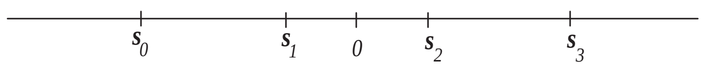
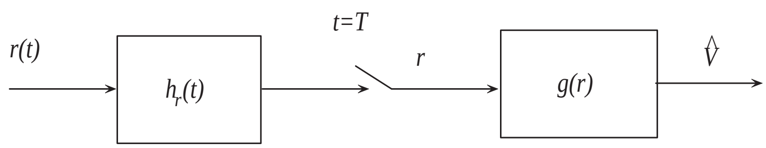
EE 7620 Solutions to Problem Set # 4 11,03,08

1. Let $\phi(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)$ for $0 \le t \le T$. Then we get the following signal vectors.

$$s_i(t) = d\left(i + \frac{1-q}{2}\right)\phi(t), \iff \mathbf{s}_i = d\left(i + \frac{1-q}{2}\right)$$

(a) For $q = 4$,





where $h_r(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)$ and

$$g(r) = \begin{cases} m_0 & r \le -d \\ m_1 & -d < r \le 0 \\ m_2 & 0 < r \le d \\ m_3 & d < r \end{cases}$$

(b) For the two signals on the outside, we have

$$P(E|\mathbf{s}_0) = P(E|\mathbf{s}_{q-1}) = P(N \ge \frac{d}{2}) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

For all the other $q-2$ signals, $(i \ne 0, q-1)$

$$P(E|\mathbf{s}_i) = P(N \ge \frac{d}{2} \text{ or } N \le -\frac{d}{2}) = 2P(N \ge \frac{d}{2}) = 2Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

Thus

$$P(E) = \frac{1}{q}\sum_{i=0}^{q-1} P(E|\mathbf{s}_i) = \frac{2(q-1)}{q} Q(\frac{d}{\sqrt{2N_0}}) \approx 2Q(\frac{d}{\sqrt{2N_0}})$$

(c)

$$\begin{aligned} E_{ave} &= \frac{1}{q}\sum_{i=0}^{q-1} E_i = \frac{1}{q}\left[\left(\frac{-q+1}{2}\right)^2 d^2 + \left(\frac{-q+3}{2}\right)^2 d^2 + \cdots + \left(\frac{q-1}{2}\right)^2 d^2\right] \\ &= \frac{1}{q}\frac{d^2}{2}\sum_{i=1}^{q/2}(2i-1)^2 \\ &= \frac{d^2}{2q}\frac{4(q/2)^3 - q/2}{3} = \frac{d^2}{12}(q^2-1) \end{aligned}$$

(d) Let A_k denote the amplitude of the signal in the kth interval $[(k-1)T, kT]$. Then

$$A_k \in \{(d(i+1/2-q/2), i = 0,1,2,\cdots,q-1\}$$

Assuming that all values of A_k are equally likely we get $E[A_k] = 0$. Thus the power spectral density of the transmitted signal is given by

$$S_s(f) = \frac{E[A_k^2]}{T}|\Phi(f)|^2$$

where $\Phi(f)$ is the Fourier transform of $\phi(t)$ and thus is given by

$$|\Phi(f)| = \frac{\sqrt{T}}{2}\left|\frac{\sin(\pi(f-f_0)T)}{\pi(f-f_0)T} + \frac{\sin(\pi(f+f_0)T)}{\pi(f+f_0)T}\right|$$

The 3 dB bandwidth is approximately given by $W = \frac{1}{T}$.

(e) Binary data is encoded into q-ary symbols using a K-to-1 encode. These symbols are sent to the q-ary modulator which produces a q-ary ASK signal of duration $T = KT_b = K/R_b$. At the receiver the q-ary demodulator demodulates the signal into one of the q-ary symbols and a 1-to-K decoder decodes this symbol into K bits.

(f) $E_b = \frac{E_{ave}}{K}$. Thus

$$\frac{E_b}{N_0} = \frac{d^2(q-1)^2}{12KN_0}$$

$$\frac{R_b}{W} = \frac{1/T_b}{1/T} = K$$

$$P(E) = \frac{2(q-1)}{q} Q\left[\sqrt{\frac{6K}{q^2-1}\frac{E_b}{N_0}}\right]$$

(g) We can use the Gray code.

$$\mathbf{s}_0 \iff (0000), \mathbf{s}_1 \iff (0001), \mathbf{s}_2 \iff (0011), \mathbf{s}_3 \iff (0010),$$

$$\mathbf{s}_4 \iff (0110)\mathbf{s}_5 \iff (0100), \mathbf{s}_6 \iff (0101), \mathbf{s}_7 \iff (0111),$$

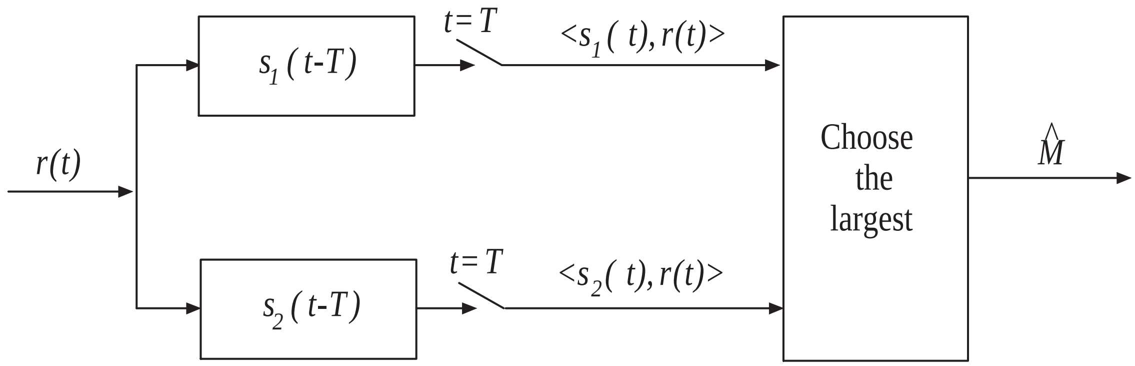
$$\mathbf{s}_8 \iff (1111)\mathbf{s}_9 \iff (1101), \mathbf{s}_{10} \iff (1100), \mathbf{s}_{11} \iff (1110),$$

$$\mathbf{s}_{12} \iff (1010), \mathbf{s}_{13} \iff (1011), \mathbf{s}_{14} \iff (1001), \mathbf{s}_{15} \iff (1000)$$

(h) The following table provides the R_b/W and the E_b/N_0 (needed to get $P_b(E) \leq 10^{-5}$.

2. (a) The signal space is not one-dimensional (it is at least two dimensional). Therefore we need at least two filters even if we find an ON basis. Thus we can use the signals themselves to compute the correlations.

q	R_b/W	E_b/N_0	E_b/N_0 (dB)
2	1	9.1	9.6
4	2	23.7	13.7
8	3	67.5	18.3
16	4	207	23.2
32	5	670	28.2



where

$$\hat{M} = \begin{cases} m_0 & \langle r(t), s_0(t) \rangle \ \geq \ \langle r(t), s_1(t) \rangle \\ m_1 & \text{otherwise.} \end{cases}$$

(b) Since there are only two signals, $P(E) = Q\left[\frac{d}{\sqrt{2N_0}}\right]$, and since Q function is monotone decreasing we should maximize d. But

$$d^2 = ||s_0(t)||^2 + ||s_1(t)||^2 - 2\langle s_0(t), s_1(t) \rangle = 2E - 2\langle s_0(t), s_1(t) \rangle$$

Now

$$\begin{aligned} \langle s_0(t), s_1(t) \rangle &= \frac{2E}{T}\int_0^T \cos(2\pi f_0 t)\cos(2\pi f_0 t + 2\pi h t/T)\, dt \\ &= \frac{E}{T}\int_0^T \cos(2\pi h t/T)\, dt + \frac{E}{T}\int_0^T \cos[2\pi(2f_0 + h/T)t]\, dt \\ &\approx E\frac{\sin(2\pi h)}{2\pi h} \end{aligned}$$

where the last approximation follows because $f_0 >> 1$ and as a result the second integral will be zero or approximately zero. Therefore

$$d^2 = 2E\left[1 - \frac{\sin(2\pi h)}{2\pi h}\right]$$

Differentiating with respect to h to find the maximum we get $h \approx .75$.

(c)

$$d^2 = 2E\left[1 - \frac{\sin(3\pi/2)}{3\pi/2}\right] = 2E\left[1 + \frac{2}{3\pi}\right]$$

$$P_b(E) = Q\left[\frac{d}{\sqrt{2N_0}}\right] = Q\left[\sqrt{\frac{E(1+\frac{2}{3\pi})}{N_0}}\right]$$

To get $P_b(E) = 10^{-5}$, we need $\frac{E}{N_0}(1+\frac{2}{3\pi}) = 18.2$ or $\frac{E}{N_0} = 15$. Since $E = E_b$, we get $\frac{E_b}{N_0} = 15$ or $\frac{E_b}{N_0} = 11.76$ dB. The bandwidth is given by

$$W = \frac{1}{2T} + \frac{.75}{T} + \frac{1}{2T} = \frac{7}{4T}$$

Since $T = T_b$, we get $W = \frac{7}{4T_b} = \frac{7R_b}{4}$. Thus $\frac{R_b}{W} = \frac{4}{7}$.