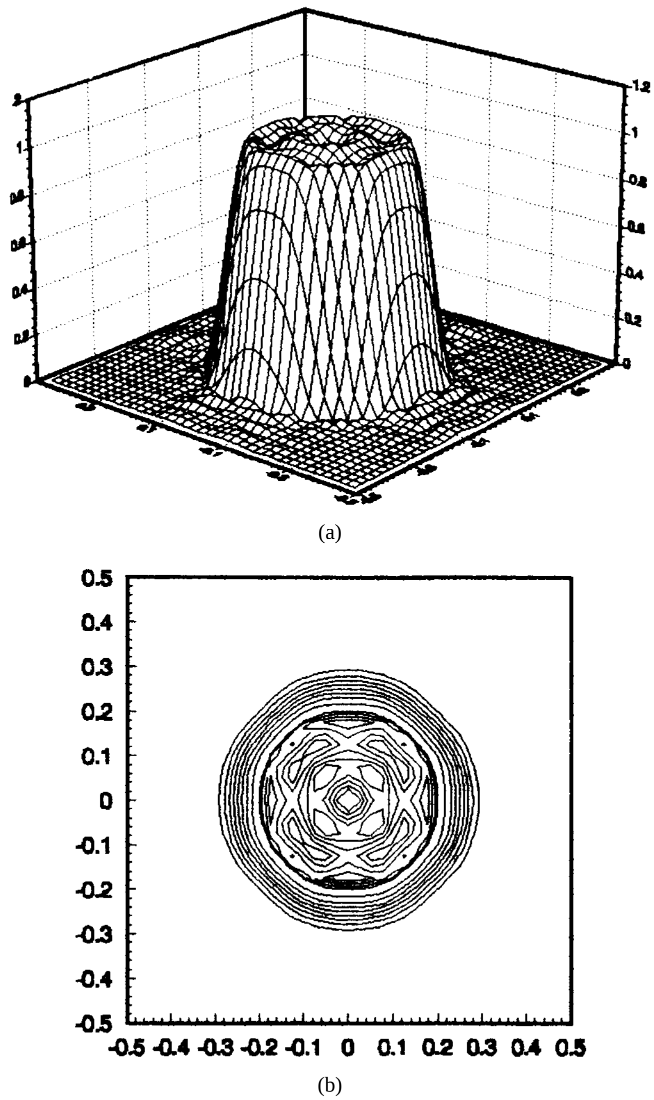

Fig. 3. Circular low-pass filter of size 15×15 designed by the NNO method, taken as (21). Plot of frequency response. (b) Contour plot.

approach. The better NNO design approach other than THNN-based algorithm will be available soon if the potential of this kind of method is fully exploited. Though the filter design examples are limited in the 2-D FIR category, the NNO design approach is of course not restricted within this kinds of filters. Other works done by us such as 1-D FIR filters [9], 1-D and 2-D IIR filters design results will be reported successively.

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Comments on "New Conditions for Global Stability of Neural Networks with Application to Linear and Quadratic Programming Problems"[1]

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***Abstract*— This letter makes the following comments on the results about global stability of neural networks presented in Forti and Tesi in the above paper: 1) the assumption of all neuron activation functions to vanish at the origin, which is utilized in the proof of the result (see the above paper [p. 357, Section III, Th. 3]) implying the existence and uniqueness of the network equilibrium point, can be actually *omitted*; 2) in the infinite sector case, the result of global asymptotic stability (GAS) (see the above paper [p. 359, Section IV, Th. 5]) remains true with respect to the class of increasing (*not* necessarily strictly) activations, as in the finite sector case. Consequently, a result about absolute stability (ABST) of neural networks, which can represent a generalization of the existing related ones, is also obtained.**

I. INTRODUCTION

In this letter, we will consider neural networks described by the system of nonlinear differential equations

$$dx/dt = -Dx + Tg(x) + I \tag{N}$$

as in the above paper,[1] where $x = (x_1, \cdots, x_n)^t \in \Re^n$ (^{t} means transposition), $D = \mathrm{diag}\,(d_1, \cdots, d_n)$ is a constant $n \times n$ diagonal matrix with $d_i > 0, i = 1, \cdots, n, T = (T_{ij})_{n\times n}$ is a constant $n \times n$ matrix, $I = (I_1, \cdots, I_n)^t \in \Re^n$ is a constant vector, $g(x) = (g_1(x_1), \cdots, g_n(x_n))^t$: $\Re^n \rightarrow \Re^n$ is a locally Lipschitz continuous nonlinear mapping. The assumption $g(0) = 0$ from the above paper[1] is *not* made here.

The matrix T is referred to as the interconnection matrix. The functions g_i represent the neuron input–output activations, while I_i

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describe constant inputs to the neural network. The diagonal entries of D model neuron self-inhibitions.

As in the above paper,[1] it is assumed that g belongs to the class $\{G_m\}$ defined by the property that $g \in \{G_m\}$ if for $i = 1, \cdots, n,$ the functions g_i: $\Re \to \Re$ are monotonic increasing (*not* necessarily strictly). If there exist constants $\overline{G}_i, 0 < \overline{G}_i < +\infty, i = 1, \cdots, n,$ such that the incremental ratio for g_i satisfies

$$0 \leq \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \leq \overline{G}_i$$

for each $\xi_1, \xi_2 \in \Re, \xi_1 \neq \xi_2$ and for $i = 1, \cdots, n,$ we say that $g \in \{G_m\}$ in the *finite sector case*. If otherwise, the incremental ratio for g_i is nonnegative but is allowed to take arbitrarily large positive values, we say that $g \in \{G_m\}$ in the *infinite sector case*.

In the case of the neuron activations to be locally Lipschitz continuous satisfying $g(0) = 0,$ and increasing in the finite sector case and *strictly* increasing in the infinite sector case, in the above paper[1] the authors have given an elegant and strict analysis of the existence, uniqueness, and GAS of the neural network equilibrium point. Compared with the previous related work referenced in Forti and Tesi's paper, which only concerns the case of the neuron activations to be bounded and strictly increasing, the GAS results in the above paper[1] are applicable to some important engineering problems where the solving neural networks for which the neuron activations are unbounded (but not necessarily surjective) and/or have infinite intervals with zero slope, as in the widely employed piecewise linear neural models. As such, the analysis results obtained by Forti and Tesi are applicable to a large class of neural networks including the additive neural network model [1], the Hopfield network [2], and the cellular neural network (CNN) model [3]. Moreover, these results were successfully applied in the above paper[1] to analyze GAS for linear and quadratic programming neural networks [4] in the general case of practical importance where the constraint amplifiers are dynamical, which extends the stability result in the case of memoryless constraint amplifiers in [4]. Note that the assumption of memoryless constraint amplifiers is unrealistic in any practical neural circuit implementation [4].

In this letter, we will show that the above-mentioned two assumptions from Forti and Tesi of $g(0) = 0$ and the *strictness* in the monotonic increasing property of activations in the infinite sector case can be actually *omitted*, perserving the soundness of the results about the existence, uniqueness and GAS of the network equilibrium point [see the above paper, p. 357, Section III, Corollary 1; p. 358, Section IV, Th. 4; and p. 359, Section IV, Th. 5]. That is, the existence, uniqueness and GAS results of the network equilibrium point in Forti and Tesi's paper really holds in the more general case of the neuron activations to be *any* Lipschitz continuous and increasing (*not* necessarily strictly) functions, which is important for applications of neural networks to a wider range of engineering problems than before. Furthermore, from the existence, uniqueness and GAS results of the network equilibrium point in the general case, an ABST result of neural networks, which can represent a generalization of the existing related ones [5], [6], follows consequently.

II. Proof of the Assertion that the Above-Mentioned Two Assumptions on the Neuron Activations can be Omitted

The following definitions from Forti and Tesi's paper are introduced in this letter.

Definition 1: $x^e \in \Re^n$ is an equilibrium point of system (N) if it is a constant solution of system (N), i.e., it satisfies the algebraic equation $-Dx^e + Tg(x^e) + I = 0$. The equilibrium x^e is said to be globally asymptotically stable (GAS) if it is locally stable in the sense of Lyapunov and globally attractive.

Definition 2: A map H: $\Re^n \to \Re^n$ is a homeomorphism of $\Re^n$ onto itself if H is C^0, H is one-to-one, H is onto and the inverse map H^{-1} is C^0.

Definition 3: A map H: $\Re^n \to \Re^n$ is a diffeomorphism of $\Re^n$ onto itself if H is C^1, H is a homeomorphism of $\Re^n$ onto itself and H^{-1} is C^1.

Definition 4: A real square matrix A is said to be Lyapunov diagonally stable (LDS) [respectively, Lyapunov diagonally semistable (LDSS)] if there exists a diagonal matrix $\alpha > 0$ such that the symmetric part of αA satisfies $[\alpha A]^S = (\alpha A + A^t \alpha)/2 > 0$ (respectively, $[\alpha A]^S \geq 0$).

The result below has been actually proved in the above paper.[1]

Theorem A: We are referring to the above paper [p. 357, Section III, Th. 3]. Let $g \in C^0$ (respectively, $g \in C^1$) and $g(0) = 0$. Suppose that either: 1) $g \in \{G_m\}$ in the finite sector case and $-T + D\overline{G}^{-1} \in$ LDS, where $\overline{G} = \mathrm{diag}\,(\overline{G}_1, \cdots, \overline{G}_n) > 0,$ or 2) $g \in \{G_m\}$ in the infinite sector case and $-T \in$ LDSS. Then, the map $\hat{H}$: $\Re^n \to \Re^n$ defined by $\hat{H}(x) = -Dx + Tg(x) (x \in \Re^n)$ is a homeomorphism (respectively, diffeomorphism) of $\Re^n$ onto itself.

Some commonly used activations, such as $g_i(x_i) = 1/[1 + \exp(-x_i)] (x_i \in \Re, i = 1, \cdots, n),$ can*not* satisfy the condition $g(0) = 0$ in Theorem A. Fortunately, the condition $g(0) = 0$ in Theorem A can be actually omitted, by a simple map translation from g to $g - g(0),$ as shown in the proof of the following result about the existence and uniqueness of the equilibrium point of system (N).

Theorem 1: Under the same hypotheses *except* $g(0) = 0$ as in Theorem A, system (N) has a unique equilibrium point for each $I \in \Re^n$.

Proof: From Theorem A, it is inferred that the map $\tilde{H}$: $\Re^n \to \Re^n$, defined by $\tilde{H}(x) = -Dx + Tf(x) (x \in \Re^n),$ is a homeomorphism of $\Re^n$ onto itself, where the nonlinear map $f(x) = g(x) - g(0)$: $\Re^n \to \Re^n$ satisfies $f(0) = 0$ and $f \in C^0$ (respectively, $f \in C^1$) if $g \in C^0$ (respectively, $g \in C^1$). Hence, by Definition 2, the equation $\tilde{H}(x) = -Tg(0) - I \in \Re^n$, i.e., $-Dx + Tg(x) + I = 0,$ has a unique solution for each $I \in \Re^n$.

Theorem 1 can be considered as an extended version of the above paper [p. 357, Section III, Corollary 1] in which the assumption $g(0) = 0$ for the neuron activations was made.

Because $-Tg(0) - I,$ as a map of $I \in \Re^n$, is C^1, it is still true that (see the above paper[1]) the unique equilibrium point of system (N) has the additional property of being a C^0 map of the input vector $I \in \Re^n$ if $g \in C^0$ or even a C^1 map if $g \in C^1$.

Further, the results about GAS of neural networks, in the finite and infinite sector cases, are obtained in Forti and Tesi's paper, respectively, based on the existence and uniqueness results of the network equilibrium point.

Theorem B: We are referring to the above paper [p. 358, Section IV, Th. 4]. Suppose that $g \in \{G_m\}$ in the finite sector case and $-T + D\overline{G}^{-1} \in$ LDS. Then, for each $I \in \Re^n$, system (N) has a unique equilibrium point which is GAS.

Theorem C: We are referring to the above paper [p. 359, Section IV, Th. 5]. Suppose that $g \in \{G_m\}$ in the infinite sector case and that $g_i, i = 1, \cdots, n,$ are strictly increasing. If $-T \in$ LDSS, then for each $I \in \Re^n$, system (N) has a unique equilibrium point which is GAS.

While the GAS result of neural networks, in the finite sector case, is obtained for $g \in \{G_m\}$ in Theorem B, it holds, in the infinite sector case, only with respect to the smaller class of *strictly* increasing activations as in Theorem C. Interestingly, making use of the *same* Lyapunov function as in the above paper,[1] which is utilized in the proof of the GAS result in the finite sector case, it can be shown that, in the infinite sector case, the GAS result also holds for $g \in \{G_m\}$ as in the finite sector case.

We need a lemma as follows.

Lemma 1 [p. 59, 7, Th. VII]: Let $V(x)$ be a scalar function with continuous first partial derivatives. If the region $\Omega_l (l > 0)$, defined by $V(x) < l$, is bounded and contains the origin, and

$$dV(x)/dt < 0 \quad \text{for all } x \neq 0 \text{ in } \Omega_l$$

then the origin is asymptotically stable, and above all, every solution in Ω_l tends to the origin as $t \to +\infty$.

Theorem 2: Suppose that g is locally Lipschitz continuous, $g \in \{G_m\}$ and $-T \in \mathrm{LDSS}$. Then, for each $I \in \Re^n$, system (N) has a unique equilibrium point which is GAS.

Proof: Since $-T \in \mathrm{LDSS}$, from Theorem 1, system (N) has a unique equilibrium point x^e. As in the above paper,[1] by means of the coordinate translation $z = x - x^e$, (N) can be put into the equivalent form

$$dz/dt = -Dz + TG(z) \tag{$\tilde{\mathrm{N}}$}$$

where $G(z) = (G_1(z_1), \cdots, G_n(z_n))^t$ and $G_i(z_i) = g_i(z_i + x_i^e) - g_i(x_i^e), i = 1, \cdots, n$. We have $G(0) = 0, G$ is locally Lipschitz continuous, and $G \in \{G_m\}$. Now, we prove that the unique equilibrium point $z = 0$ of system $(\tilde{\mathrm{N}})$ is GAS under the assumption $-T \in \mathrm{LDSS}$, which means that there exists $\alpha = \mathrm{diag}\,(\alpha_1, \cdots, \alpha_n) > 0$ such that $[\alpha(-T)]^S \geq 0$.

For *any* given $z^0 \in \Re^n$, consider the candidate Lyapunov function of the generalized Lur'e-Postnikov type as in the above paper[1]

$$V(z) = \tfrac{1}{2}\, z^t D^{-1} z + k \sum_{i=1}^{n} \alpha_i \int_0^{z_i} G_i(\rho)\, d\rho$$

where k is a positive number being dependent on $z^0 \in \Re^n$ and determined later. Computing the time derivative of $V(z)$ along solutions of system $(\tilde{\mathrm{N}})$, we have

$$\begin{aligned} dV(z)/dt &= [D^{-1}z + k\alpha G(z)]^t[-Dz + TG(z)] \\ &= -z^t z + z^t D^{-1} TG(z) - kG^t(z)\alpha Dz - kG^t(z) \\ &\quad \cdot [\alpha(-T)]^S G(z) \\ &\leq -z^t z + z^t D^{-1} TG(z) - kG^t(z)\alpha Dz \\ &\quad \text{for } z \in \Re^n. \end{aligned} \tag{1}$$

Let l be any positive number such that $l > V(z^0) \geq 0$. Without loss of generality, $l = V(z^0) + 1 > 0$ is selected. Let the set $\Omega_l = \{z \in \Re^n | V(z) < l\}$, it is obvious that Ω_l is an open subset in $\Re^n$, which contains the origin $z = 0$ and $z = z^0$ as its interior points. In fact, from the properties of G, all the points in the form $\lambda z^0, \lambda \in [0, 1]$, are interior in Ω_l. Let $\delta = \sqrt{2ld_{\max}} > 0, B_\delta = \{z \in \Re^n \,|\, ||z|| \leq \delta\} (\delta > 0), d_{\max} = \max_{1 \leq i \leq n} d_i > 0$, where $||\cdot||$ denotes the Euclidean norm of a vector defined by $||z|| = \sqrt{\Sigma_{i=1}^n z_i^2}$, then $\Omega_l \subseteq B_\delta$. Hence, Ω_l is also bounded.

Because G is locally Lipschitz continuous, we know that the first partial derivatives of $V(x)$ are continuous, and that there exist two positive numbers $\delta_1 > 0$ and κ_1 such that $\delta_1 < \delta$ and

$$|G_i(z_i)| \leq \kappa_1 |z_i| \quad \text{for } |z_i| < \delta_1 \text{ and } i = 1, \cdots, n. \tag{2}$$

Let $M_1 = \max_{1 \leq i \leq n} \sup_{|z_i| \leq \delta} |G_i(z_i)| + 1 > 0$, then

$$|G_i(z_i)| \leq \left(\frac{M_1}{\delta_1}\right) |z_i| \quad \text{for} \quad \delta_1 \leq |z_i| \leq \delta \quad \text{and} \quad i = 1, \cdots, n. \tag{3}$$

Let $\mu = \max\,(\kappa_1, M_1/\delta_1) > 0$, from inequalities (2) and (3), we obtain

$$|G_i(z_i)| \leq \mu |z_i| \quad \text{for } |z_i| \leq \delta \text{ and } i = 1, \cdots, n.$$

From this, it follows that

$$G^t(z)\alpha Dz \geq (\chi_{\min}/\mu) G^t(z) G(z) \quad \text{for } z \in \Omega_l \tag{4}$$

where $\chi_{\min} = \min_{1 \leq i \leq n} (\alpha_i d_i) > 0$.

It is easy to see that

$$\begin{aligned} -z^t z + z^t D^{-1} TG(z) &\leq (\tfrac{1}{4}) G^t(z) T^t D^{-2} TG(z) \\ &\leq (\theta/4) G^t(z) G(z) \quad \text{for } z \in \Re^n \end{aligned} \tag{5}$$

where $\theta = \lambda_{\max}(T^t D^{-2} T) \geq 0$, and $\lambda_{\max}(\cdot)$ denotes the maximum real eigenvalue of a real symmetric matrix.

Let k be any positive number such that $k > \mu\theta/(4\chi_{\min}) \geq 0$. Without loss of generality, $k = \mu(\theta + 4)/(4\chi_{\min}) \geq \mu/\chi_{\min} > 0$ is chosen. Then, from inequalities (1), (4), and (5)

$$dV(z)/dt \leq -G^t(z) G(z) \quad \text{for } z \in \Omega_l. \tag{6}$$

Now, let $z \in \Omega_l$ and $z \neq 0$. If $G(z) = 0$, then, from inequality (1), $dV(z)/dt \leq -z^t z < 0$. If otherwise, i.e., $G(z) \neq 0$, then from inequality (6) we also have $dV(z)/dt < 0$. Thus, from Lemma 1, the origin $z = 0$ is locally asymptotically stable and every solution initiating at any interior point of Ω_l, especially $z = z^0 \in \Re^n$ *arbitrarily* given beforehand, tends to the origin as $t \to +\infty$. This also implies the global attractivity of the origin $z = 0$. Therefore, the equilibrium $z = 0$ of system $(\tilde{\mathrm{N}})$ is GAS. The proof of Theorem 2 is thus completed.

From Theorems 1 and 2, we know that the existence, uniqueness and GAS results of the network equilibrium point, see the above paper [p. 357, Section III, Corollary 1], Theorems B and C remain to be shown in the general case of the neuron activations being locally Lipschitz continuous and increasing.

III. An Implied Absolute Stability Result

From the results in Theorems 1 and 2, it can be seen that, if $-T \in \mathrm{LDSS}$, then system (N) has a unique equilibrium point which is GAS for each $g \in \{G_m\}$ being locally Lipschitz continuous and for each $I \in \Re^n$. According to the definition of ABST of system (N) [5], [6], it is known that system (N) is absolutely stable (ABST) with respect to the class of activations being locally Lipschitz continuous and increasing. In [5] and [6], the ABST results are obtained under the assumption that $g \in \mathcal{S}$, defined by the property of $g_i(x_i)$ being a bounded C^1 function with positive derivative for $x_i \in \Re$ and $i = 1, \cdots, n$. The sufficient condition for absolute stability of neural networks presented in [6] is the same as one in Theorem 2, i.e., $-T \in \mathrm{LDSS}$, which includes the sufficient part of the necessary and sufficient condition of T to be negative semidefinite for ABST of neural networks with symmetric interconnection matrices [5]. Thus, the ABST result of neural networks obtained in the letter can be regarded as a generalization of the ABST ones in [5] and [6], in the sense that the class $g \in \mathcal{S}$ of activations [5], [6] can be extended to the larger one in which g is locally Lipschitz continuous and increasing.

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