A Parametric Optimization Approach to $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Strong Stabilization *

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September 2001, Revised May 2002

Abstract

This paper presents an approach for designing stable MIMO $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers by directly computing the norm constrained stable transfer matrices Q in the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ suboptimal controller parameterizations. This is done by first converting the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ strong stabilization problems into some nonlinear unconstrained optimization problems through explicit parameterization of the norm constrained Q's for any fixed order. Then, a two-stage numerical search is carried out by using a combination of a genetic algorithm and a quasi-Newton algorithm in order to reach an optimal solution. The effectiveness of the proposed algorithms is illustrated through some benchmark numerical examples.

Keywords: Strong Stabilization, $\mathcal{H}_\infty$ Control, $\mathcal{H}_2$ Control, Genetic Algorithm

1 Introduction

Over the last four decades, the development of modern control theory and design techniques has made it possible for control engineers, at least theoretically, to design sophisticated and high performance robust controllers for highly complicated interconnected systems. For practical considerations, it is always desirable and in many cases necessary to make sure that the controllers are stable themselves without sacrificing much of the desired performance since unstable controllers tend to be highly sensitive to model uncertainties, unmodelled nonlinearities and sensor/actuator faults. However, there is no guarantee from the current state of the art design techniques that the controllers obtained through these techniques are stable themselves. A necessary and sufficient condition for the existence of a stable stabilizing (*strong stabilizing*) controller for a given plant is the so-called parity interlacing property (p.i.p.) and procedures for designing strong stabilizing controllers are outlined in Vidyasagar (1985) and Youla *et al.* (1974). However, integrating these strong stabilization procedures into the state of the art robust control design techniques seems to be very difficult.

*This research was supported in part by grants from AFOSR (F49620-99-1-0179), LEQSF, NASA, and FAI(UASLP).

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It is well-known that the controllers obtained from $\mathcal{H}_2$/LQG optimal control theory are not guaranteed to be stable themselves. In Arelhi *et al.* (1997), Halevi (1994), and Wang and Bernstein (1993), this problem has been analyzed and alternative algorithms have been proposed. The $\mathcal{H}_\infty$ strong stabilization problem has also been addressed in Barabanov (1995) and Özbay (1995). Since the $\mathcal{H}_\infty$ suboptimal controller is in general not unique, it is reasonable to expect that even if the $\mathcal{H}_\infty$ central controller is unstable, there might still be a stable controller that could satisfy the $\mathcal{H}_\infty$ norm bound when the p.i.p. condition is satisfied. In Zeren and Özbay (1997, 1999, 2000) and Choi *et al.* (2000), an approach for designing stable $\mathcal{H}_\infty$ controller has been suggested based on the parameterization of all suboptimal $\mathcal{H}_\infty$ controllers. This approach converts conservatively the stable $\mathcal{H}_\infty$ controller design problem into another 2-block standard $\mathcal{H}_\infty$ problem. Recently, in Campos-Delgado and Zhou (2001) a method to alleviate this conservativeness has been suggested by introducing some weighting functions.

In this paper, a new method is explored by using the direct optimization of the free transfer matrices in the suboptimal $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controller parameterizations. Since the free transfer matrices are norm constrained and can be of any order, the direct optimization can be extremely difficult. This obstacle is removed by explicitly parameterizing the norm constrained transfer matrices in state space. To avoid to be trapped in the local minima during the optimization process, a two-stage optimization scheme is employed where a genetic algorithm is used in conjunction with a gradient-based approach. The basic principles of genetic algorithms were first proposed by Holland (1975) and since then many successful applications of these algorithms in various fields have been reported. For example, some reported applications in control include active noise control (Tang *et al.*, 1995), systems model reduction (Li *et al.*, 1997), weighting function design for $\mathcal{H}_\infty$ loop-shaping (Whidborne *et al.*, 1995), etc. Genetic algorithms along with many other evolutionary schemes are inspired by the natural selection criteria where the stronger organisms are likely to survive after generations. Thus, a parallelism can be drawn with an optimization problem where the evolution period is considered as the optimization time and the most fitted organism in the population will represent the optimal solution. Genetic algorithms present two main characteristics: a multi-directional (random) search and an information exchange among best solutions. These properties can generate new search directions in order to avoid local minima.

The rest of the paper is organized as follows. First, the notation used in the paper is presented in Section 2. Next, in Section 3 a description of the $\mathcal{H}_\infty$ strong stabilization problem and the equivalent optimization problem are outlined. Similarly, the characterization of the $\mathcal{H}_2$ strong stabilization problem is detailed in Section 4. Section 5 presents some numerical examples and in Section 6 some conclusions are drawn.

2 Notation

All notations used in this paper are fairly standard. In particular, let $G(s)$ be an MIMO proper transfer matrix with a state-space realization (A, B, C, D). Then this realization will be denoted by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The set of all proper, rational and stable transfer matrix will be denoted by $\mathcal{RH}_\infty$ and the $\mathcal{H}_\infty$ norm is defined by

$$\|G\|_\infty = \sup_{Re(s)>0} \bar{\sigma}[G(s)] = \sup_{\omega\in\mathcal{R}} \bar{\sigma}[G(j\omega)].$$

Let $\mathcal{RH}_2$ denote the space of all rational, strictly proper and stable transfer matrices. The $\mathcal{H}_2$ norm is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} Trace[G^*(j\omega)G(j\omega)]d\omega.$$

The $\mathcal{H}_2$ norm of a stable rational transfer matrix can be calculated using state space representation. For example, suppose $G(s)$ is a stable transfer matrix with a state space realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. Then

$$\|G\|_2^2 = Trace(B^*QB) = Trace(CPC^*) \tag{1}$$

where P and Q are the controllability and observability Gramians which can be obtained by solving the following Lyapunov equations

$$AP + PA^* + BB^* = 0 \qquad A^*Q + QA + C^*C = 0. \tag{2}$$

Consider a feedback system described by the block diagram in Figure 1 where the generalized plant G and the controller K are assumed to be real-rational and proper with $y(t) \in \mathcal{R}^{p_2}$ and $u(t) \in \mathcal{R}^{m_2}$. Let G be partitioned accordingly as

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array}\right] \tag{3}$$

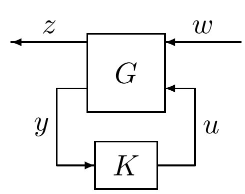

Figure 1: LFT Representation

Then the transfer function from w to z is given by

$$T_{zw} = \mathcal{F}_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

and $\mathcal{F}_l(\cdot, \cdot)$ is called a lower linear fractional transformation.

3 $\mathcal{H}_\infty$ Strong Stabilization

3.1 $\mathcal{H}_\infty$ Strong Stabilization

Consider a feedback system described in Figure 1. Then the following are two well known standard $\mathcal{H}_\infty$ control problems:

- Optimal $\mathcal{H}_\infty$ control: *find all internally stabilizing controllers K(s) such that* $\|\mathcal{F}_l(G, K)\|_\infty$ *is minimized.*

- Suboptimal $\mathcal{H}_\infty$ control: *given* $\gamma > 0$, *find all internally stabilizing controllers* $K(s)$, *if there is any, such that* $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$.

It should be noted that the optimal $\mathcal{H}_\infty$ controller is unique for SISO systems (i.e. $p_2 = m_2 = 1$). However, this is in general not true for MIMO systems in both optimal and suboptimal cases. It has been shown in Doyle *et al.* (1989) that all stabilizing controllers $K(s)$ satisfying the *suboptimal* condition can be parameterized by $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$ such that $K = \mathcal{F}_l(M_\infty, Q)$ with

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] = \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right] \tag{4}$$

where Q is an $m_2 \times p_2$ transfer matrix and M_∞ is an $(p_2 + m_2) \times (p_2 + m_2)$ transfer matrix constructed from the solutions of two Riccati equations (Doyle *et al.*, 1989; Zhou *et al.*, 1996). In the case of $Q = 0$, the solution, $K = M_{11}$, is called the *central controller*. Note that the central controller could be unstable itself since there is no guarantee that M_∞ is itself stable even though the closed-loop stability is maintained.
From practical considerations, it is always desirable to have stable controllers in any closed-loop control systems. A *strong stabilization* problem is to design a stabilizing controller $K \in \mathcal{RH}_\infty$ such that some performance specifications are satisfied. Consequently, if the performance specifications can be measured in terms of some $\mathcal{H}_\infty$ norms, the corresponding strong stabilization problem is called the $\mathcal{H}_\infty$ strong stabilization problem.

- $\mathcal{H}_\infty$ Strong Stabilization: *given $\gamma > 0$, find a stabilizing controller $K \in \mathcal{RH}_\infty$ if possible such that $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$.*

It can be seen from the parameterization of all suboptimal $\mathcal{H}_\infty$ controllers that in order to find a stable stabilizing K, it is enough to find a $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$ such that Q stabilizes M_∞. It is not hard to see that this can be achieved if and only if Q stabilizes $M_{22} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_2 \\ \hline \hat{C}_2 & \hat{D}_{22} \end{array}\right]$. This gives the next result.

Lemma 1 *Assume that a solution to the suboptimal $\mathcal{H}_\infty$ control problem exists for a given $\gamma > 0$, i.e., there exists a $K = \mathcal{F}_l(M_\infty, Q)$ with $Q \in \mathcal{RH}_\infty$ and $\|Q\|_\infty < \gamma$ such that $\|\mathcal{F}_l(G, K)\|_\infty < \gamma$. Then, the $\mathcal{H}_\infty$ Strong Stabilization is solvable if and only if there is a $Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right]$ of some suitable order, with $\|Q\|_\infty < \gamma$, such that*

$$A_k = \left[\begin{array}{cc} \hat{A} + \hat{B}_2 R^{-1} D_q \hat{C}_2 & \hat{B}_2 R^{-1} C_q \\ B_q \hat{R}^{-1} \hat{C}_2 & A_q + B_q \hat{R}^{-1} \hat{D}_{22} C_q \end{array}\right] \tag{5}$$

is stable where $R = I - D_q \hat{D}_{22}$ and $\hat{R} = I - \hat{D}_{22} D_q$.

Hence, the $\mathcal{H}_\infty$ strong stabilization problem becomes of finding a $Q \in \mathcal{RH}_\infty$ such that

1. $\|Q\|_\infty < \gamma$,
2. A_k is stable.

3.2 State Space Parameterization of $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$

In general, it is very difficult to find a Q to satisfy both conditions described in the last section since Q can have any dimension. Thus any approach to obtain Q directly must deal with the infinite dimensionality problem. However, from a practical point of view, it is very desirable to keep the dimension of Q to be as low as possible. In order to find a suitable lower order $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < \gamma$, it is desirable to have more explicit characterization of this transfer matrix. In Steinbuch and Bosgra (1991), a parameterization for an $\mathcal{H}_\infty$ norm bounded strictly proper and stable transfer matrix is presented. Now, the result is extended to the proper case in the following lemma.

Lemma 2 *Let $\gamma > 0$ and let Q be a stable transfer matrix of degree n_q and $\|Q\|_\infty < \gamma$, then Q can be represented as* $Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right]$ *with* $A_q = A_{q_k} + A_{q_s}$, *for some* $A_{q_k} = -A_{q_k}^* \in \mathcal{R}^{n_q \times n_q}$, $B_q \in \mathcal{R}^{n_q \times p_2}$, $C_q \in \mathcal{R}^{m_2 \times n_q}$, $D_q \in \mathcal{R}^{m_2 \times p_2}$, *and*

$$
\begin{aligned}
A_{q_s} &= \frac{1}{2}\left[-B_q R^{-1} D_q^* C_q - C_q^* D_q R^{-1} B_q^* - B_q R^{-1} B_q^* - C_q^*(I + D_q R^{-1} D_q^*) C_q\right] && (6)\\
\bar{\sigma}(D_q) &< \gamma && (7)
\end{aligned}
$$

where $R = \gamma^2 I - D_q^* D_q$.

Proof. Assume that $Q = \left[\begin{array}{c|c} \hat{A}_q & \hat{B}_q \\ \hline \hat{C}_q & D_q \end{array}\right] \in \mathcal{RH}_\infty$ is a n_q^{th} order observable realization and $\|Q\|_\infty < \gamma$, then according with the Bounded Real Lemma (Zhou *et al.*, 1996; Zhou and Doyle, 1998), $\bar{\sigma}(D_q) < \gamma$ and $\exists Y > 0$ such that

$$
Y(\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q) + (\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q)^* Y + Y \hat{B}_q R^{-1} \hat{B}_q^* Y + \hat{C}_q^*(I + D_q R^{-1} D_q^*) \hat{C}_q = 0 \quad (8)
$$

where $R = \gamma^2 I - D_q^* D_q$. Since $Y > 0$, there exists a Cholesky factorization of $Y = T^* T$. Now T is invertible and can be used as a similarity transformation on Q

$$
Q = \left[\begin{array}{c|c} T\hat{A}_q T^{-1} & T\hat{B}_q \\ \hline \hat{C}_q T^{-1} & D_q \end{array}\right] =: \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right]
$$

Thus, the Riccati equation in (8) becomes

$$
A_q + A_q^* + B_q R^{-1} D_q^* C_q + C_q^* D_q R^{-1} B_q^* + B_q R^{-1} B_q^* + C_q^*(I + D_q R^{-1} D_q^*) C_q = 0 \quad (9)
$$

Furthermore, A_q can be decomposed into a symmetric part A_{q_s} and a skew symmetric part A_{q_k} where

$$
A_{q_s} = (A_q + A_q^*)/2 \qquad A_{q_k} = (A_q - A_q^*)/2
$$

Consequently, the skew symmetric part A_{q_k} disappears from equation (9) and the symmetric part is given in equation (6). □

Note that when $D_q = 0$, we have (Steinbuch and Bosgra, 1991)

$$
A_{q_s} = -\frac{1}{2}(B_q B_q^* / \gamma^2 + C_q^* C_q) \quad (10)
$$

Using the previous parameterization, the $\mathcal{H}_\infty$ strong stabilization problem can be converted to an optimization problem where the matrices A_{q_k}, B_q, C_q and D_q are free parameters.

3.3 Optimization Problem

The corresponding optimization problem is then defined as

$$l_{opt} = \min_{A_{q_k}, B_q, C_q, D_q} P(D_q) * e^{\max\{real[\lambda(A_k)]\}} \tag{11}$$

where

$$P(D_q) = \begin{cases} C & \bar{\sigma}(D_q) \geq \gamma \\ 1 & \text{otherwise} \end{cases}$$

A_q is constructed following the result of Lemma 2, and $C \gg 1$ is a chosen constant. Note that the proposed optimization problem has a positive cost function for any combination of parameters and the $\mathcal{H}_\infty$ strong stabilization problem is solved if $l_{opt} < 1$. In the optimization algorithm, predefined ranges of variations for the elements of A_{q_k}, B_q and C_q are first established. These intervals are chosen according with the maximum and minimum elements of the corresponding generalized plant G. However, it is usually hard to establish a range of variation for the elements of D_q in order to satisfy (7). Therefore, it is decided to limit the elements of D_q to be only $< \gamma$ and include a penalty function $P(\cdot)$ to discard the combinations that violate (7). Note that C must be chosen sufficiently large to avoid the possibility of having $l_{opt} < 1$ and $\bar{\sigma}(D_q) \geq \gamma$.
Assume that the degree of Q is predefined to n_q, then according with the dimensions of the generalized plant (3) in the original $\mathcal{H}_\infty$ problem, the number of variables of each component of Q is $\frac{(n_q-1)n_q}{2}$ for

$$A_{q_k} = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n_q} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n_q} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n_q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n_q} & -a_{2n_q} & -a_{3n_q} & \cdots & 0 \end{bmatrix}$$

$n_q \times p_2$ for B_q, $m_2 \times n_q$ for C_q, and $m_2 \times p_2$ for D_q. Since Q is an $m_2 \times p_2$ transfer matrix, the total number of variables in the optimization scheme will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2 + m_2 p_2$. In order to implement the optimization in a systematic scheme, the variables are aligned into a vector format

$$X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{(n_q-1)n_q} & b_{11} & \cdots & b_{n_q p_2} & c_{11} & \cdots & c_{m_2 n_q} & d_{11} & \cdots & d_{m_2 p_2} \end{bmatrix}^T$$

Consequently, the optimization in (11) is carried out with respect to the variables vector X.

4 $\mathcal{H}_2$ Strong Stabilization

4.1 $\mathcal{H}_2$ Strong Stabilization

Similar to the $\mathcal{H}_\infty$ strong stabilization, if the performance criterion is now measured in terms of some $\mathcal{H}_2$ norms of some closed-loop transfer matrices, then the corresponding strong stabilization problem is called the $\mathcal{H}_2$ *Strong Stabilization* problem. More specifically, without loss of generality, consider the feedback system in Figure 1 with $D_{11} = 0$ and $D_{22} = 0$. The $\mathcal{H}_2$ strong stabilization is stated as follows:

- $\mathcal{H}_2$ Strong Stabilization: *find a stabilizing $K \in \mathcal{RH}_\infty$ that minimizes $\|T_{zw}\|_2$.*

It is well known that all stabilizing controllers for the generalized plant G can be written as $K = \mathcal{F}_l(M_2, Q)$ with

$$M_2 = \left[\begin{array}{c|cc} \hat{A}_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{12}$$

where $\hat{A}_2 = A + B_2F_2 + L_2C_2$ and F_2 and L_2 can be constructed from the solutions of two related Riccati equations. It is then clear that the closed-loop $\mathcal{H}_2$ norm is given by

$$\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 + \|Q\|_2^2$$

where the transfer matrices G_c and G_f together with the two Riccati equations are defined in Zhou *et al.* (1996) or Zhou and Doyle (1998). Consequently, $Q = 0$ represents the optimal solution to the standard $\mathcal{H}_2$ problem. As in the $\mathcal{H}_\infty$ parameterization, M_2 and Q are $(p_2 + m_2) \times (p_2 + m_2)$ and $m_2 \times p_2$ transfer matrices respectively. Let $\gamma_{opt}^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2$. Then for any $\gamma > \gamma_{opt}$, all suboptimal $\mathcal{H}_2$ controllers satisfying $\|T_{zw}\|_2 < \gamma$ can be parameterized as $K = \mathcal{F}_l(M_2, Q)$ with $Q \in \mathcal{RH}_2$ and $\|Q\|_2^2 < \gamma^2 - \gamma_{opt}^2$.
Now it is clear that the $\mathcal{H}_2$ strong stabilization problem can be stated as

$$\min_{Q \in \mathcal{RH}_2} \|Q\|_2$$

such that

$$\begin{bmatrix} \hat{A}_2 & B_2C_q \\ -B_qC_2 & A_q \end{bmatrix} \textit{ is Hurwitz} \tag{13}$$

where $Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & 0 \end{array}\right]$ with $A_q \in \mathcal{R}^{n_q \times n_q}$, $B_q \in \mathcal{R}^{n_q \times p_2}$, and $C_q \in \mathcal{R}^{m_2 \times n_q}$. The condition (13) is needed since Q must stabilize M_2. To find a suitable $Q \in \mathcal{RH}_2$ with $\|Q\|_2 < \sqrt{\gamma^2 - \gamma_{opt}^2}$, it is desirable to have more explicit characterizations of these $\mathcal{H}_2$ norm bounded analytic functions. Using the definition of the $\mathcal{H}_2$ norm given by (1) and (2), a parameterization for all $Q \in \mathcal{RH}_2$ follows.

Lemma 3 *Assume that $Q \in \mathcal{RH}_2$ has degree n_q, then Q can be written in the following canonical form*

$$Q = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & 0 \end{array}\right]$$

with $A_q = A_{q_s} + A_{q_k}$ for some $A_{q_k} = -A_{q_k}^* \in \mathcal{R}^{n_q \times n_q}$ and $A_{q_s} = -\frac{1}{2}C_q^*C_q$ such that A_q is stable. Furthermore,

$$\|Q\|_2^2 = Trace(B_q^*B_q).$$

Proof. Assume that $Q = \left[\begin{array}{c|c} \hat{A}_q & \hat{B}_q \\ \hline \hat{C}_q & 0 \end{array}\right] \in \mathcal{RH}_2$ is a n_q^{th} order observable realization, then the observability Gramian L_o satisfies the following equation

$$\hat{A}_q^*L_o + L_o\hat{A}_q + C_q^*C_q = 0$$

Since $L_o > 0$, there exists a Cholesky factorization of $L_o = T^*T$. Now T is invertible and can be used as a similarity transformation on Q

$$Q = \left[\begin{array}{c|c} T\hat{A}_qT^{-1} & T\hat{B}_q \\ \hline \hat{C}_qT^{-1} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & 0 \end{array}\right]$$

Thus, the Lyapunov equation becomes

$$A_q + A_q^* + C_q^* C_q = 0$$

Furthermore, A_q can be decomposed into a symmetric part A_{q_s} and a skew symmetric part A_{q_k} where

$$A_{q_s} = (A_q + A_q^*)/2 \qquad A_{q_k} = (A_q - A_q^*)/2$$

Consequently, the symmetric part is given by $A_{q_s} = -\frac{1}{2} C_q^* C_q$ and the $\mathcal{H}_2$ norm of Q is given by $\|Q\|_2^2 = Trace(B_q^* B_q)$. □

Using the previous result, the $\mathcal{H}_2$ strong stabilization problem is now formulated as

$$\min_{A_{q_k}, B_q, C_q} Trace(B_q^* B_q) \tag{14}$$

such that

$$\begin{bmatrix} \hat{A}_2 & B_2 C_q \\ -B_q C_2 & -\frac{1}{2} C_q^* C_q + A_{q_k} \end{bmatrix} \; is\ Hurwitz$$

4.2 Optimization Problem

Comparing to the $\mathcal{H}_\infty$ formulation, the $\mathcal{H}_2$ strong stabilization problem involves more constraints since $\mathcal{H}_2$ minimization is now incorporated in addition to the controller stability. So, the optimization scheme becomes more complex. In order to solve (14), the following optimization scheme is proposed:

$$l_{opt} = \min_{A_{q_k}, B_q, C_q} Trace(B_q^* B_q) + J(A_k) \tag{15}$$

where

$$J(A_k) = \begin{cases} M e^{\max\{real[\lambda(A_k)]\}} & \max\{real[\lambda(A_k)]\} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_k = \begin{bmatrix} \hat{A}_2 & B_2 C_q \\ -B_q C_2 & -\frac{1}{2} C_q^* C_q + A_{q_k} \end{bmatrix}$$

Here, the constant M must be chosen sufficiently large such that $M > Trace(B_q^* B_q)$ is true at all time. Therefore, $l_{opt} < M$ means that the strong stabilization condition is satisfied. In this formulation, a discontinuous penalty function $J(\cdot)$ is introduced in order to enforce the strong stabilization restriction over the minimization of the $\mathcal{H}_2$ norm. Similar to the $\mathcal{H}_\infty$ case, intervals of variations are set for the elements of the matrices A_{q_k}, B_q and C_q. These intervals are chosen according with the elements of the corresponding matrices in the generalized plant G. The number of variables to be optimized in (15) will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2$ which is smaller than in the $\mathcal{H}_\infty$ case since Q is now strictly proper.

5 Numerical Examples

The optimization problems (11) and (15) are highly nonlinear and present multi-modal characteristics. Therefore, it was decided to use a gradient-based algorithm in conjunction with an evolutionary optimization such as a genetic algorithm to solve these problems. In this way, the genetic algorithm was applied first to search the whole parameters space and find a minimum solution. The best solution coming from the genetic algorithm was used then as a starting point for the gradient-based search which was carried out by using the *Optimization Toolbox* of MATLAB (Grace, 1992). Therefore, a local search was established to obtain the optimal solution. This two-stage optimization outperformed the use of each one of the algorithms separately. The solutions to the optimization problems (11) and (15) for the following numerical examples are computed with a Sun Microsystems Ultra 5 workstation. The constants C and M in (11) and (15) are given the values 1×10^3 and 1×10^6 respectively.

5.1 Example 1

This example is taken from Zeren and Özbay (1999) where a SISO mixed sensitivity minimization problem is considered. The generalized plant G is given by

$$G = \begin{bmatrix} W_1 & W_1 P \\ 0 & W_2 \\ I & P \end{bmatrix}$$

where

$$P(s) = \frac{(s+5)(s-1)(s-5)}{(s+4s+5)(s-20)(s-30)}, \quad W_1(s) = \frac{1}{(s+1)}, \quad W_2(s) = 0.2$$

The plant P is unstable and non-minimum phase, but the p.i.p. condition is satisfied. The $\mathcal{H}_\infty$ optimal performance is 34.24 and the optimal controller has a right half plane pole at 0.1791. The optimization for the $\mathcal{H}_\infty$ strong stabilization (11) is carried out. The results are summarized in Table 1. A 1^{st} order Q is used to achieve strong stabilization. (Actually, a constant Q can also achieve strong stabilization in this case as shown in the table.) Thus, the optimization (11) involves only 3 parameters. The optimization takes on an average 51.19 sec. and 4.255×10^7 flops. The evolution of the cost function during the genetic optimization is shown in Figure 2. In the first stage, the minimum cost achieved is 1.0001. After the local search, the cost is optimized to 0.9974. Thus, a stable controller is obtained. Figure 3 shows the frequency response of the stable controller obtained through the optimization. Note that the low and high frequency gains are raised by the stable controller compared to the optimal one. During the optimization, the degree of Q is also increased but no improvements are observed. The result using the proposed optimization (11) is close to the optimal performance and a clear advantage is seen comparing to the result previously published in Zeren and Özbay (1999).

	Optimal	Proposed Optimization		Result in
		0^{th} order Q	1^{st} order Q	Zeren and Özbay (1999)
γ	34.24	44.3	34.44	42.5

Table 1: The smallest γ achieved $\mathcal{H}_\infty$ strong stabilization for Example 1

		Order of Q			
	Optimal	1^{st}	2^{nd}	3^{rd}	4^{th}
$\|T_{zw}\|_2$	98.358	98.366	98.367	98.390	98.470

Table 2: Closed-loop performance for $\mathcal{H}_2$ strong stabilization in Example 1

Now, if the penalty in the control signal is reduced, i.e. $W_2(s) = 0.1$, the $\mathcal{H}_2$ optimal controller is unstable with a real pole at 0.1137. The optimization scheme (15) is run and the results are

presented in Table 2. The degree of Q is varied from 1^{st} to 4^{th}, for these four cases the strong stabilization condition is preserved and the closed-loop $\mathcal{H}_2$ norm is maintained extremely close to the optimal. On an average, 53.33 sec and 7.45×10^7 flops are needed to obtain a solution to (15). The evolution of the cost function in the first stage and the frequency response of the optimal stable controller are shown for a 2^{nd} order Q in Figures 4 and 5 respectively. From Figure 4, it is noticed that the initial population already provided a stable controller and the genetic optimization is able to reduce the cost function to 198.52. The gradient-based optimization reduces this cost further to 192.90.
It should be intuitive that as the order of Q is increased the $\mathcal{H}_2$ performance of the closed-loop system should improve or at least should not be deteriorated. However, the results in Table 2 show the opposite. Note that for Q of 4^{th} order, the number of parameters in the optimization is 14. In this case, the parameter space is very large and it is highly probable that the optimization could get trapped in a local minima. In order to verify this analysis, the population size and mutation rate are increased in the genetic algorithm. Then, for Q of 4^{th} order the optimal performance is almost the same as that achieved by the 1^{st} order Q. As it is expected, the algorithm now takes longer time and more flops to reach a solution. In summary, the optimization problem becomes very complex when more than 10 parameters are involved and the probability to reach only a local minimum increases. The same observations hold for the next example as well as many other examples we have experimented .

5.2 Example 2

A benchmark problem in $\mathcal{H}_2$ strong stabilization was taken from Ganesh and Pearson (1989) in order to compare the performance of the suggested algorithm with the existing ones. The realization of the generalized plant G is given by

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \quad B_1 = \begin{bmatrix} 35 & 0 \\ -61 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 52.915 & 8.944 \\ 0 & 0 \end{bmatrix} \quad D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D_{22} = [0]$$

The generalized plant is stable, so the p.i.p. condition is obviously satisfied. The optimal $\mathcal{H}_2$ performance is 493.8 and the optimal controller has a unstable pole at 18.7. In Ganesh and Pearson (1989), the optimum $\mathcal{H}_2$ performance with a stable controller (4^{th} order) was computed as 622.73. The resulting optimal controller has two poles at the origin, but if the stability boundary is moved back to $s = -0.5$ (i.e. suboptimal controller), the $\mathcal{H}_2$ cost is now 628.40. The best $\mathcal{H}_2$ performance with a stable controller obtained in Kapila and Haddad (1995) is 803.91. In Corrado *et al.* (1997), new results were reported for this example. The closed-loop $\mathcal{H}_2$ cost was 627.31 and 622.20 for a second and fourth order controllers respectively. Table 3 shows the results obtained by running the $\mathcal{H}_2$ optimization algorithm (15) with a 1^{st} order Q (i.e. 3^{rd} order controller). An average of 26.59 sec and 1.94×10^7 flops were needed to reach a solution. Numerical calculation also shows that increasing the order of Q does not seem to improve the result.

6 Conclusions

Optimization schemes are presented to solve the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ strong stabilization problems. Parameterizations of norm bounded $\mathcal{H}_\infty$ and $\mathcal{H}_2$ functions are used to limit the number of variables

	Optimal	Order of Q 1^{st}	Ganesh and Pearson	Corrado *et al.*	Kapila and Haddad
$\|T_{zw}\|_2$	493.8	627.36	622.73	622.20	803.91

Table 3: Closed-loop performance for $\mathcal{H}_2$ strong stabilization in Example 3

and restrict the optimizations. The resulting schemes are highly nonlinear and present multi-modal characteristics. For this reason, a two-stage algorithm is used in the optimization process. Numerical examples show the success of the optimization schemes to design stable controllers. In many cases, the stable controllers achieve closed-loop performance close to the optimal. However, it is not possible to establish the achievable performance with these techniques and this issue has to be explored iteratively. It is noted that only low order Qs are needed in the numerical examples. Thus, the orders of resulting controllers are comparable to those of the generalized plants. More numerical examples can be found in a conference version of this paper (Campos-Delgado and Zhou, 2002).

References

Arelhi, R., M.A. Johnson and J. Wilkie (1997). LQG Stable Stabilizing Control: Some Recent Results. *Proceedings of the 36th Conference on Decision and Control*, San Diego, California. pp. 1431-1436.

Barabanov, A. E. (1995). Design of $\mathcal{H}_\infty$ Optimal Stable Stabilizing Compensator. *Proceedings of the 35th Conference on Decision and Control*, Kobe, Japan. pp. 734-738.

Bernstein, D.S. and W.M. Haddad (1989). LQG Control with and $\mathcal{H}_\infty$ Performance Bound: A Riccati Equation Approach. *IEEE Transactions on Automatic Control.* Vol. 34, No. 3, pp. 293-305.

Campos-Delgado, D.U. and K. Zhou (2001). $\mathcal{H}_\infty$ Strong Stabilization. *IEEE Transactions on Automatic Control*, vol. 46, No. 12, pp. 1968-1972.

Campos-Delgado, D. U. and K. Zhou (2002). $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Strong Stabilization by Numerical Optimization. *Proceedings of IFAC Congress*, to appear.

Choi, Y. and W.K. Chung (2000). On the Stable $\mathcal{H}_\infty$ Controller Parameterization under Sufficient Condition. Submitted to *IEEE Transactions on Automatic Control.*

Corrado, J.R., R.S. Erwin, D.S. Bernstein and W.M. Haddad (1997). Stable $\mathcal{H}_2$-Optimal Controller Synthesis. *Proceedings of the American Control Conference*, Paper # 0-7803-3835-9/97.

Doyle, J.C., K. Glover, P.K. Khargonekar and B.A. Francis (1989). State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems. *IEEE Transactions on Automatic Control.* Vol. 34, No. 8, pp. 831-847.

Ganesh, C. and J.B. Pearson (1989). $\mathcal{H}_2$-Optimization with Stable Controllers. *Automatica*, Vol. 25, pp. 629-634.

Grace, A. (1992). *Optimization Toolbox: User's Guide.* The MathWorks, Inc..

Halevi, Y. (1994). Stable LQG Controllers. *IEEE Transactions on Automatic Control.* Vol. 39, No. 10, pp. 2104-2106.

Holland, J.H. (1975). *Adaptation in Natural and Artificial Systems.* MIT Press.

Kapila, V. and W.M. Haddad (1995). $\mathcal{H}_2$ and Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stable Stabilization. *Proceedings of the 34th Conference on Decision and Control*, New Orleans, Louisiana. pp. 1911-1915.

Li, Y., K.C. Tan and M. Gong (1997). Global Structure Evolution and Local Parameter Learning for Control System Model Reductions. *Evolutionary Algorithms in Engineering Applications.* Springer-Verlag, pp. 345-360.

Mitchell, M. (1996). *An Introduction to Genetic Algorithms*, The MIT Press, Cambridge, Massachusetts.

Özbay, H. (1995). Stable $\mathcal{H}_\infty$ Controller Design for the Longitudinal Dynamics of an Aircraft. *NASA Technical Memorandum 106847.*

Saif, A.A., D. Gu and I. Postlethwaite (1997). Strong Stabilization of MIMO Systems via $\mathcal{H}_\infty$ Optimization. *Systems and Control Letters.* Vol. 32, pp. 111-120.

Steinbuch, M. and O.H. Bosgra (1991). Robust Performance in $\mathcal{H}_2/\mathcal{H}_\infty$ Optimal Control. *Proceedings of the 30th Conference on Decision and Control*, London, England. pp. 549-550.

Tang, K.S., K.F. Man, C.Y. Chan, S. Kwong and P.J. Fleming (1995). GA Approach to Multiple Objective Optimization for Active Noise Control. *IFAC Algorithms and Architecture for Real-Time Control.* pp. 13-19.

Wang, Y.W. and D.S. Bernstein (1993). $\mathcal{H}_2$-Suboptimal Stable Stabilization. *Proceedings of the 32nd Conference on Decision and Control.* San Antonio, Texas. pp. 1828-1829.

Whidborne, J.F., D.W. Gu and I. Postlethwaite (1995). Algorithms for Solving the Method of Inequalities - A Comparative Study. *Proceeding of the American Control Conference.*

Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach.* Cambridge, MA, MIT Press.

Youla, D.C., J.J. Bongiorno and N.C. Lu (1974). Single-loop Feedback Stabilization of Linear Multivariable Dynamical Plants. *Automatica.* Vol. 10, pp. 159-173.

Zeren, M. and H. Özbay (1997). On Stable $\mathcal{H}_\infty$ Controller Design. *Proceedings of the American Control Conference.* Albuquerque, New Mexico. pp. 1302-1306.

Zeren, M. and H. Özbay (1999). On the Synthesis of Stable $\mathcal{H}_\infty$ Controllers. *IEEE Transactions on Automatic Control.* Vol. 44, No. 2, pp. 431-435.

Zeren, M. and H. Özbay (2000). On the Strong Stabilization and Stable $\mathcal{H}_\infty$ Controller Design Problems for MIMO Systems. *Automatica.* Vol. 36, pp. 1675-1684.

Zhou, K., J.C. Doyle and K. Glover (1996). *Robust and Optimal Control.* Prentice Hall, Upper Saddle River, New Jersey.

Zhou, K. and J.C. Doyle (1998). *Essentials of Robust Control.* Prentice Hall, Upper Saddle River, New Jersey.

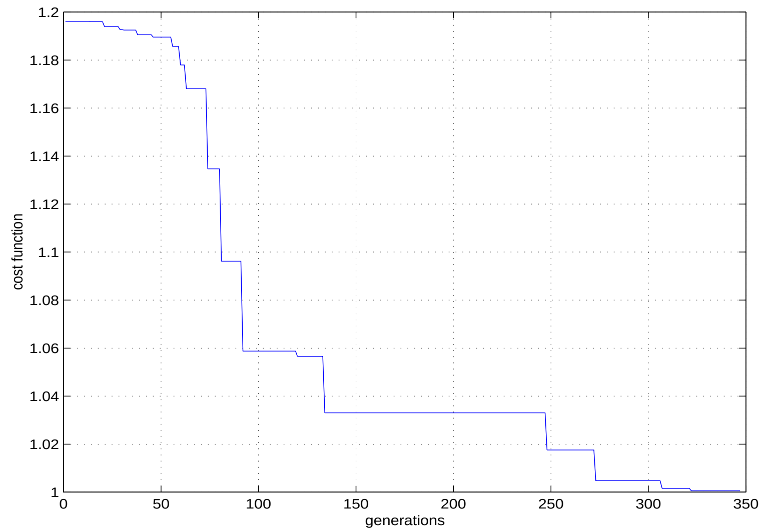

Figure 2: Evolution of Cost Function During Optimization for $\mathcal{H}_\infty$ Design in Example 1

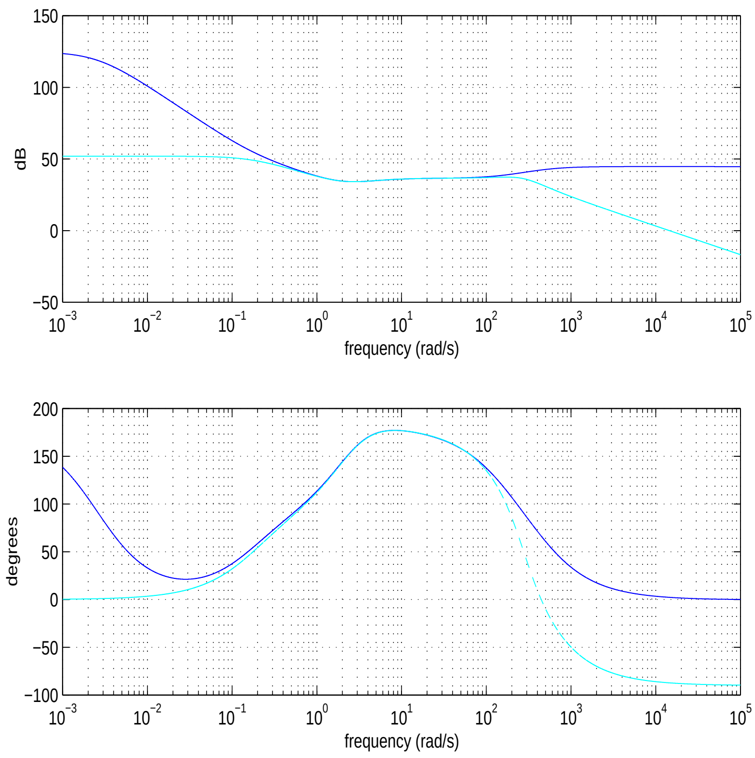

Figure 3: Controllers for $\mathcal{H}_\infty$ Design in Example 1 (a) $'-'$ Strong Stabilizing Controller, (b) $'--'$ Optimal Controller (Unstable).

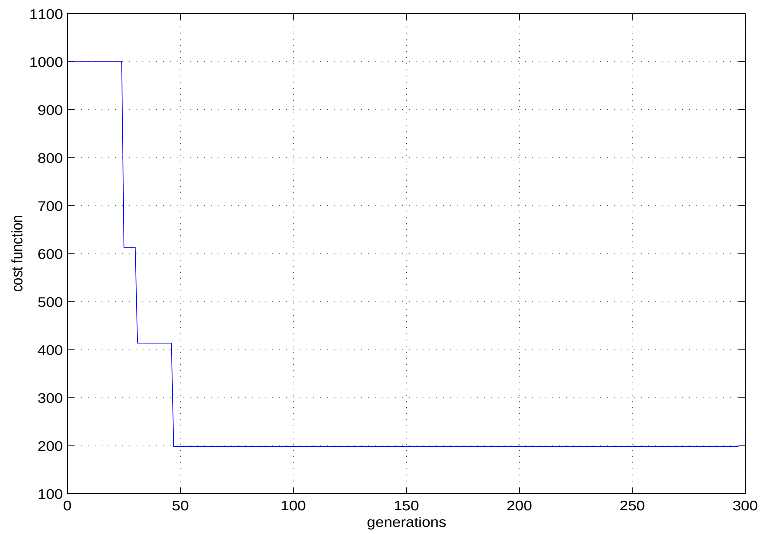

Figure 4: Evolution of Cost Function During Optimization for $\mathcal{H}_2$ Design in Example 1 with a 2^{nd} order Q.

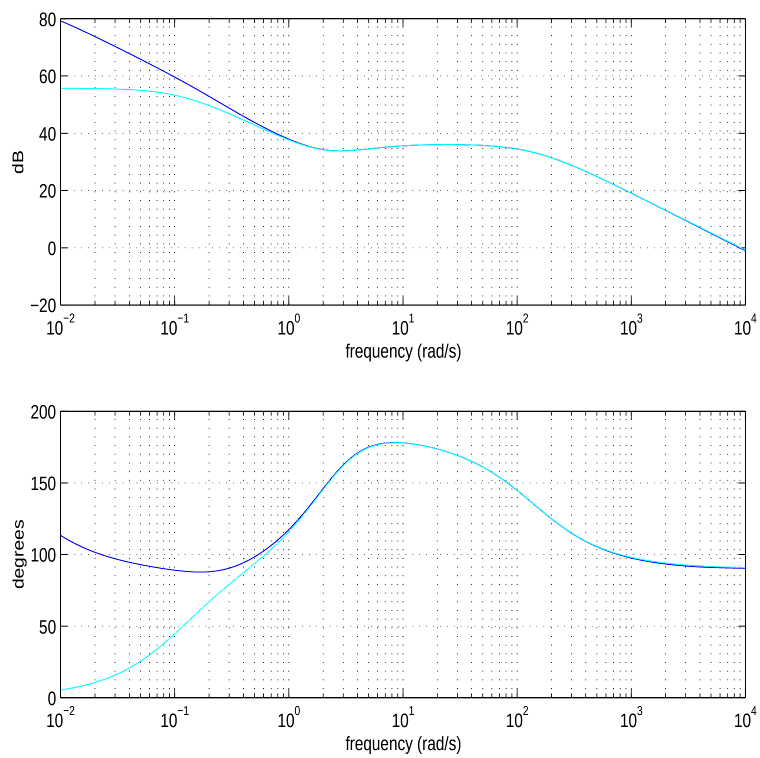

Figure 5: Controllers for $\mathcal{H}_2$ Design in Example 1 with a 2^{nd} order Q (a) $'-'$ Strong Stabilizing Controller, (b) $'--'$ Optimal Controller (Unstable).