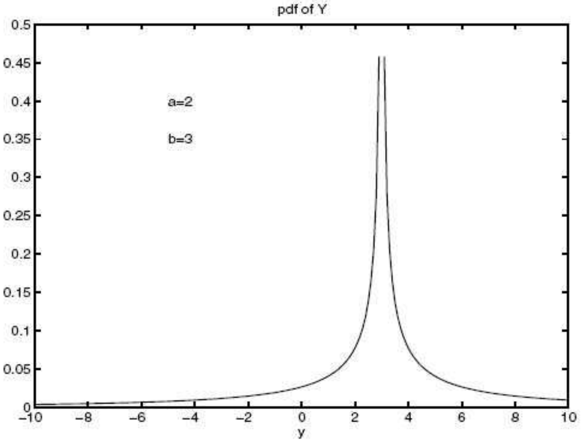
EE 7620 Solutions to Problem Set # 1

1. Relationship (2-1-44) gives :

$$p_Y(y) = \frac{1}{3a[(y-b)/a]^{2/3}}\ p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

X is a Gaussian r.v. with zero mean and unit variance, i.e., $p_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Hence

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}[(y-b)/a]^{2/3}}e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$



2. (a) We have

$$\psi_Y(v) = E\left[e^{jvY}\right] = E\left[e^{jv\sum_{i=1}^{n} X_i}\right] = E\left[\prod_{i=1}^{n} e^{jvX_i}\right] = \prod_{i=1}^{n} E\left[e^{jvX}\right] = [\psi_X(v)]^n$$

Now

$$\psi_X(v) = 1 - p + pe^{jv}$$

Thus

$$\psi_Y(v) = \left(1 - p + pe^{jv}\right)^n$$

(b)

$$\begin{aligned} E[Y] &= -j\frac{d\psi_Y(v)}{dv}|_{v=0} = np \\ E[Y^2] &= -\frac{d^2\psi_Y(v)}{dv^2}|_{v=0} = n^2p^2 + np(1-p) \end{aligned}$$

3. (a)

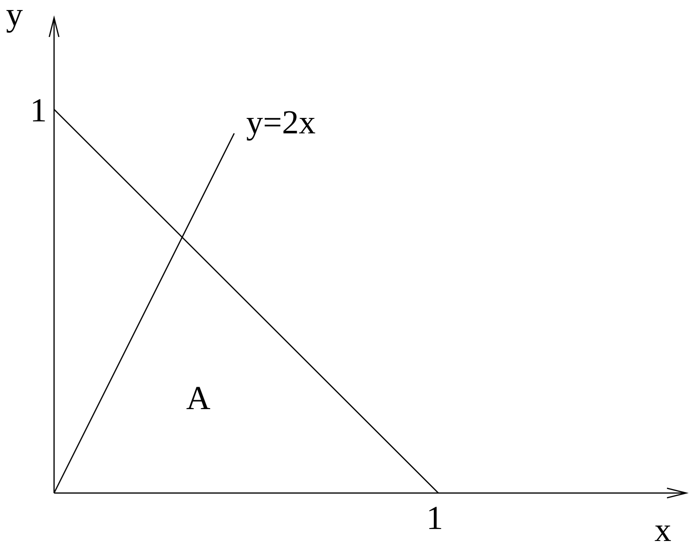
$$\int\int p_{XY}(x,y)dxdy = 1$$

Then

$$\int_{y=0}^{1}\int_{x=0}^{1-y} Cdxdy = 1 \quad \Longrightarrow \quad C = 2.$$

(b)

$$P(2X \geq Y) = \int\int_A p_{XY}(x,y)dxdy = \int\int_A 2dxdy = 2\text{area}(A) = 2(1/3) = 2/3.$$



(c)

$$P(X \leq \alpha) = 0 \quad \text{for} \quad \alpha < 0$$

If $0 \leq \alpha < 1$,

$$P(X \leq \alpha) = \int\int_B 2dxdy = 2\alpha - \alpha^2.$$

For $\alpha \geq 1$, $P(X \leq \alpha) = 1$. Therefore,

$$F_X(\alpha) = \begin{cases} 0 & \alpha < 0 \\ 2\alpha - \alpha^2 & 0 \leq \alpha < 1 \\ 1 & \alpha \geq 1. \end{cases}$$

From $F_X(\alpha)$ we can evaluate $p_X(\alpha)$.

$$p_X(\alpha) = \begin{cases} 2-2\alpha & 0 \le \alpha < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) By symmetry we have

$$F_Y(\beta) = \begin{cases} 0 & \beta < 0 \\ 2\beta - \beta^2 & 0 \le \beta < 1 \\ 1 & \beta \ge 1. \end{cases}$$

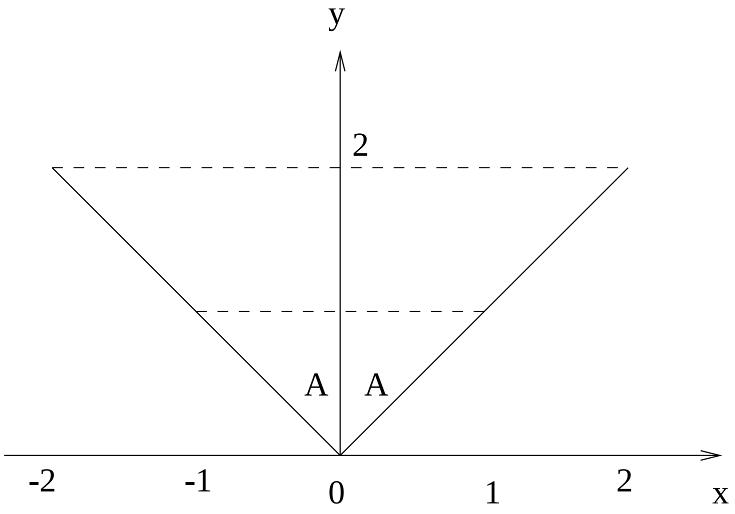
From $F_Y(\beta)$ we can evaluate $p_Y(\beta)$.

$$p_X(\beta) = \begin{cases} 2-2\beta & 0 \le \beta < 1 \\ 0 & \text{otherwise} \end{cases}$$

We see that $p_{XY}(x,y) \neq p_X(x)p_Y(y)$. Therefore, X and Y are dependent.

4. (a)

$$F_X(x|C) = P(X \le x | 0 \le Y < 1) = \frac{P(X \le x, 0 \le Y < 1)}{P(0 \le Y < 1)}$$



Now

$$P(C) = \int\int_A p_{XY}(x,y)dxdy = .25\text{area}(A) = .25.$$

For $x < -1$, $P(X \le x, 0 \le Y < 1) = 0$.
For $-1 \le x < 0$,

$$P(X \le x, 0 \le Y < 1) = (1+x)^2/8.$$

For $0 \le x < 1$,

$$P(X \le x, 0 \le Y < 1) = 1 - (1-x)^2/8.$$

And for $x \geq 1$, $P(X \leq x, 0 \leq Y < 1) = P(0 \leq Y < 1) = .25$. Thus

$$F_X(x|C) = \begin{cases} 0 & x < -1 \\ (1+x)^2/2 & -1 \leq x < 0 \\ 1-(1-x)^2/2 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Then we get

$$p_X(x|C) = \begin{cases} 1+x & -1 \leq x < 0 \\ 1-x & 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$E[X|C] = \int x p_X(x|C) dx = 0$$

5. The power spectral density of WGN process is given by

$$S_{XX}(f) = \int \phi_X X(\tau) e^{-j2\pi f\tau}\, d\tau = \frac{N_0}{2}$$

The power spectral density of output process is given by

$$S_{YY}(f) = S_{XX}(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Thus the noise power in the output process is

$$\phi_{YY}(0) = \int S_{YY}(f)\, df = \frac{N_0}{2} \int |H(f)|^2$$

Thus

$$\phi_{YY}(0) = N_0 B$$

6. We know that Y_1 and Y_2 are jointly Gaussian. So we calculate the mean vector and the covariance matrix. $EY_1 = 2EX_1 + EX_2 + 3 = 3$ and $EY_2 = -1$.

$$\text{var}(Y_1) = 4\text{var}(X_1) + \text{var}(X_2) = 5 \quad \text{and} \quad \text{var}(Y_2) = \text{var}(X_1) + \text{var}(X_2) = 2$$

$$\text{cov}(Y_1, Y_2) = E[(Y_1 - 3)(Y_2 + 1)] = E[(2X_1 + X_2)(X_1 - X_2)] = 1$$

Then

$$p_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi\sqrt{9}} \exp\{-\frac{1}{18}[2(y_1-3)^2 - 2(y_1-3)(y_2+1) + 5(y_2+1)^2]\}$$

7. We will denote the discrete-time process by the subscript d and the continuous-time (analog) process by the subscript a. Also, f will denote the analog frequency and f_d the discrete-time frequency.

(a) The autocorrelation of $\{X_n\}$ is given by

$$\phi_d(k) = E[X^*(n)X(n+k)] = E[X^*(nT)X(nT+kT)] = \phi_a(kT).$$

Hence, the autocorrelation function of the sampled signal is equal to the sampled autocorrelation function of $\{X(t)\}$.

(b)

$$\begin{aligned}\phi_d(k) &= \phi_a(kT) = \int_{-\infty}^{\infty} S_a(f) e^{j2\pi fkT}\, df \\ &= \sum_{i=-\infty}^{\infty} \int_{(2i-1)/2T}^{(2i+1)/2T} S_a(f) e^{j2\pi fkT}\, df \\ &= \int_{-1/2T}^{1/2T} \sum_{i=-\infty}^{\infty} \left[S_a(f + \frac{i}{T}) \right] e^{j2\pi fkT}\, df\end{aligned}$$

Let $f_d = fT = \frac{f}{f_s}$. Then

$$\phi_d(k) = \int_{-1/2}^{1/2} \left[\frac{1}{T} \sum_{i=-\infty}^{\infty} S_a\left(\frac{f_d + i}{T}\right) \right] e^{j2\pi f_d k}\, df_d \tag{1}$$

Now we know that

$$\phi_d(k) = \int_{-1/2}^{1/2} S_d(f_d) e^{j2\pi f_d k}\, df_d \tag{2}$$

Comparing (1) and (2) we get

$$S_d(f_d) = \frac{1}{T} \sum_{i=-\infty}^{\infty} S_a\left(\frac{f_d + i}{T}\right) \tag{3}$$

(c) From (3) we get that

$$S_d(f_d) = \frac{1}{T} S_a\left(\frac{f_d}{T}\right)$$

if and only if

$$S_a(f) = 0 \;\; \forall\, f : |f| > 1/2T$$

Otherwise the sum of the shifted copies of $S_a(f)$ will overlap and cause aliasing.

Note: This problem describes the sampling theorem for random processes.