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Congested Banyan Network Analysis Using Congested-Queue States and Neighboring-Queue Effects

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Abstract: A banyan network analysis technique is presented which more accurately models a congested network than other reported techniques. The analysis is based on the observation that a full queue (within the switching modules making up the network) causes traffic to back up. For a short time after becoming full the free space in the queue is limited to no more than one slot. A queue in such a condition is called *congested.* Because of blocking the arrival rate to a congested queue is higher; this tends to maintain congestion. The arrival rate to a congested queue's dual is lower as is the service rate for queues feeding the congested queue. These effects are captured in the analysis. The state model used for a queue encodes congested as well as normal operation. Further, the model codes whether the connected next-stage and dual queues are congested. Network throughput computed with the model is closer to that obtained from simulations than other banyan analyses appearing in the literature, including those designed to model congestion. Further, the queue-occupancy distributions are much closer than other analyses, suggesting that the analysis better models conditions in congested banyan networks.

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1 Introduction

Because of their regular structure and their performance-determining role in computation and communication systems, banyan networks are a tempting target of analysis. Several analyses have been published since the networks were described by Peace [12] and Lawrie [9]. Early analyses considered unbuffered or single-buffered banyans [1,6]. Later work considered finite-buffered networks, the type of network considered here. Yoon, Lee, and Liu described one such analysis [15] in which each stage is represented by a single queue modeled by a Markov chain. This analysis, to be referred to as Yoon's analysis here, works well at low and moderate offered-traffic rates. However the predictions are less accurate at rates causing congestion. For many applications this is acceptable, as when congestion is rare. For others, performance at congestion determines system performance, as in a parallel computer running a communication-intensive algorithm. In these cases an accurate model of the network under congestion would help in estimating performance. An accurate model could also help in the development of new network designs or traffic management schemes.

Recent work specifically models banyan networks under congestion. One approach was to develop a model in which the service rate of a queue is dependent upon the fate of the head packet (next packet to be sent) in the previous cycle. The rationale is that a blocked head packet will more likely be blocked again in the next cycle. Such approaches have been used by Lin and Kleinrock [10], Mun and Youn [11], and Hsiao *et al* [4] for finite queues and earlier by Theimer *et al* [14] and Hsiao *et al* [5] for single-slot queues (single-buffered networks).

In the simplest of these analyses, Lin and Kleinrock's [10], the queues are modeled as by Yoon [15], however an effective service rate is used in place of the service rate computed as in [15]. This service rate is derived by considering two cases: the probability of service for a packet first arriving at the head slot and the probability of service for a packet that was blocked by a full queue in the previous cycle. For a packet that first arrives, the stationary next-stage queue-full probability is used in computing the service rate. For a packet which had been blocked, the knowledge that the queue was full is used in computing the service rate. The two service rates are combined to obtain the effective service rate. This analysis is for networks using nonblocking crossbars with queues at the module outputs: any number of packets entering a module can enter a queue if there is space. (The analysis presented here is for blocking crossbars.) This analysis does account for the effect of a full queue on performance, but it does so by adjusting the overall service rate. Unlike the model described here, it does not model the higher arrival rate when a queue is full, which has a strong effect on the queue state distribution.

The analyses presented by Mun and Youn [11], Theimer *et al* [14], and Hsiao *et al* [4] model each queue with several sets of states. Each set corresponds to a possible fate of the head packet in the previous cycle. A different service rate is found for each set, thus modeling, for example, the lower service probability of a packet that had been blocked. The results reported by these investigators come closer to simulated results than those reported by Yoon *et al* [15].

The models reported in [14] and [5] are for single-slot queues, so do not apply to the networks considered here. In [4] the model is incompletely reported, and so could not be reproduced by this investigator. The model described by Mun and Youn does not always give throughput values as close to simulation as the model described here. Further, Mun and Youn's model does not predict queue-occupancy distributions as closely.

The analysis described here is designed to capture congestion's salient features, based upon simulations of congested-networks. The central feature of the model is the congested queue. A queue becomes congested in a cycle in which it is full and a packet destined for the queue is blocked. The arrival rate to a congested queue will be higher than normal because of the packets it blocks; the higher arrival rate tends to prolong congestion. Congestion ends when the queue has one slot free and no packet is ready to move into the queue.

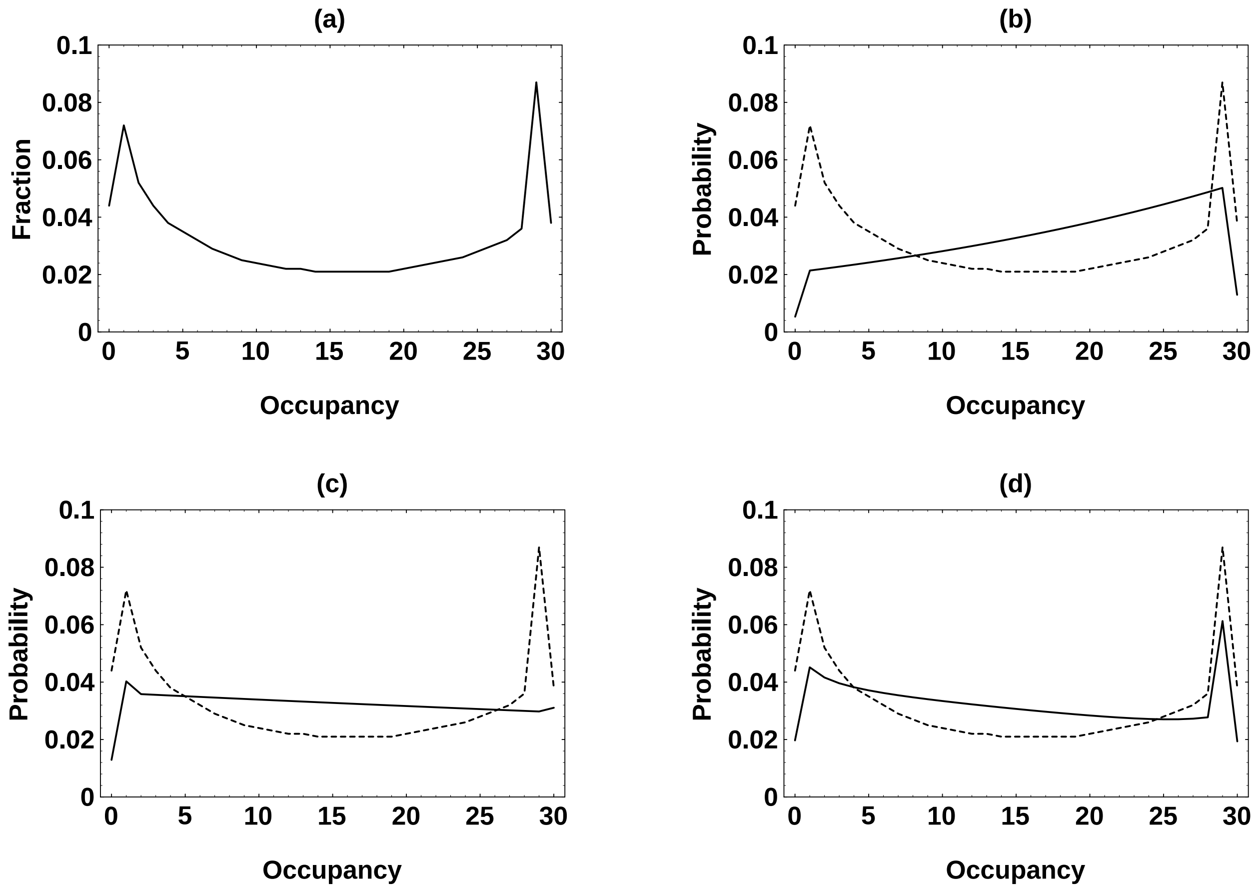

Figure 1. Queue occupancy (number of packets in queue) distributions in the second stage of an 8-stage, 30-slot-buffer banyan network obtained from *(a)* simulation, and simulation (dashed) plotted with results obtained *(b)* using Yoon's analysis, *(c)* using Mun and Youn's analysis, and *(d)* using the analysis described here.

The effect of congestion can be seen in the queue state distribution plot for a simulated banyan network appearing in Figure 1*(a)*. The plot shows the fraction of time a second-stage queue spends at each level of occupancy, from 0 packets (empty), to 30 packets (full for this network). Because of the congestion effect there is a clear peak at 29 packets. (The peak appears at 29, not 30, packets because a packet will not enter a queue that was full at the beginning of a cycle.) The throughput at and near congestion is determined by this part of the queue state distribution, so modeling it accurately is important.

The queue state distribution for the same network, obtained using the analysis method of Yoon *et al* (adapted for local flow control) appear in Figure 1*(b)*. The distribution does not have the form of Figure 1*(a)*, although the maximum does occur at 29 packets. Note that the probability that a queue holds 29 packets is lower than the corresponding quantity observed in simulation.

The queue state distribution for the same network obtained using the analysis method of Mun and Youn (adapted for local flow control) appear in Figure 1*(c)*. Again, the form does not match that observed in simulation. The full-queue probabilities are higher than Yoon's model predicts; a possible reason for its greater accuracy.

The state distribution obtained through the analysis described here is plotted in Figure 1*(d)*. There is a peak at 29, as observed in simulations. The form of the state distribution appears more like that obtained from simulations than the distributions obtained using the methods of Yoon *et al* and Mun and Youn. Further, the throughput predictions are more accurate than the previous analyses for many configurations.

The analysis described here also accounts for a congested queue's effect on its *dual.* (A queue's dual [defined for all but the first stage] in a network using 2×2 crossbars is the queue in the same stage connected to the same previous-stage crossbar.) The arrival rate to a congested queue's dual is lower (as is the service rate for queues feeding the congested queue).

The state model described here encodes information about the queue and its neighbors. The neighbors considered are the previous-stage queues which feed the queue, the next-stage queues to which the queue is connected (considered as a unit) and the dual queue in the same stage. The state of a queue (in all but the first and last stages) encodes whether or not the dual and next-stage queues are congested. If a queue is congested then its state also encodes the destinations of any packets in the heads of the previous-stage queues.

The part of the state coding previous-stage queues is used in modeling congestion. The part coding next-stage queues models the propagation of congestion towards the inputs. The part coding the dual queue models the lower arrival rate when the dual is congested. (A simpler model which gives similar results omits the state of the dual queues. That model is not described in detail here.) Arrival and service rates are computed for these conditions. In addition different arrival rates are computed for states in which the queue is empty, is full (but not congested), and has one slot free.

The remainder of the report is organized as follows. In Section 2 the banyan network being modeled is described. In Section 3 a description of congestion and the state model for the network is given. In Section 4 details on parts of the model involving head slots is presented. In Section 5 state transition probabilities are derived; flow-related probabilities appear in Section 6. Results appear in Section 7; conclusions appear in Section 8.

2 Preliminaries

The network being modeled, which will be called the $(n, 2, d)$ *basic network,* is an n-stage banyan network consisting of 2×2 modules. The stages are numbered from 0 to $n-1$, with 0 being the first stage. Each stage consists of 2^{n-1} modules. The network has 2^n inputs which are connected to the first-stage module inputs by links; for all but the last stage, module outputs are connected by links to module inputs in the next stage. Module outputs in the last stage are connected to network outputs. For a banyan network the links can be connected in any way as long as there is exactly one path between every input/output pair [3]. See [13] for an elementary introduction and [8] a discussion of the topology of banyan networks.

Each module consists of 2 d-slot queues, connected to a 2×2 crossbar switch. Module inputs connect to the queues; queue outputs connect to the crossbar; crossbar outputs connect to the module outputs. A queue's *feeder queues* are the queues in the previous-stage module to which the queue links connect. First-stage queues do not have feeder queues. A *packet* is the unit of communication; it consists of a destination (a network output) and data. Each queue slot can hold one packet (all packets are the same size). The queues use a first-in, first-out service discipline; the slot containing, or that would contain, the packet being served is called the *head slot*, a packet in the head slot is called the *head packet.*

The network operates in a synchronous clocked fashion; time is divided into *cycles.* In cycle t a packet can move from a queue in one stage to a queue in the next stage if: 1) it is at the head slot, 2) it is granted use of the appropriate link, and 3) the queue in the next stage has at least one free slot during the cycle. The packet which is granted use of a link is chosen randomly from those contending for its use. The packet meeting these conditions at cycle t will be in the next-stage queue starting at cycle $t + 1$. A packet at a network input will move into the network

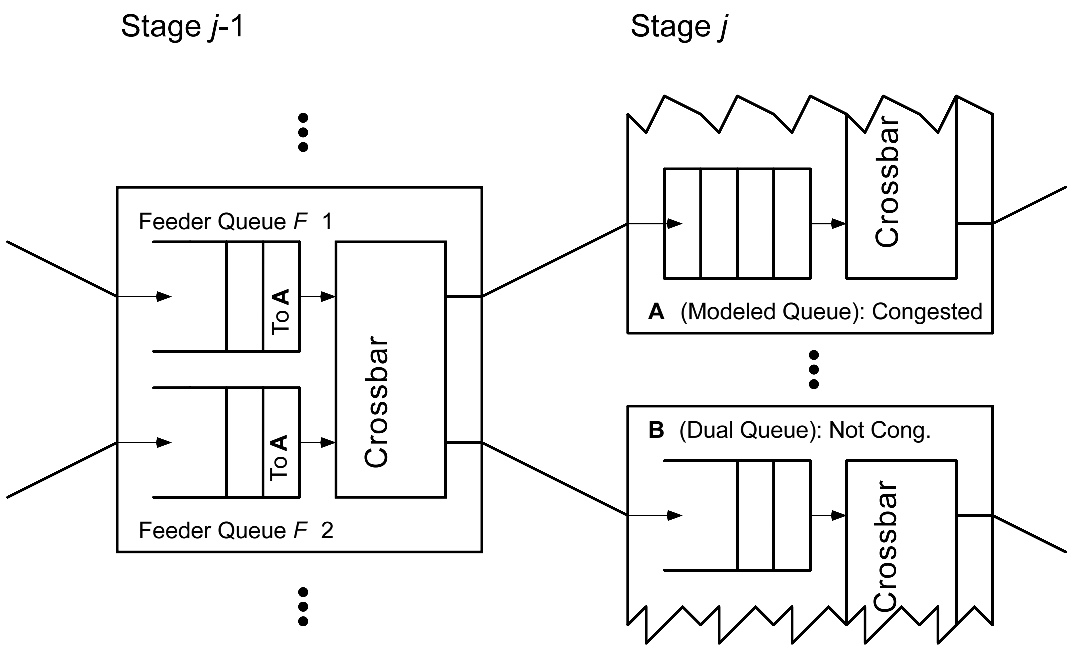

Figure 2. Illustration of congestion. Queue **A**, being congested, is full much of the time. This causes packets bound for **A** to accumulate at the head of the feeder queues, $F1$ and $F2$, prolonging congestion. Queue **B** is not congested

in cycle t if there is at least one slot free in the corresponding queue in the first stage. A packet will move from a queue in the last stage to a network output if it is in the head slot and is granted the appropriate output.

The number of packets arriving at each input at each cycle is modeled by independent, identically distributed Bernoulli random variables. The symbol λ will be used to denote the input arrival rate. Destinations are randomly chosen and are uniformly distributed over the outputs. An arriving packet which does not enter the network is dropped.

Reference will be made to Yoon's analysis method. This refers to an analysis technique which is similar to that presented by Yoon in [15], except that packets move as described above, as is done by Ding [2].

The *flow rate* through a point in a network is the mean or expected number of packets passing through the point per cycle. The *throughput* of a network is the mean network-output flow rate. In the model described here the *arrival rate* to a queue refers to the (sometimes conditional) probability that at least one packet is ready to enter the queue in a cycle. The *service rate* of a non-empty queue is the (sometimes conditional) probability that a packet in the head slot can proceed.

3 Congestion Model

The analysis presented here differs from previous analyses in that the arrival rate to a queue depends upon the state of the queue, most importantly on whether the queue is congested. Under the model a queue can change from a full state to what is called a congested state, in which the arrival rate is much higher. The higher arrival rate is due to the accumulation of packets bound for the congested queue. The higher arrival rate is a consequence of, and tends to prolong, the congested state. The reason for the higher arrival rate is that packets bound for the congested queue are more frequently blocked, so they tend to accumulate at the heads of the queues which feed the congested queue. This situation is illustrated in Figure 2 and formally defined below.

Definition 1: *A queue in stage $0 < j < n$ of an $(n, 2, d)$ basic network is said to be in the* congestion-start condition *if it contains d packets and the head slot of at least one feeder queue contains a packet which is to pass through the queue.*

Definition 2: *A queue in stage $0 < j < n$ of an $(n, 2, d)$ basic network is said to be in the* congestion-end condition *if it contains $d - 1$ packets and no feeder-queue head slot has a packet which is to pass through the queue.*

Definition 3: *A queue in stage $0 < j < n$ of an $(n, 2, d)$ basic network is said to be* congested *in cycle t if there exists a cycle $t_s < t$ such that the queue was at the congestion-start condition in cycle t_s and for all $t_s < \tau < t$ the queue was not in the congestion-end condition.*

An example of a queue going from an uncongested to a congested condition, and then back to an uncongested condition, appears in the table below. The table shows the history of four queues connected to a common crossbar, as illustrated in Figure 2. Queue **A** suffers congestion. Queues $F1$ and $F2$ are queue **A**'s feeder queues, queue **B** is queue **A**'s dual queue. The condition of each queue over a time interval is shown in the table. For queues $F1$ and $F2$ the next queue on the path of the head packet is shown, **A** or **B**, or **E** if the queue is empty. For queues **A** and **B** either the number of packets in the queue is shown, or if **A** is congested the entry indicates whether **A** has zero or one slots free, indicated by c_1 or c_0, respectively. An asterisk next to the entry indicates that the head packet of the respective queue will move to the next stage at the end of the cycle. Queue **A** is full but not congested up to cycle 2. At cycle 2 $F1$ offers a packet which **A** cannot accept so that congestion starts. Congestion continues until the end of cycle 8, in which there is one slot free in **A** while no packet is offered.

Table 1: A Congestion Example

Queues	Feeder		Fed		Queues	Feeder		Fed	
Cycle	$F1$	$F2$	**A**	**B**	Cycle	$F1$	$F2$	**A**	**B**
0	**A**∗	**E**	$d-1$	0	5	**A**	**A**∗	c_1∗	1∗
1	**E**	**B**∗	d	0	6	**B**∗	**A**∗	c_1∗	0
2	**A**	**E**	d∗	1∗	7	**B**∗	**E**	c_0∗	1∗
3	**A**∗	**B**∗	c_1	0	8	**E**	**B**∗	c_1	1∗
4	**A**∗	**A**	c_0∗	1	9	**A**∗	**E**	$d-1$	1

3.1 The Model

A banyan network is modeled by n independent Markov chains, one for each stage. The chains are referred to as *queue models* (or *queues* when the meaning is clear); each fully models one queue, called the *modeled queue*, and partially models other queues. A queue-model state is labeled by a three-tuple, (S, D, N). The feeder and modeled queues are described by S; symbols D and N, describe the *disposition* of the dual and next-stage queues respectively.

A queue is either congested or not congested; if not congested S is the number of packets in the queue, $S \in \{0, 1, \ldots, d\} = \mathcal{Q}_{\mathbf{N}}$. If congested $S \in \mathcal{Q}_{\mathbf{C}}$, where $\mathcal{Q}_{\mathbf{C}}$ is the set of possible *HOL system* (feeder-queue head-slot) states, to be described below. To reduce the number of states needed, the number of packets in a queue during congestion is not explicitly coded.

From the perspective of the modeled queue, the dual queue is either congested, labeled **C**; or not congested, labeled **N**. Thus, $D \in \{\mathbf{N}, \mathbf{C}\}$.

Because a congested queue will block packets bound for the dual queue the arrival rate to the dual queue will be lower when $D = \mathbf{C}$ than when $D = \mathbf{N}$. When $N = \mathbf{C}$ at least one of the two next-stage queues are congested; when $N = \mathbf{N}$ neither next-stage queue is congested. If a next-stage queue is congested the service rate is reduced. Let $\mathcal{Q}$ be the set of all possible queue states. Then $\mathcal{Q} = \{ (S, D, N) \mid S \in \mathcal{Q}_{\mathrm{S}},\ D, N \in \{\mathbf{N}, \mathbf{C}\} \}$, where $\mathcal{Q}_{\mathrm{S}} = \mathcal{Q}_{\mathbf{C}} \cup \mathcal{Q}_{\mathbf{N}}$.

As an illustration of this notation, the state of a queue in an idle network is $(0, \mathbf{N}, \mathbf{N})$. State $(5, \mathbf{N}, \mathbf{C})$ indicates a queue holding five packets with at least one of the next-stage queues (to which it connects) in the congested state, while the queue's dual is not congested. State $(\mathbf{AA}, \mathbf{C}, \mathbf{N})$ indicates that the queue and its dual are congested, while the next-stage queues to which it connects are not congested.

It will frequently be necessary to refer to subsets of $\mathcal{Q}$; the three-tuple notation will be used to describe these subsets. Let $\mathcal{S} \subseteq \mathcal{Q}_{\mathrm{S}}$ and $\mathcal{D}, \mathcal{N} \subseteq \{\mathbf{N}, \mathbf{C}\}$. Then define $(\mathcal{S}, \mathcal{D}, \mathcal{N}) = \{(S, D, N) \mid S \in \mathcal{S},\ D \in \mathcal{D},\ N \in \mathcal{N}\}$. Let $\star$ be the set of all symbols that can appear in a particular three-tuple position. Then $(\star, D, N) = \{(S, D, N) \mid S \in \mathcal{Q}_{\mathrm{S}}\}$ where $D, N \in \{\mathbf{N}, \mathbf{C}\}$. Similar subsets are defined, for example, $(S, \star, N) = \{(S, D, N) \mid D \in \{\mathbf{N}, \mathbf{C}\}\}$. As further examples, consider $(\mathcal{Q}_{\mathbf{N}}, D, N)$ and $(\mathcal{Q}_{\mathbf{C}}, D, \star)$; the former specifies a non-congested queue for specific dual and next-stage queue dispositions, the latter, a congested queue with the next-stage queue either congested or not congested.

Let random variable $\mathbf{q}(x, t) \in \mathcal{Q}$ be the state of stage-x queue at time t. Then define $p_q(x, t) \equiv \Pr[\mathbf{q}(x, t) = q]$ and $p_{\mathcal{Q}'}(x, t) \equiv \Pr[\mathbf{q}(x, t) \in \mathcal{Q}']$ where $q \in \mathcal{Q}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$. Let $P(x, t)$ be the $|\mathcal{Q}|$-element vector denoting the state distribution of queue x at time t so that $I(q)$'th element of $P(x, t)$ is $p_q(x, t)$, where $I \mid \mathcal{Q} \to \{1, 2, \ldots, |\mathcal{Q}|\}$ maps each state to a unique index. (No particular order is assumed here.) The stationary probability distribution is referred to when the time index is omitted.

Three sets of probabilities will be used to construct state-transition probabilities. One set indicates arrival and service probabilities as in conventional banyan analyses. One set is used to find transitions between states in which $S \in \mathcal{Q}_{\mathbf{C}}$. A set of transition probabilities, called the lateral transition probabilities, relates to the disposition of the dual and next-stage queues.

In the analysis the arrival rate is state dependent. Different arrival rates are used when the dual is congested and not congested. Further, a different arrival rate is used for when $S = 0$, $S \in \{1, 2, \ldots, d-2\}$, $S = d-1$, and $S = d$. Three of the arrival rates are denoted $r_{(0,D,\star)}$, $r_{(d-1,D,\star)}$, and $r_{(d,D,\star)}$. For notational simplicity the fourth arrival rate is denoted $r_{(i,D,\star)}$, for $i \in \{1, 2, \ldots, d-2\}$; all represent a single arrival rate. An explicit arrival rate is not needed for states in which $S \in \mathcal{Q}_{\mathbf{C}}$ since the HOL system is fully modeled. When the state of the dual queue is not known a composite arrival rate will be used, $r_{(y,\star,N)} = (p_{(y,\mathbf{N},N)} r_{(y,\mathbf{N},\star)} + p_{(y,\mathbf{C},N)} r_{(y,\mathbf{C},\star)}) / p_{(y,\star,N)}$ for $y \in \mathcal{Q}_{\mathbf{N}}$ and $N \in \{\mathbf{N}, \mathbf{C}\}$.

The probability that a packet in the head slot of a stage-x queue can proceed to the next stage is called the *service rate*. Two service rates are used, one for when the next-stage queues are not congested, $v_{\mathbf{N}}(x)$, and one for when they are congested, $v_{\mathbf{C}}(x)$.

The probabilities that the dual and next-stage queues will change disposition are called the *lateral transition probabilities*. The probability of a dual queue entering (leaving) congestion given that the modeled queue is not congested is denoted $l_{\text{dual-}\mathbf{C}|\mathbf{N}}$ ($l_{\text{dual-}\mathbf{N}|\mathbf{C}}$). For states where the modeled queue is congested lateral transition probabilities reflecting the dual queue are not explicitly computed.

The lateral transition probabilities reflecting next-stage queues are a function of the number of packets in the modeled queue. The probability, given that a queue is in state $(0, \star, \mathbf{N})$, of the next-stage queue entering congestion is denoted $l_{\text{next-}\mathbf{C}|\text{0-}\mathbf{N}}$. Similarly, the probability given that a queue is in state $(1, \star, \mathbf{N})$ of the next-stage queue entering congestion is denoted $l_{\text{next-}\mathbf{C}|\text{1-}\mathbf{N}}$. The probability of a queue entering congestion given that it is in state $(i, \star, \mathbf{N})$ is denoted $l_{\text{next-}\mathbf{C}|\text{s-}\mathbf{N}}$ for $1 < i \leq d$. The probability, for a queue in state $(i, \star, \mathbf{C})$, of the next-stage queue ending congestion is denoted $l_{\text{next-}\mathbf{N}|\text{0-}\mathbf{C}}$ if $i = 0$, and $l_{\text{next-}\mathbf{N}|\text{s-}\mathbf{C}}$ otherwise.

3.2 Invariants

Almost any analytical model will include simplifications, the one presented here is no exception. However, the simplifications were made in such a way as to keep several important aspects of the model internally self-consistent. The conditions which must be satisfied for this consistency are called *invariants.* There are two types of invariants, one type relates the probability that queues are in sets of states, the other relates the flow rates. The arrival and service rates as well as the lateral transition probabilities will be derived so that these invariants always hold.

These invariants are important because they insure that the analysis accurately reflects the network being modeled. The flow-rate invariant is important because it avoids any ambiguity in determining throughput (under stationary conditions). (In an analysis that violated flow-rate invariance one might compute a different throughput for each stage.) Unfortunately, maintaining invariance complicates the analysis.

From a queue model's stationary distribution one can determine the unconditional probability that the next-stage queues are congested. These must match the corresponding probabilities in the next stage. That is, $p_{(\star,\star,\mathsf{C})}(x-1) = p_{(\mathcal{Q}_\mathsf{C},\mathsf{N},\star)}(x) + p_{(\star,\mathsf{C},\star)}(x)$, for $0 < x < n$. Similarly for the dual-queue disposition, $p_{(\mathcal{Q}_\mathsf{C},\star,\star)}(x) = p_{(\star,\mathsf{C},\star)}(x)$ for $0 < x < n$.

The number of packets leaving a stage-$(x-1)$ queue while in state $(\star,\star,\mathbf{C})$ must equal the number of packets received by a stage-x queue when in state $(\mathcal{Q}_\mathsf{C},\mathbf{N},\star) \cup (\star,\mathbf{C},\star)$. Similarly, the number of packets leaving a stage-$(x-1)$ queue while in state $(\star,\star,\mathbf{N})$ must equal the number of packets received by a stage-x queue when in state $(\mathcal{Q}_\mathsf{N},\mathbf{N},\star)$.

These invariants, along with other model details, ensure that the flow rates computed for all stages are identical.

3.3 Analysis Overview

Stage-x, time-t, queue-model transition probabilities will be specified using a transition matrix, $M(x,t)$, a $|\mathcal{Q}| \times |\mathcal{Q}|$ matrix of elements $M(x,t)_{q_1 q_2} \in [0,1]$. Let $|\mathcal{Q}|$-element vector $P(x,0)$ be the initial state of the stage-x queue. Then $P(x,t+1) \equiv M(x,t)P(x,t)$ for $t \geq 0$.

A procedure is described below for finding the transition matrix for an $(n,2,d)$ basic network with offered traffic rate λ. The matrix is for stage x at time t and is computed given $P(x,t)$, $P(x-1,t)$ (if $x > 1$) $P(x+1,t)$ (if $x < n-1$) and other time-$(t-1)$ information. A network is analyzed by starting with some initial queue state distribution for each stage and then iteratively computing matrices and queue state distributions. Iteration stops when the difference in queue state distribution on subsequent iterations is sufficiently small.

Transition probabilities will be partly specified in terms of HOL-system transition probabilities. The HOL systems will be described first. Then queue transition probabilities will be given, followed by formulas for arrival rates, service rates, and lateral transition probabilities.

4 HOL System

The term *HOL (head-of-line) system* refers to the head slots of queues which directly connect to a common crossbar. The HOL system models the interaction between adjacent stages, this is central to the analysis. The queue state encodes the previous stage's HOL system state distribution for when the queue is congested. The analysis also makes use of a previous-stage HOL-system (state) distribution for when a queue is not congested; this is generated making use of independence assumptions. *HOL transition matrices* will be found for these HOL systems, these will be used to find queue transition probabilities as well as arrival rates and lateral transition probabilities.

4.1 Notation

Individual feeder queues are modeled as containing either zero packets, labeled **E**, a packet destined for the modeled queue, labeled **A**, or a packet destined for the dual queue, labeled **B**. The

HOL system itself models the two feeder queues; for computational efficiency the queues are considered indistinct. The possible state labels for the HOL system are $\mathcal{Q}_{\mathsf{C}} = \{\mathsf{AA}, \mathsf{AB}, \mathsf{AE}, \mathsf{BB}, \mathsf{BE}, \mathsf{EE}\}$, where the HOL labels consists of two feeder-queue labels. Suppose $b_1 = \mathsf{A}$ and $b_2 = \mathsf{B}$; then b_1b_2, b_2b_1, and AB are equivalent. Predicate $b_1 \in b_2b_3$ is true if $b_1 = b_2$ or $b_1 = b_3$. Corresponding equivalences and predicates are defined for all $b_1, b_2, b_3 \in \{\mathsf{A}, \mathsf{B}, \mathsf{E}\}$.

Let V be any vector; define $\sum V$ to be the sum of all elements of that vector. For example $\sum P(x,t) = 1$. Let V_1 and V_2 be any two identically dimensioned vectors. Then V_1V_2 $(V_1 + V_2)$ indicate an element-wise product (sum) of the two vectors. Let a be a real number, then aV is the vector obtained by multiplying each element by a. Let V be a vector in which $\sum V > 0$. Define $\langle V \rangle = \left(\sum V\right)^{-1} V$.

For all $Q \subseteq \mathcal{Q}$ let $V(Q)$ be a $|\mathcal{Q}|$-element vector of elements $V_i(Q)$ in which

$$V_{I(q)}(Q) = \begin{cases} 1, & \text{if } q \in Q; \\ 0, & \text{if } q \notin Q. \end{cases}$$

Expressions of the form $\sum P(x,t)V(Q)$ indicate some aspect of the state of queue x at time t. For example, $\sum P(x,t)V(\star, \mathsf{N}, \mathsf{C})$ is the probability that at time t the dual of queue x will not be congested and at least one of the next-stage queues will be congested. For $Q \subseteq \mathcal{Q}_{\mathsf{C}}$ define $V(Q)$ to be a similar $|\mathcal{Q}_{\mathsf{C}}|$-element vector.

Sets $\mathcal{Q}_{\mathsf{A}} = \{\mathsf{AA}, \mathsf{AB}, \mathsf{AE}\}$, $\mathcal{Q}_{\mathsf{B}} = \{\mathsf{AB}, \mathsf{BB}, \mathsf{BE}\}$, $\mathcal{Q}_{\overline{\mathsf{A}}} = \{\mathsf{BB}, \mathsf{BE}, \mathsf{EE}\}$, and $\mathcal{Q}_{\overline{\mathsf{B}}} = \{\mathsf{AA}, \mathsf{AE}, \mathsf{EE}\}$ will be used with V. Additional vectors will be defined for finding the probability of a particular head slot being occupied.

Let $V^{X'}$ be the vector of elements in which

$$V^{X'}_{(q,\star,\star)} = \begin{cases} 1, & \text{if } q = XX; \\ \frac{1}{2}, & \text{if } X \in q \text{ and } q \neq XX; \\ 0, & \text{otherwise,} \end{cases}$$

for $X \in \{\mathsf{A}, \mathsf{B}, \mathsf{E}\}$. Let V^{ps} be the vector of elements in which $V^{\mathrm{ps}}_{(q,D,N)} = 2$, if $D = \mathsf{N}$; or $= 1$, if $D = \mathsf{C}$.

4.2 Independent HOL System

The *independent HOL-system (state) distribution function,* P^{hi}, maps $p_{\mathrm{sm}} \in [0,1]$ to a $|\mathcal{Q}_{\mathsf{C}}|$-element vector that contains the state distribution of a HOL system in which the states of the queues are independent, destinations are uniformly distributed over the two module outputs, and the probability of each queue being occupied is p_{sm}.

Call the module outputs A and B. The state of a head slot having a packet bound for these outputs is labeled A and B, respectively. The state of an empty head slot is labeled E. Since the queues are independent and the destinations are uniformly distributed over the outputs, the probability that a queue head is in state s is given by

$$f_2(s) = \begin{cases} p_{\mathrm{sm}}/2, & \text{if } s \in \{\mathsf{A}, \mathsf{B}\}; \\ (1 - p_{\mathrm{sm}}), & \text{if } s = \mathsf{E}. \end{cases}$$

Accounting for both queues and the fact that the queues are indistinct, the HOL-system distribution is given by $P^{\mathrm{hi}}_{s_1s_2} = f_1(s_1,s_2)f_2(s_1)f_2(s_2)$ for all $s_1s_2 \in \mathcal{Q}_{\mathsf{C}}$, where

$$f_1(s_1,s_2) = \begin{cases} 1, & \text{if } s_1 = s_2; \\ 2, & \text{otherwise.} \end{cases}$$

4.3 HOL-System Transition Matrix

Several HOL-system transition matrices will be used, all can be computed from the following four quantities: the probability that a head slot is empty, $p_{\rm ar}$; the probability that a packet departing a head slot is replaced, $p_{\rm mo}$; and the probability that a packet bound for queue A (B) is not blocked, $p_{\text{sp-}\mathsf{A}}$ ($p_{\text{sp-}\mathsf{B}}$). The *HOL-system (state) transition matrix function, H*, maps these four quantities to a $|\mathcal{Q}_{\mathsf{C}}| \times |\mathcal{Q}_{\mathsf{C}}|$ transition matrix, $H(p_{\rm ar}, p_{\rm mo}, p_{\text{sp-}\mathsf{A}}, p_{\text{sp-}\mathsf{B}})$ in which matrix element q_1q_2, $q_1, q_2 \in \mathcal{Q}_{\mathsf{C}}$, is the probability of a HOL-system transition from q_1 to q_2. Computing these transition probabilities is straightforward, albeit tedious.

A HOL-system transition matrix is used to find the HOL-system distribution in the next cycle given the distribution in the current cycle. The matrix is computed using four quantities: the probability of an arrival to an empty queue, $p_{\rm ar}$, the probability of a packet in a head slot given that a packet left that head slot in the previous cycle, $p_{\rm mo}$, and the probability that a packet bound for queue A (B) is not blocked, $p_{\text{sp-}\mathsf{A}}$ ($p_{\text{sp-}\mathsf{B}}$).

At each cycle the head slot of a non-empty queue contending for a module output can either win or loose the contention. These outcomes will be denoted W and L respectively. An empty queue or the sole queue with a head packet bound for a particular output are also said to win. Let the triple (b, a, o) indicate the state and contention outcome of a queue with $b \in \{\mathsf{A}, \mathsf{B}, \mathsf{E}\}$, the current head-slot, $a \in \{\mathsf{A}, \mathsf{B}, \mathsf{E}\}$, the next state of the head slot, and $o \in \{\mathsf{W}, \mathsf{L}\}$, the contention outcome. Any triple consistent with the description above is called valid. The set of valid triples is

$$U_1 = \{\, (b, a, \mathsf{W}) \mid b, a \in \{\mathsf{A}, \mathsf{B}, \mathsf{E}\} \,\} \cup \{\, (b, b, \mathsf{L}) \mid b \in \{\mathsf{A}, \mathsf{B}\} \,\}.$$

Consider pairs of triples, $((b_1, a_1, o_1), (b_2, a_2, o_2))$, indicating the two queue heads in a HOL system. Such pairs will be considered valid if each triple is valid and there are either no losers or there is exactly one looser and $b_1 = b_2$. The set of valid pairs, U_2, is

$$\begin{aligned} U_2 = \{\, &((b_1, a_1, o_1), (b_2, a_2, o_2)) \mid ((b_1, a_1, o_1), (b_2, a_2, o_2)) \in U_1 \times U_1 \wedge \\ &[(b_1 = b_2 \ \wedge\ b_1 \in \{\mathsf{A}, \mathsf{B}\} \ \wedge\ (o_1, o_2) = (\mathsf{W}, \mathsf{L})) \vee \\ &(b_1 \neq b_2 \ \vee\ b_1 = \mathsf{E} \ \vee\ b_2 = \mathsf{E}) \ \wedge\ o_1 = o_2 = \mathsf{W}] \,\}, \end{aligned}$$

where $\vee$ indicates disjunction, $\wedge$ indicates conjunction, and $U_1 \times U_1 = \{(u_1, u_2) \mid u_1, u_2 \in U_1\}$. Since the description of HOL-system transition probabilities uses U_2 its properties will be formally stated.

Lemma: For $((b_1, a_1), (b_2, a_2)) \in (\{\mathsf{A}, \mathsf{B}, \mathsf{E}\} \times \{\mathsf{A}, \mathsf{B}, \mathsf{E}\}) \times (\{\mathsf{A}, \mathsf{B}, \mathsf{E}\} \times \{\mathsf{A}, \mathsf{B}, \mathsf{E}\})$, the cardinality $|\{\, ((b_1, a_1, o_1), (b_2, a_2, o_2)) \mid ((b_1, a_1, o_1), (b_2, a_2, o_2)) \in U_2 \,\}| = 1$, that is, there is exactly one pair in U_2 for each valid set of consecutive queue states.

Proof: The set $U_1 \times U_1$ contains all possible pairs when the outcomes (o_1 and o_2) are ignored. For every pair either predicate $b_1 = b_2 \in \{\mathsf{A}, \mathsf{B}\}$ or $(b_1 \neq b_2 \ \vee\ b_1 = \mathsf{E} \ \vee\ b_2 = \mathsf{E})$ will be true. Further, the predicates $(o_1, o_2) = (\mathsf{W}, \mathsf{L})$ and $o_1 = o_2 = \mathsf{W}$ can each be true for exactly one pair of o_1, o_2. Thus there is a single pair in U_2 for each $((b_1, a_1), (b_2, a_2))$.□

Let $f_3 \mid U_1 \to [0, 1]$ map a transition $(b, a, o) \in U_1$ to the probability that the queue undergoes the transition given the outcome, o. Based upon the module description, f_3 is given by

$$f_3(b, a, o) = \begin{cases} 1, & \text{if } o = \mathsf{L}; \\ p_{\rm ar}/2, & \text{if } b = \mathsf{E},\ a \in \{\mathsf{A}, \mathsf{B}\},\ o = \mathsf{W}; \\ p_{\text{sp-}b}(1 - p_{\rm mo}), & \text{if } b \in \{\mathsf{A}, \mathsf{B}\},\ a = \mathsf{E},\ o = \mathsf{W}; \\ (1 - p_{\rm ar}), & \text{if } b = a = \mathsf{E},\ o = \mathsf{W}; \\ p_{\text{sp-}b}p_{\rm mo}/2, & \text{if } b, a \in \{\mathsf{A}, \mathsf{B}\},\ a \neq b,\ o = \mathsf{W}; \\ p_{\text{sp-}b}p_{\rm mo}/2 + (1 - p_{\text{sp-}b}), & \text{if } b, a \in \{\mathsf{A}, \mathsf{B}\},\ a = b,\ o = \mathsf{W}. \end{cases}$$

The individual queue-transition probabilities given in the formula above will be used to determine a HOL-system transition probability. Let f_4 map a state transition for a HOL system consisting of a pair of distinct queues to the transition probability. The function is given by $f_4((b_1, a_1), (b_2, a_2)) = f_3(b_1, a_1, o_1) f_3(b_2, a_2, o_2)$, where $((b_1, a_1, o_1), (b_2, a_2, o_2))$ is the unique element in U_2. Queues are indistinct in the HOL-system model used here, so that the transition probabilities in the HOL-system transition matrix function are computed to account for the possible ways the transition can occur. That function, $H(p_{\rm ar}, p_{\rm mo}, p_{\textsf{sp-A}}, p_{\textsf{sp-B}})$, maps any $p_{\rm ar}$, $p_{\rm mo}$, $p_{\textsf{sp-A}}$, and $p_{\textsf{sp-B}} \in [0,1]$ to a $|\mathcal{Q}_{\mathsf{C}}| \times |\mathcal{Q}_{\mathsf{C}}|$ matrix of elements in which

$$H_{b_1 b_2\ a_1 a_2}(p_{\rm ar}, p_{\rm mo}, p_{\textsf{sp-A}}, p_{\textsf{sp-B}}) = \begin{cases} f_4((b_1, a_1), (b_2, a_2)), & \text{if } b_1 = b_2; \\ f_4((b_1, a_1), (b_2, a_2)) + f_4((b_1, a_2)(b_2, a_1)), & \text{otherwise;} \end{cases}$$

for all $b_1 b_2 \in \mathcal{Q}_{\mathsf{C}}$ and $a_1 a_2 \in \mathcal{Q}_{\mathsf{C}}$, where f_4 is defined in terms of $p_{\rm ar}$, $p_{\rm mo}$, $p_{\textsf{sp-A}}$, and $p_{\textsf{sp-B}}$ as described above.

4.4 Non-Congested HOL-System Transitions

A HOL-system distribution and transition matrix will be derived for use in computing arrival rates for queues in non-congested states $(\mathcal{Q}_{\mathsf{N}}, \mathsf{N}, \star)$. This HOL-system distribution in stage $x-1$, is denoted $P^{\rm nc}(x)$, and is given by $P^{\rm nc}(x) = P^{\rm hi}(p_{\rm sm}(x))$, where $p_{\rm sm}(x)$ is the probability of a stage-$(x-1)$ head slot being occupied given that neither next-stage queue is congested: $p_{\rm sm}(x) = 1 - p_{(0,\star,\mathsf{N})}(x-1)/p_{(\star,\star,\mathsf{N})}(x-1)$, for $0 < x < n$. The HOL-system transition matrix for the non-congested HOL-system, denoted $H^{(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x)$, is the HOL-system transition matrix given by

$$H^{(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x) = H(r_{(0,\star,\mathsf{N})}(x-1), p_{\textrm{mo-}\mathsf{N}}(x), p_{\textrm{sp-}(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x), p_{\textrm{sp-}(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x)).$$

The first argument, $r_{(0,\star,\mathsf{N})}(x-1)$, is probability of arrival into an empty queue given that the next stage is not congested. The second argument, $p_{\textrm{mo-}\mathsf{N}}(x)$, is found by subtracting from one the probability that the queue will be empty after a departure. This will happen if the queue has one packet and there is no arrival:

$$p_{\textrm{mo-}\mathsf{N}}(x) = 1 - \frac{p_{(1,\star,\mathsf{N})}(x-1)}{p_{(\mathcal{Q}_{\rm S}-\{0\},\star,\mathsf{N})}(x-1)}(1 - r_{(1,\star,\mathsf{N})}(x-1)),$$

for $0 < x < n$. Note that the probability the queue has one packet is conditioned on the probability it is not empty and the next stage is not congested.

In a simpler model values of parameters $p_{\textsf{sp-A}}$ and $p_{\textsf{sp-B}}$ would simply be the probability that the respective queues were not full, $1 - p_{(d,\mathsf{N},\star)}/p_{(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}$. (In the non-congested case queue **A** and **B** are indistinguishable.) Because the probability of arrival to a queue depends upon the state of the queue, there is a correlation between states of the HOL system and the queue. The correlation is determined in the computation of $r_{(d,\mathsf{N},\star)}$ as will be explained in Section 6. Therefore $p_{\textrm{sp-}(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x)$ is derived so that the probability of a packet leaving the HOL system, $p_{\textrm{sp-}(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x) \sum V(\mathcal{Q}_{\mathsf{B}}) P^{\rm nc}(x)$, is equal to the probability a packet will not be blocked, $\sum V(\mathcal{Q}_{\mathsf{B}}) P^{\rm nc}(x) - r_{(d,\mathsf{N},\star)} p_{(d,\mathsf{N},\star)}(x)/p_{(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x)$, given that the queue is in state $(\mathcal{Q}_{\mathsf{N}}, \mathsf{N}, \star)$. Solving yields

$$p_{\textrm{sp-}(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x) = 1 - \frac{p_{(d,\mathsf{N},\star)}(x)}{p_{(\mathcal{Q}_{\mathsf{N}},\mathsf{N},\star)}(x)} \frac{r_{(d,\mathsf{N},\star)}(x)}{\sum V(\mathcal{Q}_{\mathsf{B}}) P^{\rm nc}(x)}.$$

4.5 Congested HOL-System Transitions

Unlike the non-congested HOL system, the state distribution for the congested HOL system is part of the queue state. Two HOL-system transition matrices will be used for these cases, one for a queue which is congested when its dual is not, $H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N)}(x)$, and one for a queue which is congested with its dual, $H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{C},N)}(x)$. Except for $p_{\text{sp-}\mathbf{A}}$, the expressions for the HOL-system-transition-matrix function parameters are similar to those for the non-congested HOL system. The probability of an arrival to an empty queue in the HOL system is $r_{(0,\star,\mathbf{C})(x-1)}$. The probability that a stage-$(x-1)$ head slot is occupied given a previous-cycle departure, denoted $p_{\text{mo-}\mathbf{C}}(x)$, is given by

$$p_{\text{mo-}\mathbf{C}}(x) = 1 - \frac{p_{(1,\star,\mathbf{C})}(x-1)}{p_{(\mathcal{Q}_{\mathrm{S}}-\{0\},\star,\mathbf{C})}(x-1)}(1 - r_{(1,\star,\mathbf{C})}(x-1)\ ).$$

The derivation of $p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)}(x)$ is similar to the non-congested case. Unlike the non-congested case, the queue state is used to find the probability a packet is bound for **B** :

$$p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)}(x) \ = 1 - \frac{p_{(d,\mathbf{C},\star)}(x)}{p_{(\mathcal{Q}_{\mathbf{N}},\mathbf{C},\star)}(x)} \frac{r_{(d,\mathbf{C},\star)}(x)}{p_{(\mathcal{Q}_{\mathbf{B}},\mathbf{N},\star)}(x)\ /p_{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)}(x)}.$$

The probability of space in queue **A** must be inferred because the number of packets in a congested queue is not encoded in the queue state. This complicates the expression for $p_{\text{sp-}\mathbf{A}}$. For illustration purposes, an approximate method of computing $p_{\text{sp-}\mathbf{A}}$ will be presented followed by the exact expression.

Suppose, only for the approximate derivation, that lateral state transitions do not occur during congestion nor at the cycle before. Before becoming congested a queue has d packets. Therefore, at the first cycle of congestion the queue is full with probability $p_{\text{c-0}}(x,t) = 1 - v_N$ and has one slot free with probability $p_{\text{c-1}}(x,t) = v_N$, where $N \in \{\mathbf{N},\mathbf{C}\}$ is the disposition of the next-state queues, and t is the first cycle of congestion. When congested, the probability of transition from full to one slot free is v_N. The probability of transition from one slot to zero free is $r(1-v_N)$ and from one slot free to non-congested is $1-r$, where r is the probability of an arrival. It can easily be shown that the state distribution for this system satisfies $p_{\text{c-0}}(x,t)/p_{\text{c}}(x,t) = 1-v_N$ and $p_{\text{c-1}}(x,t)/p_{\text{c}}(x,t) = v_N$ for *all* t during which the queue is congested, where $p_{\text{c}} = p_{\text{c-0}} + p_{\text{c-1}}$. Therefore, $p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)} \approx v_N$.

The equation for $p_{\text{sp-}\mathbf{A}}$ used in the analysis is derived by finding expressions for the number of arrivals and departures for a queue during congestion. The expressions are solved for $p_{\text{sp-}\mathbf{A}}$, ensuring packet conservation. The period analyzed starts at the cycle before congestion begins; the queue has d packets at that time. The period ends at the last cycle of congestion; the queue has $d-1$ packets. Thus there is one net departure. Let $p_{\text{st}} = p_{(d,\star,\star)}r_{(d,\star,\star)}$, the probability of congestion starting. The expected congestion duration is then $p_{(\mathcal{Q}_{\mathbf{C}},\star,\star)}/p_{\text{st}}$. The total number of arrivals to the queue during congestion is $p_{\text{sp-}\mathbf{A}}p_{(\mathcal{Q}_{\mathbf{A}},\star,\star)}/p_{\text{st}}$ (this expression takes the lack of arrival at the last cycle into account). The number of departures before the first cycle is $v_\star$; the number of departures during congestion, except for the last cycle, is $v_\star(p_{(\mathcal{Q}_{\mathbf{C}},\star,\star)} - p_{(\mathcal{Q}_{\overline{\mathbf{A}}},\star,\star)}p_{\text{sp-}\mathbf{A}})/p_{\text{st}}$. For packet conservation the following equation must hold:

$$p_{\text{sp-}\mathbf{A}}p_{(\mathcal{Q}_{\mathbf{A}},\star,\star)}/p_{\text{st}} - v_\star - v_\star(p_{(\mathcal{Q}_{\mathbf{C}},\star,\star)} - p_{(\mathcal{Q}_{\overline{\mathbf{A}}},\star,\star)}p_{\text{sp-}\mathbf{A}})/p_{\text{st}} = -1.$$

The equation can be solved for $p_{\text{sp-}\mathbf{A}}$, however the packets needed are $p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,\mathbf{N})}$ and $p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,\mathbf{C})}$. A set satisfying the equation can be found by splitting the equation into two parts, one for the next-stage-congested case and the other for the next-stage-not-congested case. The resulting equations are

$$p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}(x) \ = v_N(x)\ \frac{p_{(d,\star,N)}(x)\ r_{(d,\star,N)}(x) + p_{(\mathcal{Q}_{\mathbf{C}},\star,N)}(x)}{p_{(\mathcal{Q}_{\overline{\mathbf{A}}},\star,N)}(x)\ v_N(x) + p_{(\mathcal{Q}_{\mathbf{C}},\star,N)}(x)},$$

for $N \in \{\mathbf{N},\mathbf{C}\}$ and $0 < x < n$.

The HOL-system transition matrix for $(\mathcal{Q}_{\mathbf{C}}, \mathbf{N}, N)$ is given by

$$H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N)}(x) = H(r_{(0,\star,\mathbf{C})}(x-1), p_{\text{mo-}\mathbf{C}}(x), p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}(x), p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)}(x)).$$

If the dual is congested, then the **A** and **B** queues are indistinguishable, and so the same space probability can be used. The HOL-system matrix for $(\mathcal{Q}_{\mathbf{C}}, \mathbf{C}, N)$ is given by

$$H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{C},N)}(x) = H(r_{(0,\star,\mathbf{C})}(x-1), p_{\text{mo-}\mathbf{C}}(x), p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}(x), p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}(x)).$$

4.6 To-Congestion HOL-System Transitions

A HOL-system transition matrix will be used for finding the state distribution at the first cycle of congestion. The state of the HOL system for a queue in $(d, \mathbf{N}, \star)$ at the cycle before congestion is assumed to be

$$P^{\text{dn}} = \left\langle V(\mathcal{Q}_{\mathbf{A}}, \mathbf{N}, \star) P^{\text{hi}} \right\rangle. \tag{1}$$

At the cycle before congestion queue **A** will be full, the probability of space in queue **B** is the same as in any other non-congested state. The HOL-system transition matrix is given by

$$H^{(d,\mathbf{N},\star)}(x) = H(r_{(0,\star,\mathbf{N})}(x-1), p_{\text{mo-}\mathbf{C}}(x), 0, p_{\text{sp-}(\mathcal{Q}_{\mathbf{N}},\mathbf{N},\star)}(x))$$

4.7 Congestion-Start and -End Probabilities

While a queue is congested three congestion-related transitions are possible: the queue's congestion can end and either the dual queue can become congested or the dual queue's congestion can end. Expressions will be derived which give probabilities for each of these events conditioned on the modeled queue's state and/or HOL-system transition.

Congestion at a queue ends when there is no packet bound for the queue while it has a slot free. The probability of congestion ending in queue X given that the modeled queue is in state $(S, \star, N)$ is

$$p_{\text{en-}X}(S,N) = \begin{cases} 0, & X \in S \\ p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}, & \text{otherwise,} \end{cases}$$

where $S \in \mathcal{Q}_{\mathbf{C}}$. Note that by symmetry, $p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_{\mathbf{C}},\star,N)}$ gives the probability of space in the dual queue if it is congested.

Congestion starts in the dual queue if a packet bound for the queue is blocked. Given a transition, *e.g.,* **AB** to **AE**, one can easily tell if a packet was bound for the queue; it is more difficult to determine if a packet was blocked. (In the example, the packet going to queue **B** was not blocked, the packet going to **A** may have been blocked.) Let b be the state of a queue in the HOL system before the transition and a be the state after the transition. A necessary—but not sufficient—condition for blocking is $b = a = \mathbf{B}$ since a packet bound for the dual queue could be replaced by another packet bound for the dual queue. To find the blocking probability in this case Bayes law is used. Let $f_{\text{bl}}(b, a, o)$ denote the probability that a particular packet bound for the dual queue was blocked, where $o \in \{\mathbf{W}, \mathbf{L}\}$ is the contention outcome. Then

$$f_{\text{bl}}(b,a,o) = \begin{cases} 1 - \left(\frac{2(1-p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)})}{p_{\text{mo-}\mathbf{C}} p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_{\mathbf{C}},\mathbf{N},\star)}} + 1 \right)^{-1}, & \text{if } b = a = \mathbf{B},\ o = \mathbf{W}; \\ 0, & \text{otherwise.} \end{cases}$$

The single-queue blocking probability above is used to find the blocking probability given a HOL-system transition in a way similar to the HOL-system matrix derivation above. Let

$$f_{\text{bl}}((b_1,a_1),(b_2,a_2)) = f_{\text{bl}}(b_1,a_1,o_1) + f_{\text{bl}}(b_2,a_2,o_2),$$

where $((b_1, a_1, o_1), (b_2, a_2, o_2))$ is a valid pair of contention outcomes. *E.g.,* $((\mathbf{A}, \mathbf{E}, \mathbf{W}), (\mathbf{A}, \mathbf{E}, \mathbf{W}))$ is invalid because both packets could not win contention, $((\mathbf{A}, \mathbf{E}, \mathbf{W}), (\mathbf{B}, \mathbf{E}, \mathbf{L}))$ is invalid because the one packet bound for queue **B** could not lose contention, and $((\mathbf{A}, \mathbf{E}, \mathbf{W}), (\mathbf{B}, \mathbf{A}, \mathbf{W}))$ is valid. The probability that congestion starts in the dual queue given a HOL-system transition from $q_1 = b_1 b_2$ to $q_2 = a_1 a_2$ is denoted $p_{\text{st-}\mathbf{B}}(b_1 b_2, a_1 a_2)$; it is given by

$$p_{\text{st-}\mathbf{B}}(b_1 b_2, a_1 a_2) = \begin{cases} f_{\text{bl}}((b_1, a_1), (b_2, a_2)), & \text{if } b_1 = b_2; \\ \frac{f((b_1,a_1),(b_2,a_2)) f_{\text{bl}}((b_1,a_1),(b_2,a_2)) + f((b_1,a_2),(b_2,a_1)) f_{\text{bl}}((b_1,a_1),(b_2,a_2))}{f((b_1,a_1),(b_2,a_2)) + f((b_1,a_2),(b_2,a_1))}, & \text{otherwise,} \end{cases}$$

where f is a distinct-queue HOL-system transition probability. (See f_4 in in the HOL-system matrix derivation.)

5 The Transition Matrix

The transition matrix is presented below. Transitions into congestion are described first, followed by transitions within congestion, transitions leaving congestion, and finishing with transitions between non-congested states.

5.1 Transitions into Congestion

Congestion-start transition probabilities will be specified in separate equations for the four possible dual-queue dispositions before and after the transition.

Consider the transition from the state in which the dual is not congested, $(d, \mathbf{N}, \star)$. The HOL-system distribution before transition is assumed to be P^{dn}, (1), (containing a packet bound for the queue, with the state of the two head slots independent). The transition matrix $H^{(d,\mathbf{N},\star)}$ will be used. Recall that this matrix is for cases where the queue always blocks, $p_{\text{sp-}\mathbf{A}} = 0$, and the dual queue is not congested. The product $H^{(d,\mathbf{N},\star)} P^{\text{dn}}$ gives the HOL-system distribution at the next cycle given that congestion starts.

The transition probabilities must take into account the next state of the dual queue since it can enter the congested state only if the head slots contain packets bound for the queue and its dual. This is done by first computing the probability that the dual will become congested given that the HOL system is in state **AB**, denoted p_{af},

$$p_{\text{af}}(x) \ = l_{\text{dual-}\mathbf{C}|\mathbf{N}}(x) \ \left(\frac{2}{p_{(\mathcal{Q}_\text{S} - \{0\}, \star, \mathbf{N})}(x-1)} - \frac{1}{2} \right).$$

An adjusted HOL-system distribution is then found for the case where the dual queue does not enter congestion, $(V(\mathcal{Q}_\mathbf{C}) - p_{\text{af}}(x) \ V(\{\mathbf{AB}\}))P^{\text{dn}}$. The transition probability when the dual does not enter congestion is then given by

$$\begin{aligned} M_{(d,\mathbf{N},N_1)(q,\mathbf{N},N_2)}(x) \ = \\ & l_{\text{next-}N_2|\text{s-}N_1}(x) \ r_{(d,\mathbf{N},\star)}(x) \\ & \sum V(q) \left(H^{(d,\mathbf{N},\star)}(x) \left((V(\mathcal{Q}_\mathbf{C}) - p_{\text{af}}(x) \ V(\{\mathbf{AB}\}))P^{\text{dn}} \right) \right), \end{aligned}$$

for $0 < x < n$, N_1, $N_2 \in \{\mathbf{N}, \mathbf{C}\}$, and $q \in \mathcal{Q}_\mathbf{C}$. First-stage queues do not become congested. Note that the congestion change for the next-stage queues is independent of the transition. The transition in which the dual queue becomes congested is given by

$$M_{(d,\mathbf{N},N_1)(q,\mathbf{C},N_2)}(x) \ = \begin{cases} l_{\text{next-}N_2|\text{s-}N_1}(x) \ r_{(d,\mathbf{N},\star)}(x) \ l_{\text{dual-}\mathbf{C}|\mathbf{N}}(x) \ , & \text{if } q = \mathbf{AB}; \\ 0, & \text{otherwise,} \end{cases}$$

for $0 < x < n$, N_1, $N_2 \in \{\mathbf{N}, \mathbf{C}\}$, and $q \in \mathcal{Q}_{\mathbf{C}}$.

Transitions from the non-congested state in which the dual queue is congested to the congested state are considered next. The HOL-system distribution before the transition is a mirror-image of part of the queue state. Rather than assuming a HOL-system distribution, as above, the appropriate part of the queue state will be appropriately mapped.

Define

$$\widetilde{b} = \begin{cases} \mathbf{E} & \text{if } b = \mathbf{E}; \\ \mathbf{A} & \text{if } b = \mathbf{B}; \\ \mathbf{B} & \text{if } b = \mathbf{A}, \end{cases}$$

and $\widetilde{b_1 b_2} = \widetilde{b_1}\widetilde{b_2}$, recalling that $b_1 b_2 = b_2 b_1$. The probability of HOL-system state q_1 before the transition is then $(\widetilde{q_1}, \mathbf{N}, \star)$. HOL-system transition matrix $H^{(\mathcal{Q}_{\mathbf{C}}, \mathbf{N}, N_1)}$, where N_1 is the congestion disposition of the next-stage queues, will be used to find the first congested state. The transition probability will be found by summing over all possible pre-transition HOL-system states. The transition probability is given by

$$M_{(d,\mathbf{C},N_1)(q_2,\mathbf{N},N_2)}(x) = \\ l_{\text{next-}N_2|\text{s-}N_1}(x)\, r_{(d,\mathbf{C},\star)}(x) \\ \frac{\sum_{q_1 \in \mathcal{Q}_{\mathbf{C}}} p_{(q_1,\mathbf{N},N_1)}(x)\, p_{\text{en-}\mathbf{B}}(q_1, N_1) H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N_1)}_{q_1 \widetilde{q_2}}(x)}{p_{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N_1)}(x)},$$

for $0 < x < n$, N_1, $N_2 \in \{\mathbf{N}, \mathbf{C}\}$, and $q_2 \in \mathcal{Q}_{\mathbf{C}}$. The transition probability when the dual queue remains congested is given by

$$M_{(d,\mathbf{C},N_1)(q,\mathbf{C},N_2)}(x) = \\ l_{\text{next-}N_2|\text{s-}N_1}(x)\, r_{(d,\mathbf{C},\star)}(x) \\ \frac{\sum_{q_1 \in \mathcal{Q}_{\mathbf{C}}} p_{(q_1,\mathbf{N},N_1)}(x)\, (1 - p_{\text{en-}\mathbf{B}}(q_1, N_1)) H^{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N_1)}_{q_1 \widetilde{q}}(x)}{p_{(\mathcal{Q}_{\mathbf{C}},\mathbf{N},N_1)}(x)},$$

for $0 < x < n$, N_1, $N_2 \in \{\mathbf{N}, \mathbf{C}\}$, and $q \in \mathcal{Q}_{\mathbf{C}}$.

Transitions in which the queue is congested before and after are easily specified. A single equation will be used for all combinations of next-stage and dual-queue dispositions. For a transition from (q_1, D_1, N_1) to (q_2, D_2, N_2) the following are used. The HOL-system transition matrix $H^{(\mathcal{Q}_{\mathbf{C}}, D_1, N_1)}$ specifies the HOL system's next state. A factor $(1 - p_{\text{en-}\mathbf{A}}(q_1, N_1))$ gives the probability that the queue will remain congested. A function $\Delta(D_1, D_2, N_1, q_1, q_2)$ gives the probability that the dual queue will make the transition from D_1 to D_2 given the transition. As in all transitions above, the probability that the next-stage queue makes the appropriate transition is included in the equation. For $0 < x < n$, D_1, D_2, N_1, $N_2 \in \{\mathbf{N}, \mathbf{C}\}$, and $q_1, q_2 \in \mathcal{Q}_{\mathbf{C}}$

$$M_{(q_1,D_1,N_1)(q_2,D_2,N_2)}(x) = \\ l_{\text{next-}N_2|\text{s-}N_1}(x)\, (1 - p_{\text{en-}\mathbf{A}}(q_1, N_1)) \Delta(D_1, D_2, N_1, q_1, q_2) H^{(\mathcal{Q}_{\mathbf{C}},D_1,N_1)}_{q_1 q_2}(x), \tag{2}$$

where $\Delta(D_1, D_2, N, q_1, q_2) = \begin{cases} p_{\text{st-}\mathbf{B}}(q_1, q_2), & \text{if } D_1 = \mathbf{N},\ D_2 = \mathbf{C}; \\ (1 - p_{\text{st-}\mathbf{B}}(q_1, q_2)), & \text{if } D_1 = \mathbf{N},\ D_2 = \mathbf{N}; \\ p_{\text{en-}\mathbf{B}}(q_1, N), & \text{if } D_1 = \mathbf{C},\ D_2 = \mathbf{N}; \\ (1 - p_{\text{en-}\mathbf{B}}(q_1, N)), & \text{if } D_1 = \mathbf{C},\ D_2 = \mathbf{C}. \end{cases}$

5.2 Out-of-Congestion Transitions

The expression for transitions from the congested state to the non-congested state is similar to (2). HOL-system transition matrix $H^{(\mathcal{Q}_{\mathbf{C}},D_1,N_1)}$ is again used, in this case to find the next state of the dual queue. The factor $\nu(y,N_1)$ is the probability of having $y \in \{d-2, d-1\}$ packets in the queue, given that congestion is ending.

$$M_{(q_1,D_1,N_1)(y,D_2,N_2)}(x) = l_{\text{next-}N_2|\text{s-}N_1}(x)\,\nu(y,N_1) \sum_{q_2\in\mathcal{Q}_{\mathbf{C}}} H^{(\mathcal{Q}_{\mathbf{C}},D_1,N_1)}_{q_1q_2}(x)\,p_{\text{en-}\mathbf{A}}(q_1,N_1)\Delta(D_1,D_2,N_1,q_1,q_2),$$

where $\nu(y,N_1) = \begin{cases} v_{N_1}, & \text{if } y=d-2; \\ 1-v_{N_1}, & \text{if } y=d-1. \end{cases}$

5.3 Non-congestion Occupancy Transitions

The transition-probability expressions for transitions between states in which the modeled queue is not congested are similar to the transition expressions used in other banyan-network analyses, for example [15]. The only difference is the inclusion of the lateral transition probabilities.

$$M_{(0,D_1,N_1)(1,D_2,N_2)}(x) = l_{\text{dual-}D_2|D_1}(x)\,l_{\text{next-}N_2|0\text{-}N_1}(x)\,r_{(0,D_1,\star)}(x)$$

for $0 \le x < n$ and D_1, D_2, N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(y,D_1,N_1)(y+1,D_2,N_2)}(x) = l_{\text{dual-}D_2|D_1}(x)\,l_{\text{next-}N_2|y\text{-}N_1}(x)\,r_{(y,D_1,\star)}(x)\,(1-v_{N_1}(x))$$

for $0 \le x < n$, $0 < y < d$ and D_1, D_2, N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(y,D_1,N_1)(y,D_2,N_2)}(x) = l_{\text{dual-}D_2|D_1}(x)\,l_{\text{next-}N_2|y\text{-}N_1}(x)\; r_{(y,D_1,\star)}(x)\,v_{N_1}(x) + (1-r_{(y,D_1,\star)}(x))(1-v_{N_1}(x))$$

for $0 \le x < n$, $0 \le y < d$ and D_1, D_2, N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(d,D_1,N_1)(d,D_2,N_2)}(x) = l_{\text{dual-}D_2|D_1}(x)\,l_{\text{next-}N_2|d\text{-}N_1}(x)\,(1-r_{(d,D_1,\star)}(x))(1-v_{N_1}(x))$$

for $0 < x < n$ and D_1, D_2, N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(d,\mathbf{N},N_1)(d,\mathbf{N},N_2)}(0) = l_{\text{next-}N_2|d\text{-}N_1}(0)\,(1-v_{N_1}(0))$$

for N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(y,D_1,N_1)(y-1,D_2,N_2)}(x) = l_{\text{dual-}D_2|D_1}(x)\,l_{\text{next-}N_2|y\text{-}N_1}(x)\,(1-r_{(y,D_1,\star)}(x))v_{N_1}(x)$$

for $0 \le x < n$, $0 < y \le d$ (except $x=0$, $y=d$), and D_1, D_2, N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

$$M_{(d,\mathbf{N},N_1)(d-1,\mathbf{N},N_2)}(0) = l_{\text{next-}N_2|d\text{-}N_1}(0)\,v_{N_1}(0)$$

for N_1, $N_2 \in \{\mathbf{N},\mathbf{C}\}$.

6 Flow-Related Probabilities

6.1 Arrival Rates

The arrival rates to a queue are determined by the HOL-system distributions which in turn are determined by the probabilities of a previous-stage queue being occupied. Two HOL-system distributions are used, congested and non-congested, to compute two corresponding sets of arrival rates. Each set contains an arrival rate for a queue with 0, $\{1, 2, \ldots, d-2\}$, $d-1$, and d packets. The four arrival rates are based on a relationship between HOL-system and queue states.

Consider the computation of a single arrival rate for a non-congested queue, $r_{(\mathcal{Q}_{\mathbf{N}},\mathbf{N},\star)}$. The arrival rate, the probability a packet is bound for a queue, is $\sum V(\mathcal{Q}_{\mathbf{B}})P^{\mathrm{nc}}$. To accurately model the important queue-empty and queue-full states, state-dependent arrival rates will be computed.

In a basic network the arrival probability is dependent upon whether there was an arrival in the previous cycle. For example, if there was an arrival to queue $\mathbf{A}$ in cycle t when the HOL system was in state $\mathbf{AA}$ then at cycle $t+1$ there will surely be another arrival. Let $r_{\mathrm{an},\mathbf{N}}$ denote the probability of an arrival given that there was an arrival in the previous cycle and given that the queue was in state $(\mathcal{Q}_{\mathbf{N}}, \mathbf{N}, \star)$ in the previous cycle. This quantity will be found by multiplying a normalized HOL-system distribution modeling a packet bound for $\mathbf{A}$ with a transition matrix in which the queue accepts the packet. The probability of the resulting distribution having a packet bound for $\mathbf{A}$ is then extracted:

$$r_{\mathrm{an},\mathbf{N}}(x) = \sum V(\mathcal{Q}_{\mathbf{A}})H(r_{(0,\star,\mathbf{N})}(x-1), p_{\text{mo-}\mathbf{N}}(x), 1, p_{\text{sp-}(\mathcal{Q}_{\mathbf{N}},\mathbf{N},\star)}(x))\left\langle V(\mathcal{Q}_{\mathbf{A}})P^{\mathrm{nc}}(x)\right\rangle. \quad (3)$$

Let $r_{\mathrm{na},\mathbf{N}}$ denote the probability of an arrival given that there was not an arrival in the previous cycle and given that the queue was in state $(\mathcal{Q}_{\mathbf{N}}, \mathbf{N}, \star)$ in the previous cycle. A similar formula is used to compute this quantity:

$$r_{\mathrm{na},\mathbf{N}}(x) = \sum V(\mathcal{Q}_{\mathbf{A}})H(r_{(0,\star,\mathbf{N})}(x-1), p_{\text{mo-}\mathbf{N}}(x), 1, p_{\text{sp-}(\mathcal{Q}_{\mathbf{N}},\mathbf{N},\star)}(x))\left\langle V(\mathcal{Q}_{\overline{\mathbf{A}}})P^{\mathrm{hi}(x)}\right\rangle. \quad (4)$$

Similar arrival rates, $r_{\mathrm{an},\mathbf{C}}$, and $r_{\mathrm{na},\mathbf{C}}$, are found for a non-congested queue whose dual is congested, $(\mathcal{Q}_{\mathbf{N}}, \mathbf{C}, \star)$. The HOL-system distribution here is part of the queue state. The arrival rate is being found for the uncongested dual queue, given that the dual does not become congested, nor does the queue become uncongested. The probability of another arrival is given by

$$r_{\mathrm{an},\mathbf{C}}(x) = \frac{\sum V(\mathcal{Q}_{\mathbf{B}}, \mathbf{N}, \star)\,(M(x)\ \ (V(\mathcal{Q}_{\mathbf{B}}, \mathbf{N}, \star)P(x)))}{\sum V(\mathcal{Q}_{\mathbf{C}}, \mathbf{N}, \star)\,(M(x)\ \ (V(\mathcal{Q}_{\mathbf{B}}, \mathbf{N}, \star)P(x)))}. \quad (5)$$

Similarly, the probability of an arrival given no arrival in the previous cycle is given by

$$r_{\mathrm{na},\mathbf{C}}(x) = \frac{\sum V(\mathcal{Q}_{\mathbf{B}}, \mathbf{N}, \star)\left(M(x)\ \ \left(V(\mathcal{Q}_{\overline{\mathbf{B}}}, \mathbf{N}, \star)P(x)\right)\right)}{\sum V(\mathcal{Q}_{\mathbf{C}}, \mathbf{N}, \star)\left(M(x)\ \ \left(V(\mathcal{Q}_{\overline{\mathbf{B}}}, \mathbf{N}, \star)P(x)\right)\right)}. \quad (6)$$

The arrival rates for queues with 0, $d-1$, and d packets will be found using $r_{\mathrm{an},\mathbf{N}}$, $r_{\mathrm{na},\mathbf{N}}$, $r_{\mathrm{an},\mathbf{C}}$, and $r_{\mathrm{na},\mathbf{C}}$. It is not possible to determine whether there was an arrival in the previous cycle from the queue state, however the probability that there had been an arrival can be computed. These will be used to find three of the arrival rates. Let $p_{\mathrm{a}|(S,D,N)}$ be the probability that a queue received a packet in the previous cycle given that it is in state (S, D, N). Similarly, let $p_{\mathrm{na}|(S,D,N)} = 1 - p_{\mathrm{a}|(S,D,N)}$, the probability of a queue not having an arrival given that is in state (S, D, N). (This quantity can easily be found by splitting the transition matrix into two parts.) Then

$$r_{(y,D,\star)}(x) = p_{\mathrm{a}|(y,D,\star)}(x)\ r_{\mathrm{an},D}(x) + p_{\mathrm{na}|(y,D,\star)}(x)\ r_{\mathrm{na},D}(x), \quad (7)$$

for $y \in \{0, d-1, d\}$, $0 < x < n$, and $D \in \{\mathbf{N}, \mathbf{C}\}$. Were (7) applied to all non-congested queue states flow invariance would be violated. This is so because the matching of HOL-system states to queue states implied by (7) is not perfect. For example, the HOL-system distributions assumed in (3-6) are not state-dependent. A single arrival rate for a queue having 1 to $d-2$ packets will be derived that will ensure flow invariance. The derivation will make use of $r_{(\mathcal{Q}_\mathbf{N},\mathbf{N},\star)}$ and a similarly defined quantity, $r_{(\mathcal{Q}_\mathbf{N},\mathbf{C},\star)}$. The probability that a HOL packet is bound for a non-congested queue given that the dual is congested is $r_{(\mathcal{Q}_\mathbf{N},\mathbf{C},\star)} = p_{(\mathcal{Q}_\mathbf{B},\mathbf{N},\star)}/p_{(\mathcal{Q}_\mathbf{C},\mathbf{N},\star)}$. Then

$$r_{(y,D,\star)}(x) = \frac{r_{(\mathcal{Q}_\mathbf{N},D,\star)}(x)\ p_{(\mathcal{Q}_\mathbf{N},D,\star)}(x) - \sum_{i\in\{0,d-1,d\}} p_{(i,D,\star)}(x)\ r_{(i,D,\star)}(x)}{\sum_{i=1}^{d-2} p_{(i,D,\star)}(x),}$$

for $y \in \{1, 2, \ldots, d-2\}$, $0 < x < n$, and $D \in \{\mathbf{N}, \mathbf{C}\}$.

Finally, the arrival rate to the queues in the first stage is the input arrival rate, λ, and is denoted by the symbol $r_{(i,D,N)}$ for $i \in \mathcal{Q}_\mathbf{N}$ and $D, N \in \{\mathbf{N}, \mathbf{C}\}$.

6.2 Service Rate

The computation of the arrival rate implicitly matched HOL-system states to queue states. To maintain flow invariance the service rate must be computed using the same matching. This is done by, in effect, dividing the expected number of packets entering a queue by the expected amount of time the feeder queues are not empty. The computation of $v_\mathbf{N}$ is straightforward because congested queues are not involved.

$$v_\mathbf{N}(x-1) = \frac{\sum_{i=0}^{d-1} p_{(i,\mathbf{N},\star)}(x)\ r_{(i,\mathbf{N},\star)}(x)}{p_{(\mathcal{Q}_\mathrm{S}-\{0\},\star,\mathbf{N})}(x-1),}$$

for $0 < x \le n-1$. The service rate when the next stage is congested is given by

$$v_\mathbf{C}(x-1) = \frac{\sum_{i=0}^{d-1} p_{(i,\mathbf{C},\star)}(x) r_{(i,\mathbf{C},\star)}(x) + \sum_{D\in\{\mathbf{N},\mathbf{C}\}} p_{(\mathcal{Q}_\mathbf{A},\star,D)}(x)\ p_{\text{sp-}\mathbf{A}\text{-}(\mathcal{Q}_\mathbf{C},\star,D)}(x)}{(1 - p_{(\star,\star,\mathbf{N})}(x-1) - p_{(0,\star,\mathbf{C})}(x-1))},$$

for $0 < x \le n-1$. The numerator contains a term for flow into a queue while its dual is congested and two terms for flow while it is congested and the dual may be congested. The denominator is the amount of time the queue in stage $x-1$ is offering a packet.

For the last stage the service rate equation used by [1,15] is used:

$$\begin{aligned} v_\mathbf{N}(n-1) =& v_\mathbf{C}(n-1) = \\ & \left(1 - \left(1 - \frac{1 - p_{(0,\star,\star)}(n-1)}{2}\right)^2\right) \frac{1}{(1 - p_{(0,\star,\star)}(n-1))}. \end{aligned}$$

6.3 Lateral Transition Probabilities

The lateral transition probabilities are found from the state distributions and the probabilities of certain transitions. The lateral transition probabilities are conditional probabilities of the form $\Pr[E \mid F]$, where E is the transition in the next-stage or dual queue and F is usually the queue state. Formulas for the lateral transition probabilities will be given in the form of $\Pr[E \cap F]/\Pr[F]$.

Quantity $l_{\text{dual-}\mathbf{C}|\mathbf{N}}(x)$ is the probability that the dual queue starts congestion given that the modeled queue is not congested. The probability that the dual queue starts congestion and the queue is not congested is $p_{(d,\mathbf{N},\star)}(x)\ r_{(d,\mathbf{N},\star)}(x)$. The probability that the modeled queue and the dual queue are not congested is $p_{(\mathcal{Q}_\mathbf{N},\mathbf{N},\star)}$. Dividing yields the lateral transition probability:

$$l_{\text{dual-}\mathbf{C}|\mathbf{N}}(x) = \frac{p_{(d,\mathbf{N},\star)}(x)\ r_{(d,\mathbf{N},\star)(x)}}{p_{(\mathcal{Q}_\mathbf{N},\mathbf{N},\star)}(x)},$$

for $0 < x < n$. First-stage queues do not become congested. Rather than specifying separate transition-probability equations for the first stage, transition probabilities $l_{\text{dual-}\mathbf{C}|\mathbf{N}}(0) = 0$ and $l_{\text{dual-}\mathbf{N}|\mathbf{C}}(0) = 1$, are used.

Lateral transition probability $l_{\text{dual-}\mathbf{N}|\mathbf{C}}(x)$ applies to states $(\mathcal{Q}_\mathbf{N} - \{d\}, \mathbf{C}, \star)$. It is also used in state $(d, \mathbf{C}, \star)$ when congestion is not entered. (The probability of the dual ending congestion is dependent on the probability of an arrival.) For $0 < x < n$ the transition probability is given by

$$l_{\text{dual-}\mathbf{N}|\mathbf{C}}(x) = \frac{\sum V(\mathcal{Q}_\mathbf{N}, \mathbf{N}, \star)\left(M(x)\ \left(V(\mathcal{Q}_\mathbf{C}, \mathbf{N}, \star)P(x)\right)\right)}{p_{(\mathcal{Q}_\mathbf{C},\mathbf{N},\star)}(x)\ - (1 - p_{\text{sp-}\mathbf{B}\text{-}(\mathcal{Q}_\mathbf{C},\mathbf{N},\star)}(x))p_{(\mathcal{Q}_\mathbf{B},\mathbf{N},\star)}(x)}.$$

The probabilities that the next-stage queues become congested are computed so that the probability of a feeder queue being empty at the first cycle of the next-stage congestion would match the corresponding probability in the HOL system as closely as possible. An exact match is not possible because of feeder-queue independence. Three lateral transition probabilities will be computed, one for empty queues, $(0, \star, \mathbf{N})$, one for queues with one packet, $(1, \star, \mathbf{N})$, and one for other non-congested queues, $(\mathcal{Q}_\mathbf{N} - \{0, 1\}, \star, \mathbf{N})$. For simplicity the lateral transition probabilities will be derived so that lateral transitions are independent of other transitions.

The probability of a particular feeder queue in stage $x - 1$ being empty at the first cycle of congestion is given by $p_{\text{zf}}(x) = \sum V^{\mathbf{E}'} V^{\text{ps}}\left(M(x)\ \left(V(d, \mathbf{N}, \star)P(x)\right)\right)$ for $0 < x < n$.

For simplicity, $l_{\text{next-}\mathbf{C}|0\text{-}\mathbf{N}}(x) = 0$ for $0 \le x < n-1$. The probability that the queue, which has one packet, has zero packets in the first cycle of congestion is then the product of $l_{\text{next-}\mathbf{C}|1\text{-}\mathbf{N}}$ and the probability of a departure and no arrivals. Solving for the lateral transition probability yields

$$l_{\text{next-}\mathbf{C}|1\text{-}\mathbf{N}}(x) = \frac{p_{\text{zf}}(x+1)}{p_{(1,\star,\mathbf{N})}(x)(1 - r_{(1,\star,\mathbf{N})}(x))v_\mathbf{N}(x)},$$

for $0 < x < n$. The remaining probability is then given by

$$l_{\text{next-}\mathbf{C}|\text{s-}\mathbf{N}}(x) = \frac{p_{(d,\mathbf{N},\star)}(x)\ r_{(d,\mathbf{N},\star)}(x) - p_{(1,\star,\mathbf{N})}(x) l_{\text{next-}\mathbf{C}|1\text{-}\mathbf{N}}(x);}{p_{(\mathcal{Q}_\mathbf{C}-\{0,1\},\star,\mathbf{N})}(x)},$$

for $0 \le x < n - 1$.

The probabilities of next-stage congestion ending are computed in a straightforward manner. Two cases are considered, one for when the queue is empty and one for when the queue is not empty. The probability of a particular feeder queue being empty is $V^{\mathbf{E}'} V^{\text{ps}} P(x)$. Similarly the probability of a particular feeder queue not being empty is $V^{\overline{\mathbf{E}}'} V^{\text{ps}} P(x)$, where

$$V^{\overline{\mathbf{E}}'}(q, D, N) = \begin{cases} 1, & \text{if } q \in \{\mathbf{AA}, \mathbf{AB}, \mathbf{BB}\}; \\ 1/2, & \text{if } q \in \{\mathbf{AE}, \mathbf{BE}\}; \\ 0, & \text{otherwise.} \end{cases}$$

The lateral transition probabilities are then given by

$$l_{\text{next-}\mathbf{N}|0\text{-}\mathbf{C}}(x-1) = \frac{\sum V(\mathcal{Q}_\mathbf{N}, \mathbf{N}, \star)\left(M(x)\ \left(V^{\mathbf{E}'} V^{\text{ps}} P(x)\right)\right)}{(p_{(\mathcal{Q}_\mathbf{C},\mathbf{N},\star)}(x)\ + p_{(\star,\mathbf{C},\star)}(x))p_{(0,\star,\mathbf{C})}(x-1)/p_{(\star,\star,\mathbf{C})}(x-1)}$$

and

$$l_{\text{next-}\mathbf{N}|\text{s-}\mathbf{C}}(x-1) = \frac{\sum V(\mathcal{Q}_{\mathbf{N}}, \mathbf{N}, \star)\left(M(x)\ \left(V^{\overline{\mathbf{E}}'} V^{\text{ps}} P(x)\ \right)\right)}{\left(p_{(\mathcal{Q}_{\mathbf{C}}, \mathbf{N}, \star)}(x)\ + p_{(\star, \mathbf{C}, \star)}(x)\right)\left(1 - p_{(0, \star, \mathbf{C})}(x-1)\ / p_{(\star, \star, \mathbf{C})}(x-1)\right)},$$

for $0 \leq x < n-1$.

Last-stage queues do not enter states in $(\star, \star, \mathbf{C})$. Rather than specify a separate queue model and transition probabilities for the last stage, transition probabilities $l_{\text{next-}\mathbf{N}|\star\text{-}\mathbf{C}}(n-1) = 1$ and $l_{\text{next-}\mathbf{C}|\star\text{-}\mathbf{N}}(n-1) = 0$ are used.

7 Verification

The analysis was tested by comparing its predictions to the output of a simulator. The analysis and simulator were used on a variety of network and traffic configurations, as were the analysis methods of Yoon *et al* [15] and Mun and Youn [11,16] for comparison. Throughput and queue-occupancy distributions are compared.

7.1 Methodology

A simulator was used to determine the performance of the network. The simulator precisely implements the basic network and traffic described in Section 2. Simulations were performed for 40 000 cycles; confidence intervals were computed for severe parameter sets (large queue and network sizes and heavy traffic). The confidence intervals were much smaller than the differences with the analytical models. For example, the 95% confidence interval for the throughput of $(8, 2, 50)$ networks at $\lambda = .90$ based on three runs was $[.7186, .7190]$. Simulator output included throughput and queue-occupancy distributions.

The analyses were used to determine the networks' throughput and queue occupancy distributions. The analysis methods of Yoon *et al* [15] and Mun and Youn [11,16] were adapted to use local flow control, to be comparable to the analysis presented here. (See [2,16] for a description of these variations.) Each was run for enough cycles to obtain results sufficiently close to stationary.

7.2 Comparisons

At low arrival rates all analyses predicted the simulated network throughput closely; those results are not shown here. At higher arrival rates most analyses overestimated throughput. This can be seen in Figure 3*(c)* where throughput v. arrival rate is plotted for $(8, 2, 4)$ basic networks for $.5 < \lambda < 1$. The effect of network size can be seen in Figure 3*(a)* where throughput is plotted for $(n, 2, 4)$ networks for $3 \leq n \leq 8$ with $\lambda = .9$. The analysis described here computes the throughput most closely matching simulated throughput. Mun and Youn's analysis also performs substantially better than that of Yoon *et al.* In Figure 3*(b)* throughput is plotted for $(8, 2, d)$ networks for $4 \leq d \leq 50$ with $\lambda = .9$. The analysis described here computes throughput closest to simulated values for small and large queue sizes. Mun and Youn's analysis performed better for moderate queue sizes. In all cases Yoon's analysis overestimates throughput.

A goal of the work and basis of the analysis reported here is to accurately model the queue occupancy distribution during congestion. The analysis comes much closer than any of the other analyses to doing so. In Figure 1 appear the occupancy distributions obtained from simulation and the three analyses for stage 1 of an $(8, 2, 30)$ network with $\lambda = .9$. The analysis was motivated by the observation that the state distribution of queues in simulated networks have peaks at 1 and $d-1$ packets. This can be seen in Figure 1*(a)* where the state distribution for the simulated network is plotted; the peaks are clearly visible. The state distribution obtained through the analysis described here is plotted in Figure 1*(d)*. There is a peak at $d-1$, however the effect at occupancy 1 is less pronounced. The state distribution obtained through the analysis of Yoon

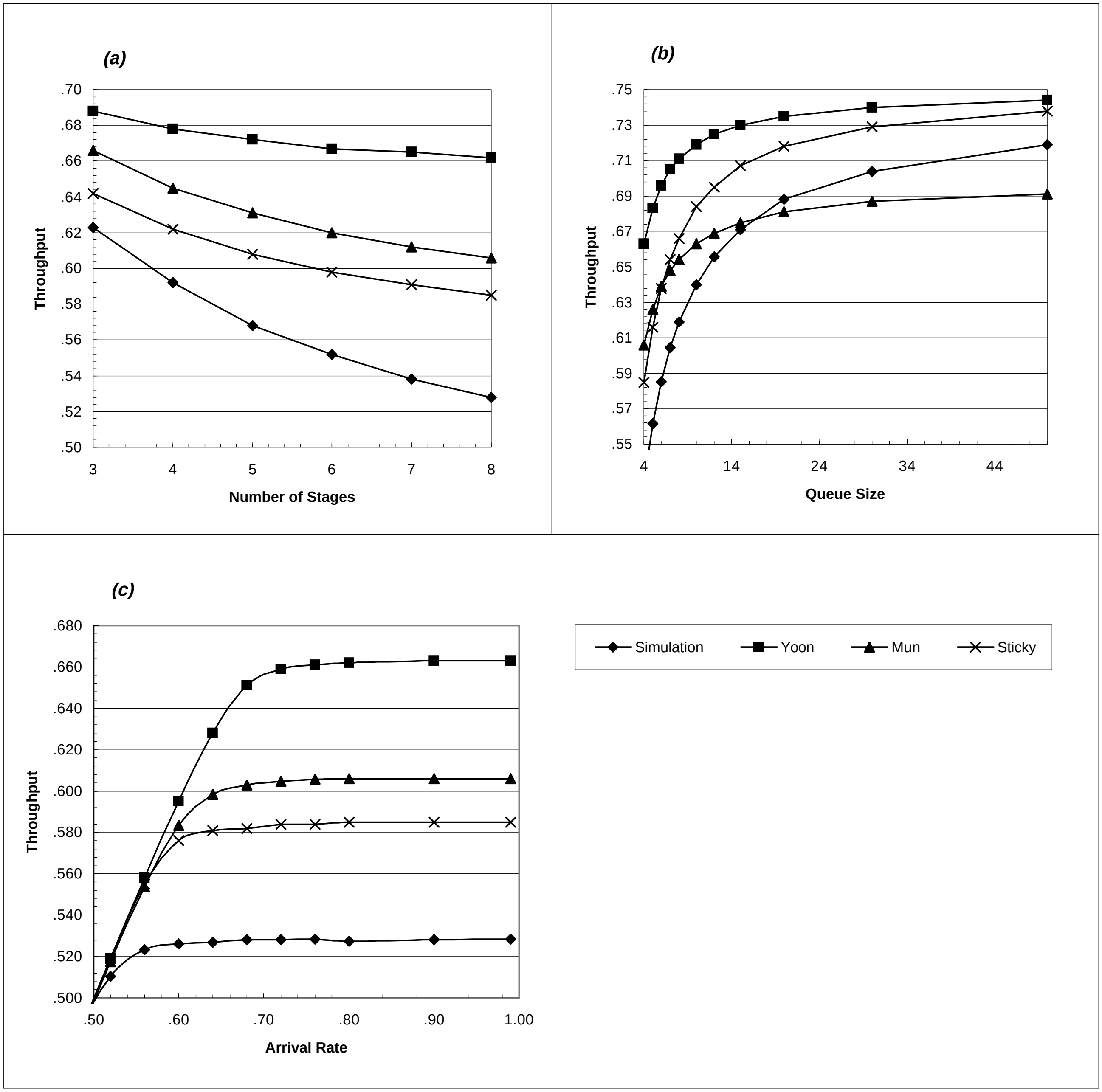

Figure 3. Throughput v. *(a)* number of stages for queue size 4; *(b)* queue size for 8-stage networks; *(c)* arrival rate for 8-stage networks using 4 slot queues.

et al is plotted in Figure 1*(b)*; here a peak at $d-1$ does not appear. This accounts for the larger throughputs obtained with this analysis.

8 Conclusions

A banyan network analysis method designed to accurately model congestion has been presented. A congested state for a queue is defined and incorporated in the queue's state model. The state model encodes the state of the queue as well as its neighbors. This allows the effect of congestion on the queue's neighbors to be modeled. State-dependent arrival and service rates are used. Comparisons to simulation and other analyses show the model can predict throughput more closely than other models. Further, it models queue occupancy distribution much more closely than other models.

Two other models based on congestion have been developed by the author, both models use fewer states. One model is similar to the one reported here, except that the state contains no information about the dual queue, resulting in about half the number of states. In fact, the predictions of that model are close to those described here. Since the model described here is a superset of the simpler model and since the model described here might lead to a more accurate model, the simpler model was not described despite its greater efficiency. The other model based on congestion is much simpler: the queue state contains no information about the dual or next-stage queues. An advantage of the model is that a queue's stationary distribution can be found in closed form [7]. (Iteration is still required to compute expected congestion duration and to solve the entire network.) This analysis does not provide as accurate predictions as the analysis described here, but it is computationally more efficient, especially for large queue sizes.

Although the predictions of the analysis described here are closer than those reported by others, there is still room for improvement. Examination of detailed simulator and analysis output reveals areas in which the simulation and analysis diverge. The traffic flow during congestion is higher in analysis than in simulation. This may be caused by insufficient correlation of congestion in adjacent stages. A revised model could increase correlation by having a next-stage-recently-congested state. A recently congested queue would likely soon be congested again. Another difference between analysis and simulation is the amount of time that a queue and its dual are simultaneously congested; it is greater in analysis. Further work then might be undertaken on a more accurate model of HOL/queue interaction during congestion.

9 References

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