EE 7620 **Problem Set # 2** **Due: 10,06,08**

1) Please attempt each problem on a new page.
2) Show all your work clearly.

1. $\{X(t)\}$ is a zero mean wide sense stationary Gaussian random process whose power spectral density is given by

$$S_X(f) = \begin{cases} 1 & |f| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\{X(t)\}$ is the input into a square law device whose output process $\{Y(t)\}$ is given by $Y(t) = [X(t)]^2$. $\{Y(t)\}$ is then the input into a linear time-invariant system with frequency response given below.

$$H(f) = \begin{cases} 1 & 1 < |f| < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let the output of the filter be denoted as $\{Z(t)\}$.

   (a) Calculate the mean of the processes $\{Y(t)\}$ and $\{Z(t)\}$.
   (b) Calculate the autocorrelation function of $\{Y(t)\}$ in terms of that of $\{X(t)\}$. Is $\{Y(t)\}$ a wide sense stationary process? If yes, find its power spectral density.
   (c) Is $\{Z(t)\}$ a wide snese stationary process? If yes, evaluate its power spectral density.
   (d) Find $E\{[Z(t)]^2\}$.

2. Given the random process $\{X(t)\}$ such that

$$X(t) = a\sin(2\pi f_0 t + \Theta)$$

where a and f_0 are constants and Θ is a random variable uniformly distributed on the interval $(-\pi, \pi)$, define a new random process $\{Y(t)\}$ such that $Y(t) = X^2(t)$.

   (a) Find the autocorrelation function of $\{Y(t)\}$.
   (b) Find the cross-correlation function of $\{X(t)\}$ and $\{Y(t)\}$.
   (c) Are $\{X(t)\}$ and $\{Y(t)\}$ wide sense stationary?
   (d) Are $\{X(t)\}$ and $\{Y(t)\}$ jointly wide sense stationary?

3. If $X(t)$ is a stationary random process having a mean value $E[X(t)] = 3$ and autocorrelation function $R_X(\tau) = 9 + 2e^{-|\tau|}$, find the mean and variance of the random variable

$$Y = \int_0^2 X(t)dt.$$

(Hint: Assume expectation and integration operations are interchangeable.)

4. Let $\{Y(t) : t \in \Re\}$ and $\{Z(t) : t \in \Re\}$ be two independent random processes each with mean zero and

$$R_Y(s,t) = 2e^{-|s-t|}\cos[2\pi f_c(s-t)]$$

and

$$R_Z(t,s) = 4 + 2e^{-2|s-t|^2}$$

Let $X(t) = Y(t)Z(t)$. Find $E[X(t)]$ and $R_X(s,t)$. Is $\{X(t)\}$ a WSS process?

5. Find the power spectral density of the random process $\{Y(t)\}$ where

$$Y(t) = X(t)\cos(2\pi f_1 t + \Theta).$$

$\{X(t)\}$ is a wide sense stationary random process with autocorrelation function

$$R_X(\tau) = \sigma^2 \frac{\sin(2\pi f_0 \tau)}{2\pi f_0 \tau}$$

where σ is a positive constant. Θ is a random variable uniformly distributed over $(0, 2\pi)$ and is independent of $\{X(t)\}$.

6. An observation R is characterized under two hypotheses according to the following probability density functions.

$$H_0 : \quad p_R(r|H_0) = \begin{cases} 2e^{-2(r-1)} & r \geq 1 \\ 0 & r < 1 \end{cases}$$

$$H_1 : \quad p_R(r|H_1) = \begin{cases} e^{-r} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

The probabilities of the two hypotheses H_0 and H_1 are $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

   (a) Find the minimum error probability test between the hypotheses H_0 and H_1. Specify the decision regions.

   (b) Find the probability of error for the decision rule of part (1).

7. A random variable X has alphabet $\{1, 2\}$ and it is given that $P(X = 1) = .5$.

   (a) We would like to estimate X from the observation of the random variable Y where $Y = NX$ and where N is a zero mean unit variance Gaussian random variable.

      i. Find the decision rule which minimizes the probability of error for X based on Y.

      ii. Find the probability of error for your decision rule.

   (b) Now suppose we have n observations $Y_1, Y_2, ..., Y_n$ where $Y_i = N_i X$ for $i = 1, 2, ..., n$, and where $N_1, N_2, ..., N_n$ are independent, identically distributed zero mean unit variance Gaussian random variables. Find the decision rule which minimizes the probability of error for X based on $Y_1, Y_2, ..., Y_n$.

8. Three random variable M, R_1 and R_2 are given with $\Omega_M = \{0,1\}$, $p_M(0) = p_0$, $p_M(1) = p_1$ and $\frac{p_1}{p_0} < e$. We also know that

$$p_{R_1R_2|M}(r_1, r_2|m) = .25e^{-|r_1-m|}e^{-|r_2|}.$$

(a) Find the optimum decision rule for M based on (R_1, R_2).

(b) Assuming $p_0 = .5$, find the probability of error for this decision rule.