EE 7620 Solutions to Problem Set # 3 Oct. 2008

Problem 4.10:

(a) To show that the waveforms $f_n(t)$, $n = 1, 2, 3$ are orthogonal, we have to prove that

$$\int_{-\infty}^{\infty} f_m(t) f_n(t)\ dt = 0, \quad forall \ \ m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \int_0^4 f_1(t) f_2(t) dt \\ &= \int_0^2 f_1(t) f_2(t) dt + \int_2^4 f_1(t) f_2(t) dt \\ &= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\ &= 0 \end{aligned}$$

Similarly:

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t) dt = \int_0^4 f_1(t) f_3(t) dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

and :

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t) dt = \int_0^4 f_2(t) f_3(t) dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \ \ i = 1, 2, 3$$

Hence, they are orthonormal.

(b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t) dt, \qquad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t) f_1(t) dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t) f_2(t) dt = \frac{1}{2}\int_0^4 x(t) dt = 0 \\ x_3 &= \int_0^4 x(t) f_3(t) dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 4.11 :

(a) As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$
$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

(b) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

(c) The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

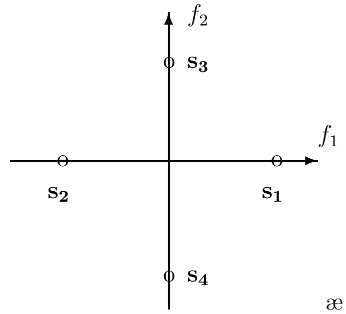
Similarly we find that :

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 4.18 :

$$\mathbf{s}_1 = \left(\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_2 = \left(-\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_3 = \left(0, \sqrt{\mathcal{E}}\right)$$
$$\mathbf{s}_4 = \left(0, -\sqrt{\mathcal{E}}\right)$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

Problem 5.1 :

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \operatorname{sgn}(t) - \operatorname{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

where $sgn(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ i rectangular pulse of unit height and width, centered at $x = 0$.

(b) The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 5.2 :

(a) The impulse response of the matched filter is :

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(b) The output of the matched filter at $t = T$ is :

$$\begin{aligned} g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\ &\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\ &= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\ &= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right] \end{aligned}$$

(c) The output of the correlator at $t = T$ is :

$$\begin{aligned} q(T) &= \int_0^T s^2(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)d\tau \end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

Problem 5.4 :

(a) The correlation type demodulator employs a filter :

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{T}} & 0 \leq \mathrm{t} \leq \mathrm{T} \\ 0 & \mathrm{o.w} \end{array} \right\}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance :

$$\sigma_n^2 \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\text{PM}(\mathbf{r}, \mathbf{s}_0)}{\text{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently :

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\text{SNR}}\right]
\end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 5.6 :

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E\left[|n(T)|^2\right]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{array}{rcl} E\left[|n(T)|^2\right] & = & \int_0^T \int_0^T E\left[z(a)z^*(b)\right] h_l(T-a)h_l^*(T-b)dadb \\ & = & 2N_0 \int_0^T |h_l(T-t)|^2 dt \end{array}$$

so we want to maximize :

$$SNR = \frac{\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2}{2N_0 \int_0^T |h_l(T-t)|^2 dt}$$

From Schwartz inequality :

$$\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt$$

Hence :

$$SNR \leq \frac{1}{2N_0}\int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

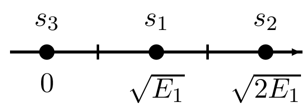
$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$

Problem 4.5

1. Note that $s_2(t) = 2s_1(t)$ and $s_3(t) = 0s_1(t)$, hence the system is PAM and a singular basis function of the form $\phi_1(t) = \frac{1}{A\sqrt{T}}s_1(t)$ would work

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 < t \leq T/3 \\ -\frac{1}{\sqrt{T}} & T/3 \leq t < T \end{cases}$$

Assuming $E_1 = A^2T$, we have $s_3 = 0$, $s_1 = \sqrt{E_1}$, $s_2 = 2\sqrt{E_1}$. The constellation is shown below.



2. For equiprobable messages the optimal decision rule is the nearest neighbor rule and the perpendicular bisectors are the boundaries of the decision regions as indicated in the figure.

3. This is ternary PAM system with the distance between adjacent pints in the constellation being $d = \sqrt{E_1} = A\sqrt{T}$. The average energy is $E_{\text{avg}} = \frac{1}{3}(0 + A^2T + 4A^2T) = \frac{5}{3}A^2T$, and $E_{\text{bavg}} = E_{\text{avg}}/\log_2 3 = \frac{5}{3\log_2 3}A^2T$, from which we obtain

$$d^2 = \frac{3\log_2 3}{5}E_{\text{bavg}} \approx 0.951 E_{\text{bavg}}$$

The error probability of the optimal detector is the average of the error probabilities of the three signals. For the two outer signals error probability is $P(n > d/2) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$ and for the middle point s_1 it is $P(|n| > d/2) = 2Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$. From this,

$$P_e = \frac{4}{3}Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{0.951E_{\text{bavg}}}{2N_0}}\right) = 43Q\left(\sqrt{0.475\frac{E_{\text{bavg}}}{2N_0}}\right)$$

4. $R = R_s \log_2 M = 3000 \times \log_2 3 \approx 4755$ bps.

Problem 4.6

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.