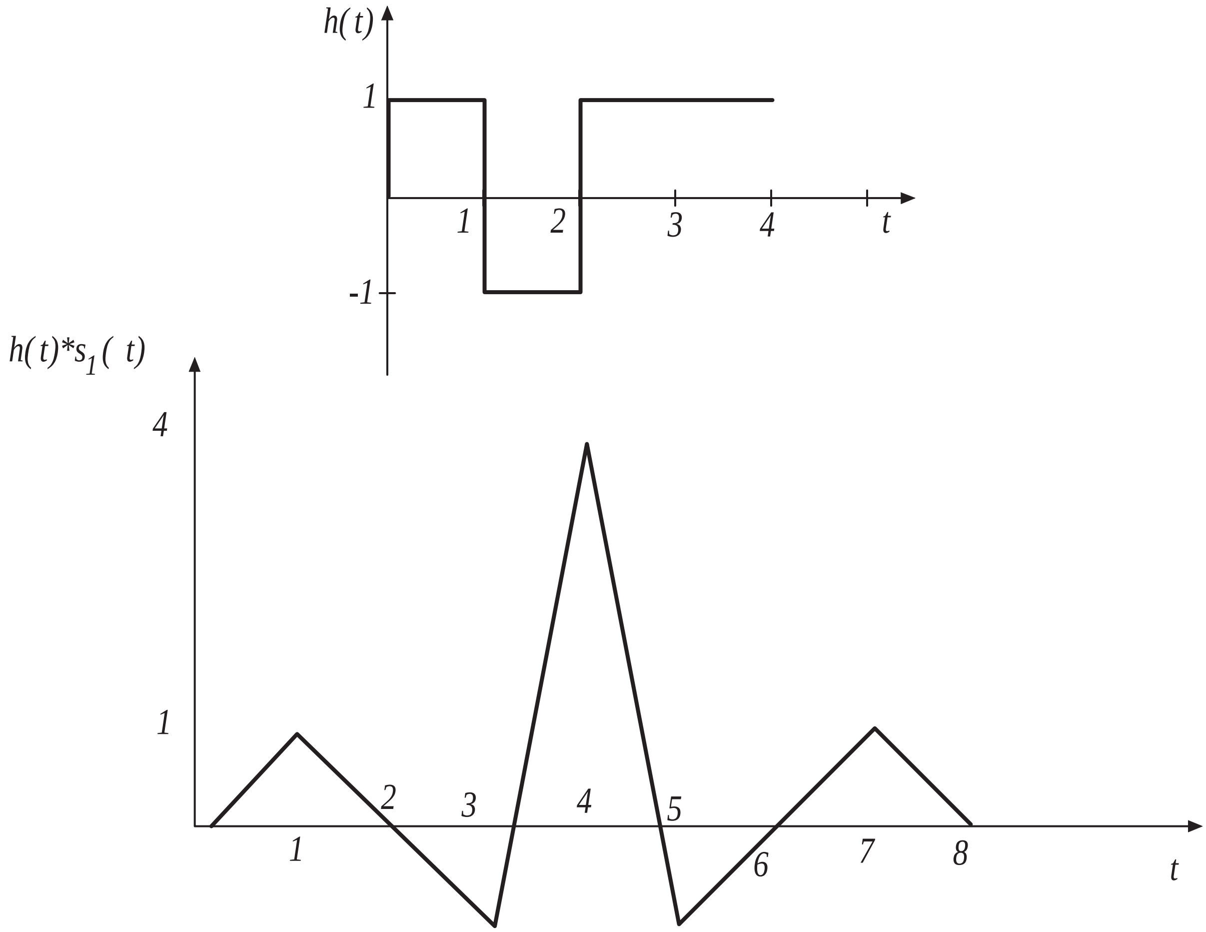
$$g(r) = \begin{cases} 2 & 0 \leq r \leq 4 \\ -2 & -4 \leq r \leq 0 \end{cases}$$

(b) In this case

$P(E|M=2) = P(-4 \leq R < 0|M=2) = 0$ and $P(E|M=-2) = P(0 \leq R < 4|M=-2) = 0$

Thus $P(E) = 0$.

Problem 2



We have $P(E) = Q\left[\frac{d}{\sqrt{2N_0}}\right]$ where $d = ||s_1(t) - s_2(t)|| = \sqrt{5}$. Thus $P(E) = Q\left[\sqrt{\frac{5}{2N_0}}\right]$.

Problem 3

1. We know that $m_X = E[X(t)] = 0$. Then

$$E[Y(t)] = E[X(t-T)] = m_X$$

$$R_Y(t+\tau,t) = E[Y(t+\tau)Y(t)] = E[X(t+\tau-T)X(t-T)] = R_X(\tau)$$

   This show that $\{Y(t)\}$ is a WSS process.

2.

$$E[Z(t)] = E[X(t^2)] = m_X$$

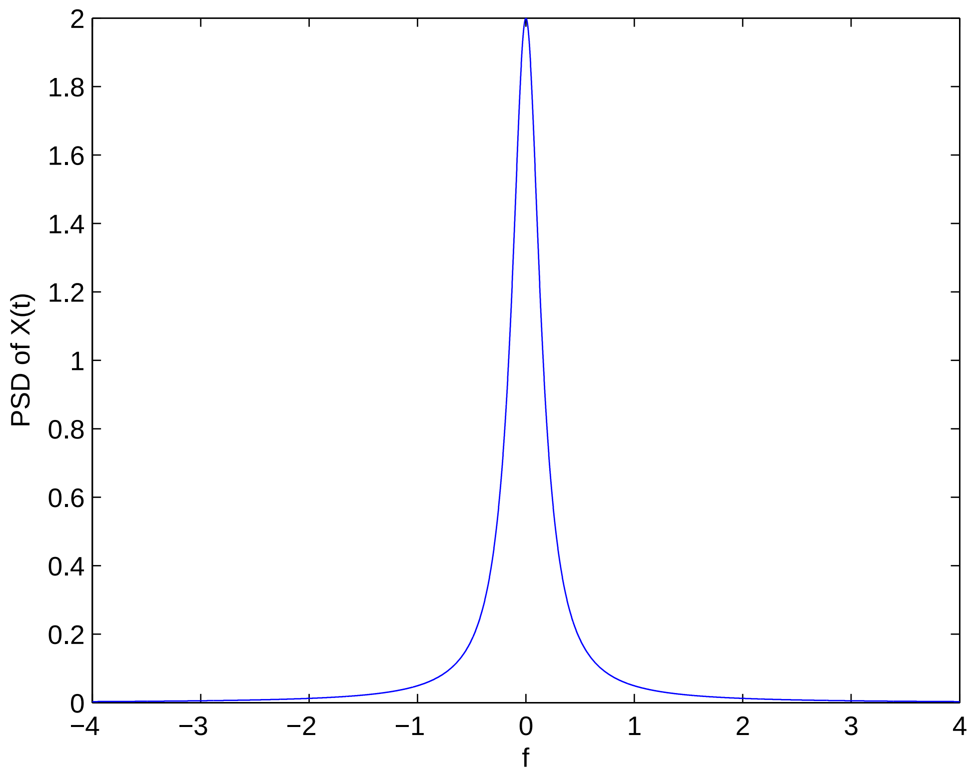
$$R_Z(t+\tau,t) = E[Z(t+\tau)Z(t)] = E[X(t+\tau)^2X(t^2)] = R_X(2t\tau+\tau^2)$$

   It can be seen that in general $R_Z(t+\tau,t)$ depends on t and thus the process is not WSS.

3. We have $S_V(f) = S_X(f)|H(f)|^2$. Now

$$S_X(f) = \mathcal{F}(R_X(\tau)) = \frac{2}{1+4\pi^2 f^2}$$

This is plotted below. Now $S_V(f) = S_X(f)$ for $2 < |f| < 4$ and zero otherwise. To find the



PDF of $V(0)$ we know that $\{V(t)\}$ is a Gaussian random process. Therefore we only need the mean and variance of $V(0)$.

$$E[V(0)] = m_X \int_{-\infty}^{\infty} h(t)\, dt = 0$$

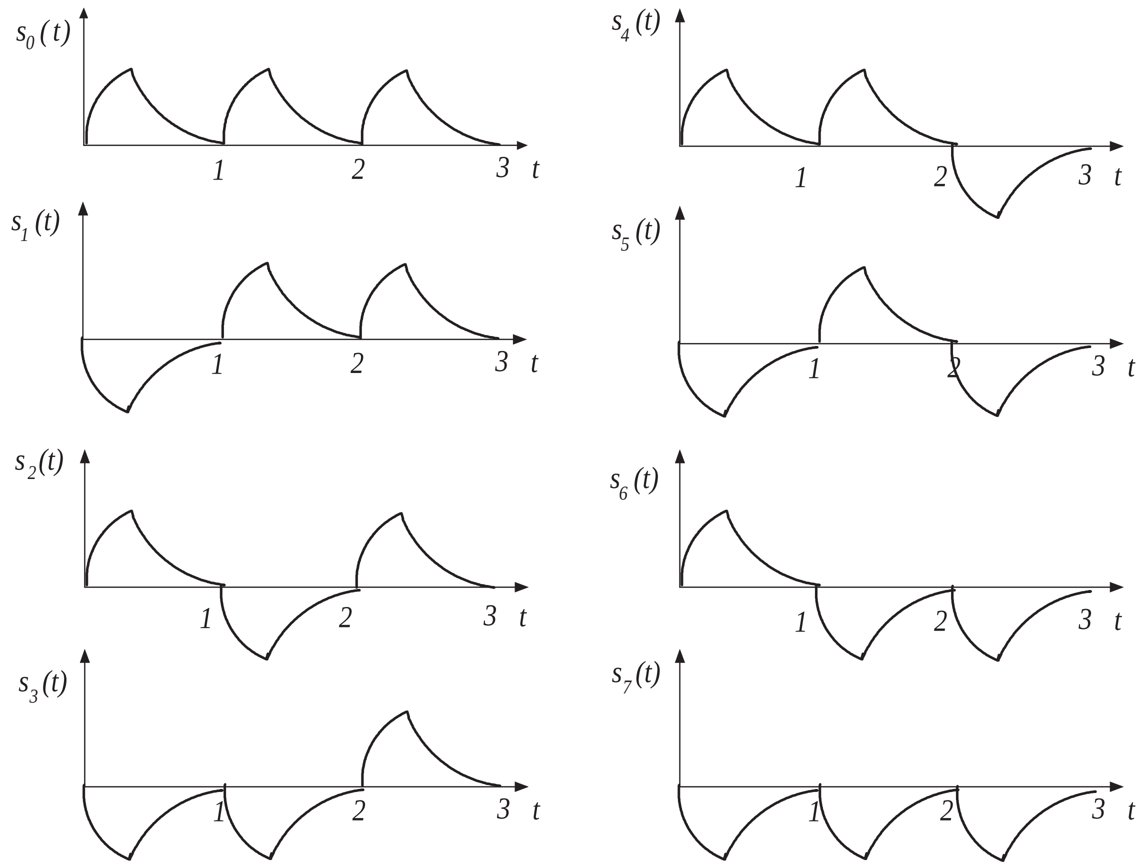
$$E[V(0)]^2 = R_V(0) = \int_{-\infty}^{\infty} S_V(f)\, df = 2\int_2^4 S_X(f)\, df = \sigma^2$$

where σ^2 is a constant. Then $\text{var}[V(0)] = \sigma^2$. Thus

$$f_{V(0)}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Problem 4

1. The eight signals are plotted below.



   Note that the signal duration is $T = 3$.

2. It is clear that the set of signals given by

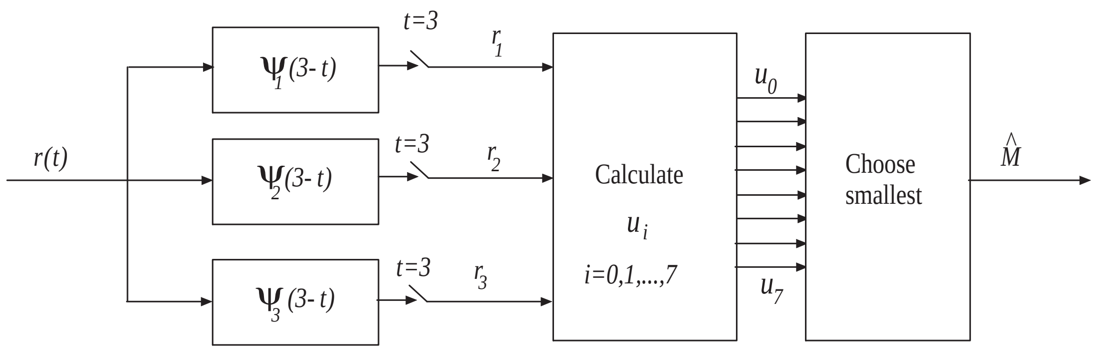
$$\psi_1(t) = \phi(t), \;\; \psi_2(t) = \phi(t-1), \;\; \psi_3(t) = \phi(t-2),$$

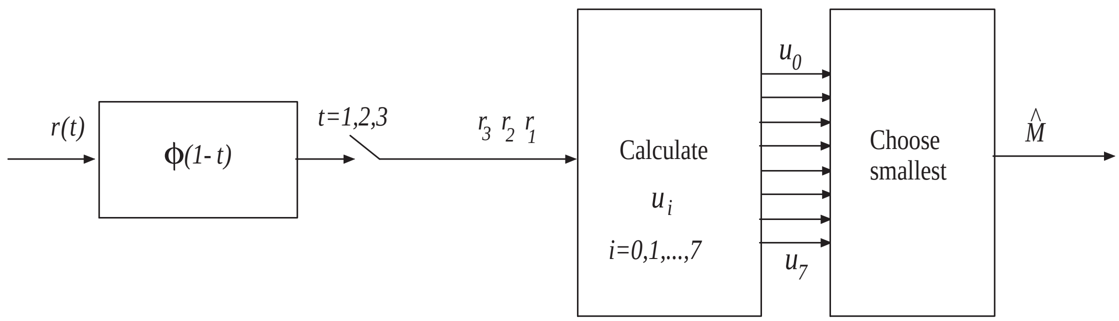
   forms an O.N. basis for the signal set. The vector representation of the signals are then given by

$$\mathbf{s}_0 = (1,1,1), \;\; \mathbf{s}_1 = (1,1,-1), \;\; \mathbf{s}_2 = (1,-1,1), \;\; \mathbf{s}_3 = (1,-1,-1),$$

$$\mathbf{s}_4 = (-1,1,1), \;\; \mathbf{s}_5 = (-1,1,-1), \;\; \mathbf{s}_6 = (-1,-1,1), \;\; \mathbf{s}_7 = (-1,-1,-1)$$

   It can be seen that the vectors form the vertices of a hypercube. The optima receiver is diagrammed below where $u_i = ||\mathbf{r} - \mathbf{s}_i||^2$. Note that instead of three filters we can use one filter matched to $\phi(t)$ as shown in the second figure.





3. Since the signal set is the vertices of a hypercube, the probability of error is given by

$$P(E) = 1 - \left[1 - Q\left(\sqrt{\frac{2E}{LN_0}}\right)\right]^L$$

where $E = ||\mathbf{s}_i||^2 = 3$ and $L = 3$. Thus

$$P(E) = 1 - \left[1 - Q\left(\sqrt{\frac{2}{N_0}}\right)\right]^3$$