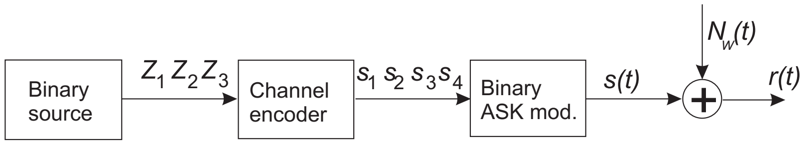
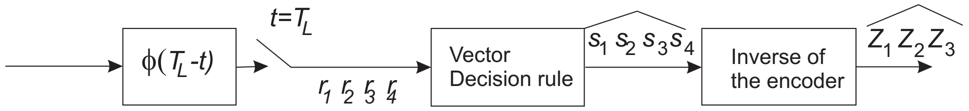
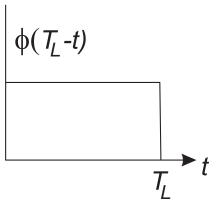
EE 7620 Solutions to Problem Set # 6 12,1,08

1. (a) $K=3, L=4$. The code rate is $R_c = K/L = 3/4$.







(b)

The binary ASK modulator multiplies each s_i by a delayed version of the a unit energy pulse such as $\phi(t)$ plotted above.

$$T_L = T/L = KT_b/L = .75T_b = \frac{.75}{R_b}.$$

The vector decision rule is

$$\hat{\mathbf{s}} = \mathbf{s}_i \text{ if } ||\mathbf{r}-\mathbf{s}_i|| \leq ||\mathbf{r}-\mathbf{s}_i|| \text{ for all } j \neq i$$

The encoding rule is given by

Z_1, Z_2, Z_3	$\mathbf{s}$
000	$\mathbf{s}_1$
001	$\mathbf{s}_2$
010	$\mathbf{s}_3$
011	$\mathbf{s}_4$
100	$\mathbf{s}_5$
101	$\mathbf{s}_6$
110	$\mathbf{s}_7$
111	$\mathbf{s}_8$

The decoding rule is the inverse of the table above.

(c)

$$P(E) \leq \frac{1}{2^K} \sum_{i=1}^{8} \sum_{j \neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

where $d_{ij} = ||\mathbf{s}_i - \mathbf{s}_j||$. Now for this signal set for each i there is one j such that $d_{ij} = 4\sqrt{E_c}$ and for all other signals $d_{ik} = 2\sqrt{2E_c}$. Thus

$$P(E) \leq \frac{1}{2^K} 2^K \left[Q\left(\frac{4\sqrt{E_c}}{\sqrt{2N_0}}\right) + (2^K - 2) Q\left(\frac{2\sqrt{2E_c}}{\sqrt{2N_0}}\right)\right] = Q\left(\frac{2\sqrt{2E_c}}{\sqrt{N_0}}\right) + 6Q\left(2\sqrt{\frac{E_c}{N_0}}\right)$$

(d) The signal set is biorthogonal with energy $4E_c$. The exact formula for $P(E)$ is given in the text.

(e) From the figure in the text $\frac{E_b}{N_0} \approx 8.4$dB.

(f) From the union bound we get

$$P(E) \leq Q\left(2\sqrt{\frac{3E_b}{2N_0}}\right) + 6Q\left(2\sqrt{\frac{3E_b}{4N_0}}\right)$$

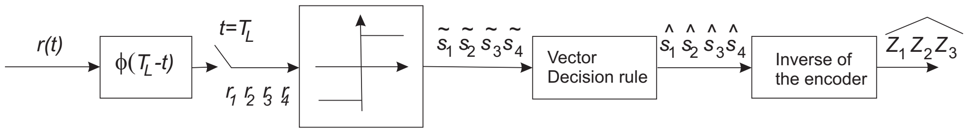
Now the first term on the right hand side is much smaller and can be ignored. Therefore setting

$$6Q\left(2\sqrt{\frac{3E_b}{4N_0}}\right) = 10^{-5} \implies 2\sqrt{\frac{3E_b}{4N_0}} = 4.65 \implies \frac{E_b}{N_0} = 7.2 \implies \frac{E_b}{N_0} = 8.57 \text{ dB}$$

(g)

$$W = \frac{1}{2T_L} = \frac{1}{2}\frac{4}{3T_b} = \frac{2}{3}R_b \implies \frac{R_b}{W} = \frac{3}{2}.$$

(h)



The vector decision rule is

$$\hat{\mathbf{s}} = \mathbf{s}_i \text{ iff } d_H(\mathbf{s}_i, \tilde{\mathbf{s}}) \leq d_H(\mathbf{s}_j, \tilde{\mathbf{s}}) \text{ forall } j \neq i$$

Ties are broken arbitrarily.

(i) By inspection $d_{\min} = 2$. The error correcting capability is $t = \lfloor \frac{d_{\min}-1}{2} \rfloor = 0$. Thus he code is not even guaranteed to correct all error patterns with one error.

(j)

$$P(E) \leq \sum_{k=t+1}^{L} \binom{L}{k} p^k (1-p)^{L-k} \approx \binom{L}{t+1} p^{t+1} (1-p)^{L-t-1} = 4p(1-p)^3$$

where $p = Q\left(\sqrt{\frac{2E_c}{N_0}}\right)$. The exact expression for error probability can aslo be found and is given by

$$P(E) = 3p - 3p^2 + p^3$$

(k)

$$P(E) \leq 4p^1(1-p)^3 = 10^{-5} \implies p \approx 2.5 \times 10^{-6}$$

From $p = Q\left(\sqrt{\frac{2E_c}{N_0}}\right) = Q\left(\sqrt{\frac{6E_b}{4N_0}}\right)$. Thus $\frac{E_b}{N_0} = 13.9$ or 11.45 dB.

(l) Comparing parts e and f we see that the union bound is pretty good. Comparing e,f with k we see that hard decision looses about 2.9 dB of $\frac{E_b}{N_0}$.

2. We can do this for both the hard decision receiver or the optimum receiver. First try part b.

(a)

$$W = \frac{1}{2T_L} = \frac{1}{2}\frac{L}{T} = \frac{1}{2}\frac{L}{KT_b} = \frac{L}{2K}R_b$$

Thus $\frac{R_b}{W} = 2\frac{K}{L} = \frac{2\times 12}{23} = 1.04$.

(b) With an optimum receiver we se the union bound.

$$P(E) \leq \frac{1}{2^K}\sum_i \sum_{j\neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

Now

$$d_{ij}^2 = \sum_{k=1}^{L}(s_{ik} - s_{jk})^2 = 4E_c d_H(\mathbf{s}_i, \mathbf{s}_j) \implies d_{ij} = 2\sqrt{E_c}\sqrt{d_H(\mathbf{s}_i,\mathbf{s}_j)} \geq 2\sqrt{E_c}\sqrt{d_{\min}}$$

$$P(E) \leq \frac{1}{2^K}\sum_i \sum_{j\neq i} Q\left(2\frac{\sqrt{E_c d_{\min}}}{\sqrt{2N_0}}\right) = (2^K - 1)Q\left(2\sqrt{\frac{E_c d_{\min}}{2N_0}}\right)$$

Now since $E_c = \frac{12}{23}E_b$,

$$P(E) \leq (2^{12}-1)Q\left(\sqrt{2\times\frac{12}{23}\times 7\frac{E_b}{N_0}}\right) = (2^{12}-1)Q\left(\sqrt{7.3\frac{E_b}{N_0}}\right)$$

Setting the upper bound to 10^{-5} we get $\frac{E_b}{N_0} = 4.7$ or 6.7 dB.

For a hard decision receiver

$$P(E) \leq \begin{pmatrix} L \\ t+1 \end{pmatrix} p^{t+1}(1-p)^{L-t-1}$$

where $p = Q\left(\sqrt{\frac{2E_c}{N_0}}\right) = Q\left(\sqrt{\frac{2KE_b}{LN_0}}\right)$ and $t = \lfloor\frac{d_{\min}-1}{2} = 3$. Thus

$$P(E) \leq \begin{pmatrix} 23 \\ 4 \end{pmatrix} p^4(1-p)^{19} \approx \begin{pmatrix} 23 \\ 4 \end{pmatrix} p^4 = 10^{-5}, \implies p \approx .006$$

$$p = Q\left(\sqrt{\frac{2\times 12\times E_b}{23\times N_0}}\right) = .006$$

This gives $\frac{E_b}{N_0} = 5.9$ or 7.7 dB.

The difference between hard decision and optimum (soft decision) receivers is about 1 dB. However it should be noted that we have used upper bounds here and not the exact values.