EE 7620 Solutions Problem Set # 2 10,06,08

1. The following fact regarding Gaussian random variables is used in the solution.

   Fact: If X_1, X_2, X_3 and X_4 are four Gaussian random variables, then

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3] \ .$$

   (a)

$$E[Y(t)] = E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df = 2$$

$$E[Z(t)] = m_Y \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

   where $h(\tau)$ is the impulse response of the filter. From Fourier transforms we know

$$\int_{-\infty}^{\infty} h(\tau)\, d\tau = H(0) = 1$$

   Thus $E[Z(t)] = 2$.

   (b)

$$R_Y(t,s) = E[Y(t)Y(s)] = E[X^2(t)X^2(s)]$$

   Using the fact above we get

$$R_Y(t,s) = E[X^2(t)]E[X^2(s)] + 2\{E[X(t)X(s)\}^2 = R_X(0)R_X(0) + 2\{R_X(t-s)\}^2$$

   Thus

$$R_Y(\tau) = 2R_X^2(\tau) + R_X^2(0)$$

   We see that the process $\{Y(t)\}$ is a WSS process.

$$S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = 2S_X(f) * S_X(f) + R_X^2(0)\delta(f)$$

   where $*$ is the convolution operation. Therefore $S_Y(f) = G(f) + R_X^2(0)\delta(f)$ where $G(f)$ is given by

$$G(f) = \begin{cases} 2 - |f| & |f| \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Since $Z(t)$ is the response of a linear time-invariant system to a WSS input, it is WSS. We know that $S_Z(f) = S_Y(f)|H(f)|^2$. Thus

$$S_Z(f) = \begin{cases} G(f) & < 1|f| < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(d)

$$E[Z^2(t)] = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)\, df = 1$$

2. (a)

$$\begin{aligned}
R_Y(t,t+\tau) &= E[Y(t)Y(t+\tau)] = E[a^2\sin^2(2\pi f_0 t+\Theta)a^2\sin^2(2\pi f_0(t+\tau)+\Theta)] \\
&= \frac{a^4}{4}E\{[1-\cos(4\pi f_0 t+2\Theta)][1-\cos(4\pi f_0(t+\tau)+2\Theta)]\} \\
&= \frac{a^4}{4}\{1-E[\cos(4\pi f_0 t+2\Theta)]-E[\cos(4\pi f_0(t+\tau)+2\Theta)] \\
&\quad +E[\cos(4\pi f_0 t+2\Theta)\cos(4\pi f_0(t+\tau)+2\Theta)]\} \\
&= \frac{a^4}{4}+\frac{a^4}{4}E[\frac{1}{2}\cos(4\pi f_0\tau)+\frac{1}{2}\cos(4\pi f_0(2t+\tau)+4\Theta)] \\
&= \frac{a^4}{4}[1+\frac{1}{2}\cos(4\pi f_0\tau)]
\end{aligned}$$

Therefore, $R_Y(t,t+\tau)$ depends only on τ.

$$R_Y(\tau) = \frac{a^4}{4}[1+\frac{1}{2}\cos(4\pi f_0\tau)]$$

(b)

$$\begin{aligned}
R_{XY}(t,t+\tau) &= E[X(t)Y(s)] = E[a\sin(2\pi f_0 t+\Theta)a^2\sin^2(2\pi f_0(t+\tau)+\Theta)] \\
&= \frac{a^3}{2}E\{\sin(2\pi f_0 t+\Theta)[1-\cos(4\pi f_0(t+\tau)+2\Theta)]\} \\
&= -\frac{a^3}{2}E\{\sin(2\pi f_0 t+\Theta)\cos(4\pi f_0(t+\tau)+2\Theta)\} \\
&= -\frac{a^3}{4}E\{\sin(2\pi f_0(3t+2\tau)+3\Theta)\}-\frac{a^3}{4}E\{\sin(2\pi f_0(t+2\tau)+\Theta)\}=0
\end{aligned}$$

(c) $\{X(t)\}$ is WSS (Shown in class). As for $\{Y(t)\}$, $R_Y(t,s)$ only depends on $t-s$ as shown before. Next

$$m_Y(t) = E[Y(t)] = E[X^2(t)] = R_X(0) = \frac{a^2}{2}$$

since $R_X(\tau)=\frac{a^2}{2}\cos(2\pi f_0\tau)$. Thus, $m_Y(t)$ is independent of t. Therefore $\{Y(t)\}$ is also WSS.

(d) Yes. $\{X(t)\}$ and $\{Y(t)\}$ are individually WSS and since $R_{XY}(t,s)=0$, it only depends on the time difference $t-s$.

3. (a)

$$E[Y] = E[\int_0^2 X(t)dt] = \int_0^2 E[X(t)]dt = 6$$

(b)

$$\begin{aligned} E[Y^2] &= E[\int_0^2 X(t)dt \int_0^2 X(s)ds] \\ &= \int_0^2 \int_0^2 E[X(t)X(s)]dtds = \int_0^2 \int_0^2 R_X(t-s)dtds \end{aligned}$$

Making a change of variables by setting $t-s=\tau$ and $s=u$ we get

$$\begin{aligned} E[Y^2] &= \int_{\tau=-2}^{0} \int_{u=-\tau}^{2} R_X(\tau)dud\tau + \int_{\tau=0}^{2} \int_{u=0}^{2-\tau} R_X(\tau)dud\tau \\ &= \int_{\tau=-2}^{0} R_X(\tau)(2+\tau)d\tau + \int_{\tau=0}^{2} R_X(\tau)(2-\tau)d\tau \\ &= 2\int_{\tau=0}^{2} R_X(\tau)(2-\tau)d\tau \\ &= 2\int_{\tau=0}^{2} (9+2e^{-\tau})(2-\tau)d\tau \\ &= 40+4e^{-2} \end{aligned}$$

Thus $\text{var}(Y) = 4+4e^{-2}$.

4.

$$E[X(t)] = E[Y(t)Z(t)] = E[Y(t)]E[Z(t)] = 0$$

and

$$\begin{aligned} R_X(s,t) = E[X(s)X(t)] = E[Y(s)Y(t)Z(s)Z(t)] \\ = E[Y(s)Y(t)]E[Z(s)Z(t)] = R_Y(s,t)R_Z(s,t) \end{aligned}$$

Clearly $\{X(t)\}$ is WSS.

5.

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[X(t)\cos(2\pi f_1 t+\Theta)X(t+\tau)\cos(2\pi f_1(t+\tau)+\Theta)] \\ &= E[X(t)X(t+\tau)]E[\cos(2\pi f_1 t+\Theta)\cos(2\pi f_1(t+\tau)+\Theta)] \\ &= R_X(\tau)\{1/2\cos(2\pi f_1\tau) + E[\cos(2\pi f_1(2t+\tau)+2\Theta)]\} \\ &= 1/2 R_X(\tau)\cos(2\pi f_1\tau) \end{aligned}$$

Therefore, using the convolution

$$\begin{aligned} S_Y(f) &= \frac{1}{4} S_X(f) * [\delta(f-f_1)+\delta(f-f_1)] \\ &= \frac{1}{4}[S_X(f-f_1)+S_X(f+f_1)] \\ &= \frac{\sigma^2}{8f_0}[\text{rect}(\frac{f-f_1}{2f_0}) \ + \ \text{rect}(\frac{f+f_1}{2f_0})] \end{aligned}$$

where

$$\mathrm{rect}(x) = \begin{cases} 1 & -1/2 \le x \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

6.

$$H_0: \quad p_R(r|H_0) = \begin{cases} 2e^{-2(r-1)} & r \ge 1 \\ 0 & r < 1 \end{cases}$$

$$H_1: \quad p_R(r|H_1) = \begin{cases} e^{-r} & r \ge 0 \\ 0 & r < 0 \end{cases}$$

(a)

$$I_{H_0} = \{r : \frac{1}{3} p_R(r|H_0) \ge \frac{2}{3} p_R(r|H_1)\}$$

For $0 \le r \le 1$, $p_R(r|H_0) = 0$. Thus $r \in I_{H_1}$. For $r > 1$,

$$\frac{1}{3} 2e^{-2(r-1)} \ge \frac{2}{3} e^{-r}, \quad \Longrightarrow r \le 2$$

Thus for $1 \le r \le 2$, $r \in I_{H_0}$.

$$I_{H_0} = \{r : 1 \le r \le 2\}$$
$$I_{H_1} = \{r : 0 \le r \le 1\} \cup \{r : r > 2\}$$

(b)

$$P(E) = p(H_0)P(E|H_0) + p(H_1)P(E|H_1) =$$

$$\frac{1}{3}[\int_0^1 0\, dr + \int_2^\infty 2e^{-2(r-1)}\, dr] + \frac{2}{3}\int_1^2 e^{-r}\, dr = \frac{2}{3e} - \frac{1}{3e^2} \approx .2$$

7. (a) i. It can easily be seen that

$$p_{Y|X}(y|1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad \text{and} \quad p_{Y|X}(y|2) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{y^2}{8}}$$

Now the maximum likelihood rule gives $g(y) = 1$ if and only if $p_{Y|X}(y|1) \ge p_{Y|X}(y|2)$. After simplification this gives $g(y) = 1$ if and only if $y^2 \le \frac{8}{3}\ln(2)$ or $-1.36 \le y \le 1.36$.

ii. We have

$$p(C|X=1) = \int_{-1.36}^{1.36} p_{Y|X}(y|1)dy = 1 - 2Q(1.36)$$

and thus $P(E|X=1) = 2Q(1.36)$. Similarly,

$$P(E|X=2) = \int_{-1.36}^{1.36} p_{Y|X}(y|2)dy = 1 - 2Q(.68)$$

Thus

$$P(E) = .5[P(E|X=1) + P(E|X=2)] = .5 + Q(1.36) - Q(.68)$$

(b) Let $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$. Then

$$p_{\mathbf{Y}|X}(y_1,...,y_n|1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} \quad \text{and} \quad p_{\mathbf{Y}|X}(y_1,...,y_n|2) = \prod_{i=1}^{n} \frac{1}{2\sqrt{2\pi}} e^{-\frac{y_i^2}{8}}$$

Now the maximum likelihood rule gives $g(y_1,...,y_n) = 1$ if and only if

$$p_{\mathbf{Y}|X}(y_1,...,y_n|1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} \geq p_{\mathbf{Y}|X}(y_1,...,y_n|2) = \prod_{i=1}^{n} \frac{1}{2\sqrt{2\pi}} e^{-\frac{y_i^2}{8}}$$

which gives $g(y_1,...,y_n) = 1$ if and only if $\sum_{i=1}^{n} y_i^2 \leq \frac{8n}{3} \ln(2)$.

8. (a)

$$g(r_1,r_2) = 0 \text{ iff } p_M(0) p_{R_1R_2|M}(r_1,r_2|0) \geq p_M(1) p_{R_1R_2|M}(r_1,r_2|1)$$

or iff

$$.25 p_0 e^{-|r_1|} e^{-|r_2|} \geq .25 p_1 e^{-|r_1-1|} e^{-|r_2|}$$

or $g(r_1,r_2) = 0$ iff $r_1 \leq \frac{1 - \ln p_1 + \ln p_0}{2}$.

(b) In this case $g(r_1,r_2) = 0$ iff $r_1 \leq .5$.

$$P(E) = .5P(E|M=0) + .5P(E|M=1)$$

$$= .5 \int_{.5}^{\infty} \int_{\infty}^{\infty} .25 p_{R_1R_2|M}(r_1,r_2|0) dr_2 dr_1 + .5 \int_{-\infty}^{.5} \int_{\infty}^{\infty} .25 p_{R_1R_2|M}(r_1,r_2|1) dr_2 dr_1$$

Thus

$$P(E) = .5 \int_{.5}^{\infty} \int_{\infty}^{\infty} .25 e^{-|r_1|} e^{-|r_2|} dr_2 dr_1$$

$$+.5 \int_{-\infty}^{.5} \int_{\infty}^{\infty} e^{-|r_1-1|} e^{-|r_2|} dr_2 dr_1 = .5e^{-.5}$$