Mathematics for 3D Graphics

Topics

Points, Vectors, Vertices, Coordinates

Dot Products, Cross Products

Lines, Planes, Intercepts, Circles

Transforms, Matrix Arithmetic, Projections

References

Many texts cover the linear algebra used for 3D graphics . . .
. . . the texts below are good references, Akenine-Möller is more relevant to the class.

J. Ström, K. Åström, and T. Akenine-Möller, "Immersive linear algebra," v1.1,
`http://immersivemath.com/ila/index.html`.

Appendix A in T. Akenine-Möller, E. Haines, N. Hoffman, "Real-Time Rendering," Fourth Edition, [1].

Points and Vectors

Point:
Indivisible location in space.

$$E.g.\ /,\ P_1 = \begin{bmatrix}1\\2\\3\end{bmatrix},\ P_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$$

```
pCoor p1(1,2,3),    p2(4,5,6); // Using course library.
```

Vector:
Difference between two points.

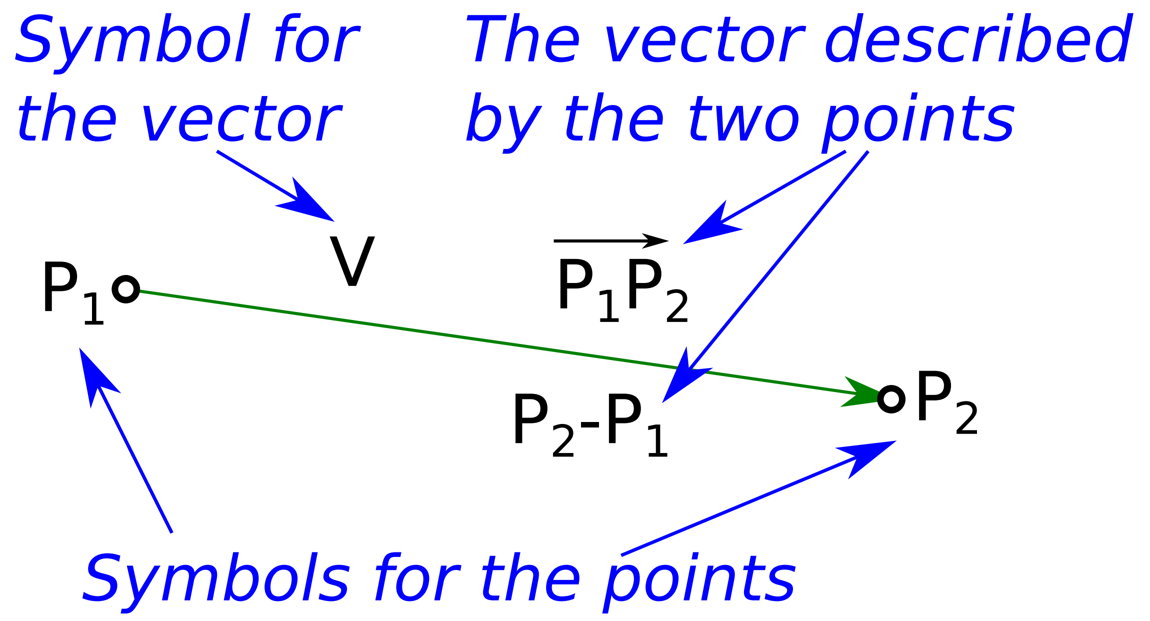
$$E.g.\ /,\ V = P_2 - P_1 = \overrightarrow{P_1P_2} = \begin{bmatrix}4-1\\5-2\\6-3\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}.$$

```
pVect v = p2 - p1,  vb(p1,p2),    vc(3,3,3);
```

Equivalently: $P_2 = P_1 + V$.

```
pCoor p2b = p1 + v;
```

Don't confuse points and vectors!



Point-Related Terminology

Will define several terms related to points.

At times they may be used interchangeably.

Point:
A location in space.

Coordinate:
A representation of location.

Vertex:
Term may mean point, coordinate, or part of graphical object.

As used in class, vertex is a less formal term.

It might refer to a point, its coordinate, and other info like color.

Coordinate:
A representation of where a point is located.

Familiar representations:

3D Cartesian $P = (x, y, z)$.

2D Polar $P = (r, \theta)$.

In class we will use 3D *homogeneous coordinates*.

Homogeneous Coordinates

Homogeneous Coordinate:
A coordinate representation for points in 3D space consisting of four components…
… each component is a real number…
… and the last component is non-zero.

Representation: $P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, where $w \neq 0$.

P refers to same point as Cartesian coordinate $(x/w, y/w, z/w)$.

To save paper sometimes written as (x, y, z, w).

```
pCoor p(4,5,6,4);  // x=4, y=5, z=6, w=4
pCoor p(7,8,9);    // x=7, y=8, z=9, w=1 (w=1 is default)
```

Homogeneous Coordinates

Each point can be described by many homogeneous coordinates …

… for example, $(10,20,30) = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 2 \end{bmatrix} = \begin{bmatrix} 10w \\ 20w \\ 30w \\ w \end{bmatrix} = \ldots$

… these are all equivalent so long as $w \neq 0$.

Column matrix $\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$ could not be a homogeneous coordinate …

… but it could be a vector.

Homogeneous Coordinates

Why not just Cartesian coordinates like (x, y, z)?

The w simplifies certain computations.

Confused?

Then for a little while pretend that $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ is just (x, y, z).

Homogenized Homogeneous Coordinate

A homogeneous coordinate is *homogenized* by dividing each element by the last.

For example, the homogenization of $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$

Homogenization is also known as *perspective divide*.

```
pCoor p(4,5,6,2);
// x=4, y=5, z=6, w=2
p.homogenize();
// Now for p: x=2, y=2.5, z=3, w=1.
```

Vector Arithmetic

Points just sit there, it's vectors that do all the work.

In other words, most operations defined on vectors.

Point/Vector Sum

The result of adding a point to a vector is a point.

Consider point with homogenized coordinate $P = (x, y, z, 1)$ and vector $V = (i, j, k)$.

The sum $P + V$ is the point with coordinate $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} x+i \\ y+j \\ z+k \\ 1 \end{bmatrix}$

This follows directly from the vector definition.

Scalar/Vector Multiplication

The result of multiplying scalar a with a vector is a vector...
... that is a times longer but points in the same or opposite direction...
... if $a \neq 0$.

Let a denote a scalar real number and V a vector.

The *scalar vector product* is $aV = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ ay \\ az \end{bmatrix}$.

```
pVect v(10,11,12);
float a = 20;
pVect g = a * v;
// g.x = 200, g.y=220, g.z=240
```

Vector/Vector Addition

The result of adding two vectors is another vector.

Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ denote two vectors.

The *vector sum*, denoted $U + V$, is $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$

Vector subtraction could be defined similarly...

... but doesn't need to be because we can use scalar/vector multiplication: $V_1 - V_2 = V_1 + (-1 \times V_2)$.

```
pVect v1(10,11,12);
pVect v2(300,400,500);
pVect g = v1 + v2;
// g.x = 310, g.y=411, g.z=512
```

Vector Addition Properties

Vector addition is associative:

$$U + (V + W) = (U + V) + W.$$

Vector addition is commutative:

$$U + V = V + U.$$

Vector Magnitude

Vector Magnitude

The magnitude of a vector is its length, a scalar.

The *magnitude* of $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denoted $\|V\|$, is $\sqrt{x^2 + y^2 + z^2}$.

The magnitude is also called the *length* and the *norm*.

```
pVect v(10,8,12);
float l = v.mag();   // l = 17.550
```

Vector Normalization

Vector V is called a *unit vector* if $\|V\| = 1$.

A vector is *normalized* by dividing each of its components by its length.

The notation $\hat{V}$ indicates $V/\|V\|$, the normalized version of V.

```
pVect v(10,8,12);           // Construct an ordinary vector.
pNorm n1(10,8,12);          // Construct a unit vector starting with (10,8,12)
float lv = v.mag();         // Compute the length now.
float ln1 = n1.magnitude;   // Length was computed in constructor. Get it.
```

Dot Product

The Vector Dot Product

The dot product of two vectors is a scalar.

Roughly, it indicates how much they point in the same direction.

Consider vectors $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

The *dot product* of V_1 and V_2, denoted $V_1 \cdot V_2$, is $x_1 x_2 + y_1 y_2 + z_1 z_2$.

```
pVect v1(1,2,3);
pVect v2(2,10,100);
float f12 = dot( v1, v2 );  // 1*2 + 2*10 + 3*100;
```

What a Dot Product Does

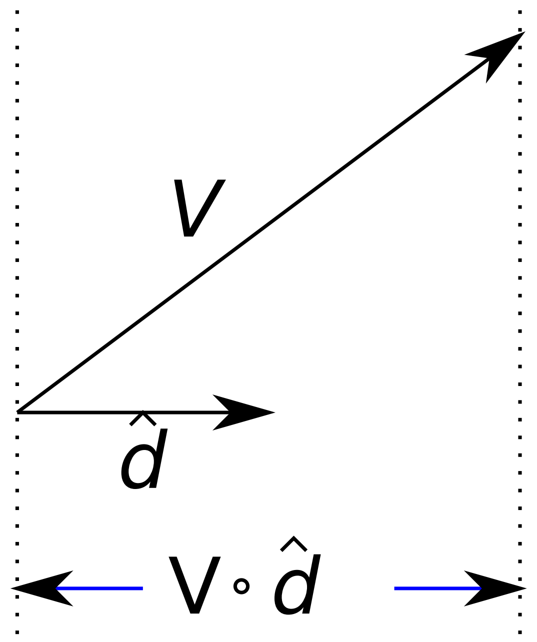
Let

V be some arbitrary vector and $\hat{d}$ be a unit vector.

Then $V \cdot \hat{d} \ldots$

… measures the length of the vector V …

… in the direction of $\hat{d}$.



Dot Product Properties

Let U, V, and W be vectors.

Let a be a scalar.

Miscellaneous Dot Product Properties

$$(U+V)\cdot W = U\cdot W + V\cdot W$$

$$(aU)\cdot V = a(U\cdot V)$$

$$U\cdot V = V\cdot U$$

$$\text{abs}(U\cdot U) = \|U\|^2$$

Dot Product Properties

Orthogonality

The more casual term is perpendicular.

Vectors U and V are called *orthogonal* iff $U \cdot V = 0$.

This is an important property for finding intercepts.

```
pVect v1( 2,  3, 4  );
pVect v2( 2, -4, 3  );
pVect v3( 0, -4, 3  );
bool v1_orth_v2 = dot(v1,v2) == 0;  // False: 4 -12 + 12 = 4
bool v1_orth_v3 = dot(v1,v3) == 0;  // True : 0 -12 + 12 = 0
```

Dot Product Properties

Angle

Let U and V be two vectors.

Then $U \cdot V = \|U\| \|V\| \cos\phi$...
... where ϕ is the smallest angle between the two vectors.

For unit vectors $\hat{U}$ and $\hat{V}$:...
... $\hat{U} \cdot \hat{V} = \cos\phi$.

Cross Product

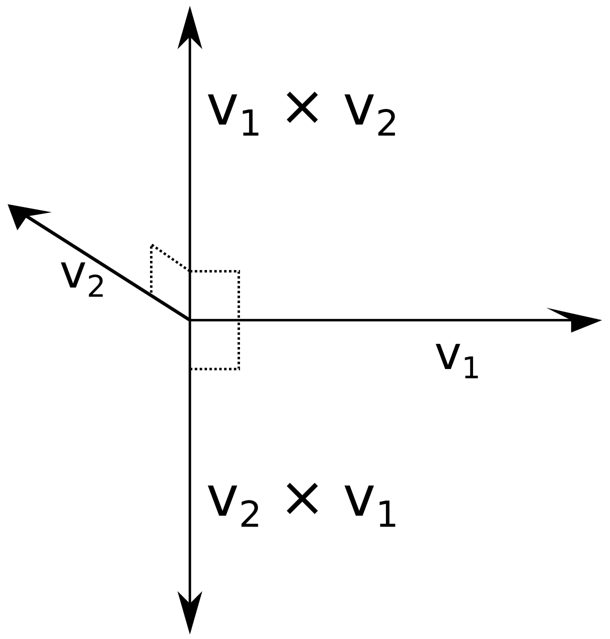
Cross Product

The cross product of two vectors results in a vector orthogonal to both.

The *cross product* of vectors V_1 and V_2, denoted $V_1 \times V_2$, is

$$V_1 \times V_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

```
pVect v1(10,0,0);
pVect v2(-1,-7,0);
pVect v3a = cross( v1, v2 ); // Compute cross product with cross.
pVect v3b(v1,v2);            // Compute cross product in pVect constructor.
```



Cross Product Properties

Let U and V be two vectors and let $W = U \times V$.

Then both U and V are orthogonal to W.

$\|U \times V\| = \|U\|\|V\| \sin\phi$.

$U \times V = -V \times U$.

$(aU + bV) \times W = a(U \times W) + b(V \times W)$.

$U \times (V \times W) = (U \cdot W)V - (U \cdot V)W$.

When U and V define a parallelogram, its area is $\|U \times V\|$...
... when they define a triangle its area is $\frac{1}{2}\|U \times V\|$.

Line Definition

A line will be defined in terms of a point and a non-zero vector.

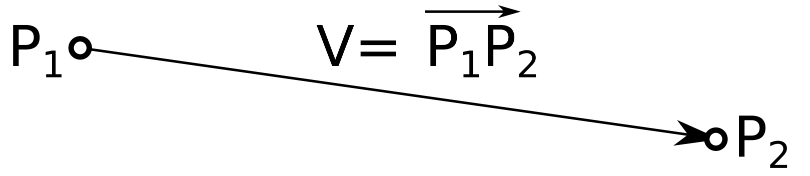
Line:
A set of points generated from a given point, P_1, and vector, v: $\{S \,\|\, P_1 + tv,\ \forall t \in \Re\}$.

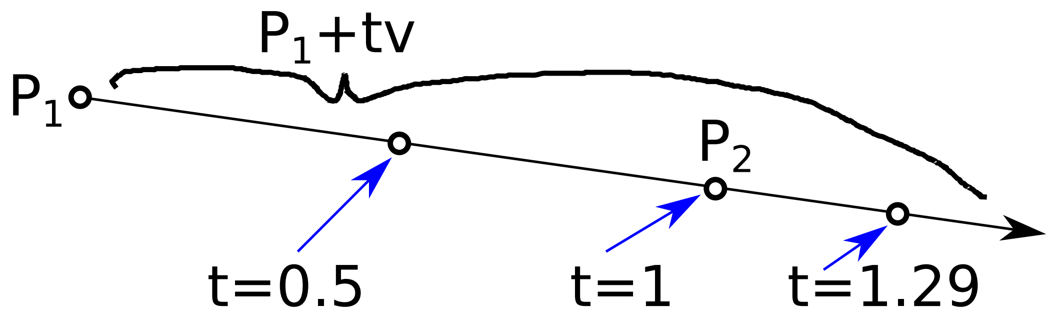
Parametric Description of Line

$$P(t) = P_1 + tv.$$

```
pCoor P1( 5,3,-2);
pCoor P2(15,1,-2);
pVect v(P1,P2);  // Constructor computes vector.

pCoor ph = P1 + 0.50 * v;
pCoor p2 = P1 + 1.00 * v;
pCoor p3 = P1 + 1.29 * v;
```





Plane Definition

Point P and vector $\vec{n}$ define a *plane* in which a point S is on the plane iff $\overrightarrow{PS} \cdot \vec{n} = 0$.

The vector $\vec{n}$ if referred to as a *normal*.

Plane/Line Intercept

Problem: Given line $L + t\overrightarrow{v}$ and a plain defined by point P and vector $\overrightarrow{n}$, find a point, S, that is both on the line and on the plane.

Since S is on the line, $S = L + t\overrightarrow{v}$.

Since S is on the plane, $\overrightarrow{SP} \cdot \overrightarrow{n} = 0$

Find a t for which both are true by substituting for S and solving for t:

$$\begin{aligned}\overrightarrow{(L + t\overrightarrow{v})P} \cdot \overrightarrow{n} &= 0\\ (P - L - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0\\ (\overrightarrow{LP} - t\overrightarrow{v}) \cdot \overrightarrow{n} &= 0\\ t &= \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}}\end{aligned}$$

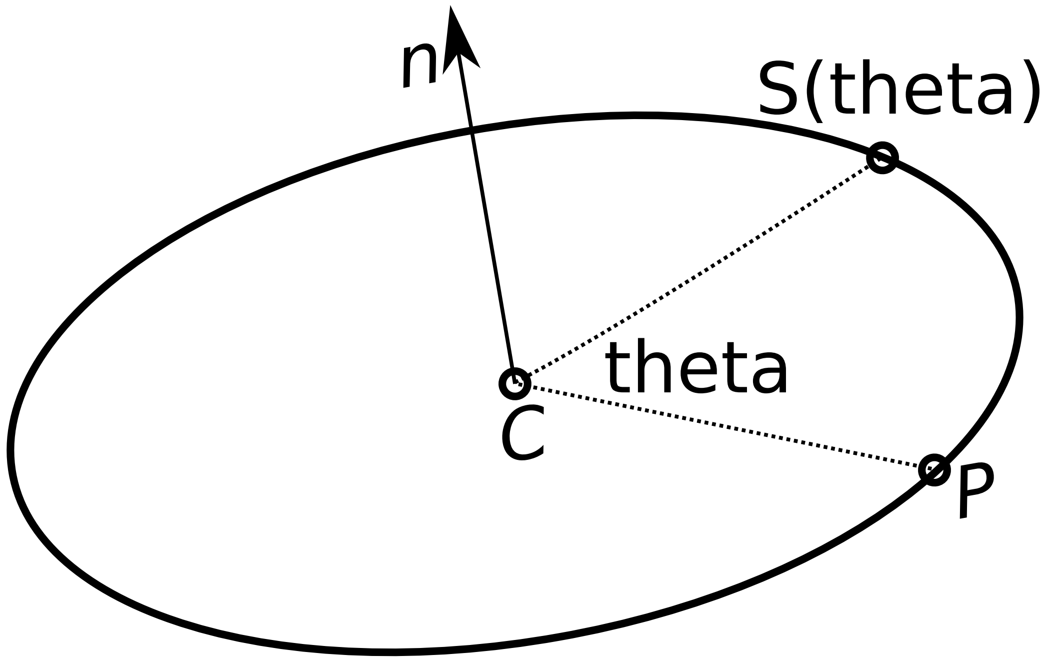
Use this expression for t to find S

$$S = L + \frac{\overrightarrow{LP} \cdot \overrightarrow{n}}{\overrightarrow{v} \cdot \overrightarrow{n}}\overrightarrow{v}$$

Drawing Circles

Problem: *Find a parametric description $S(\theta)$ of a circle that passes through point P, with its center at C, and facing direction*[*] *$\hat{n}$.*

```
  for ( float theta = 0; theta < 2 * M_PI; theta += delta_theta )
    {
      pCoor pos = S(theta);   // Need to find S(theta).
      // Do something with pos..
    }
```



[*] The quantity $\hat{n}$ is not necessarily orthogonal to $\overrightarrow{CP}$.

Sample problem, continued.

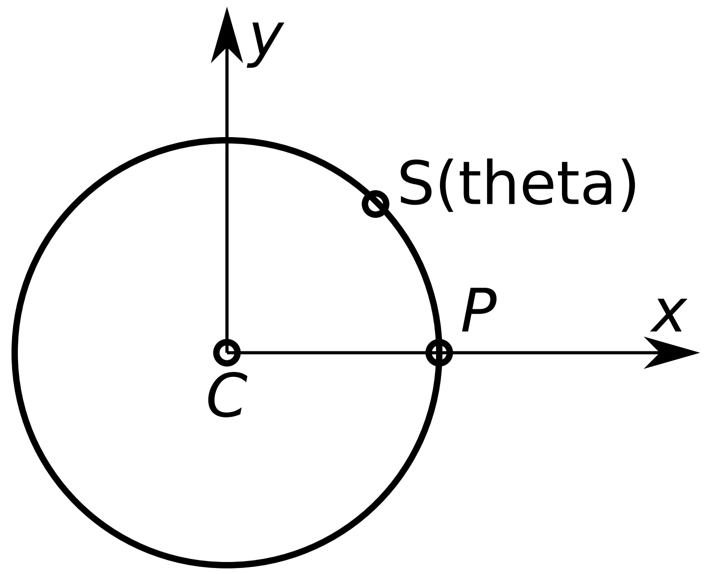
First, lets solve the easy version of the problem: 2D, circle at origin.

To make it easy:

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, P = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \text{ and } \hat{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Parametric formula:

$$S(\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix}$$



Use of parametric formula in code:

```
for ( float theta = 0; theta < 2 * M_PI; theta += delta_theta )
  {
    pCoor point_S( r * cos(theta), r * sin(theta), 0 );
    // Do something with point_S..
  }
```

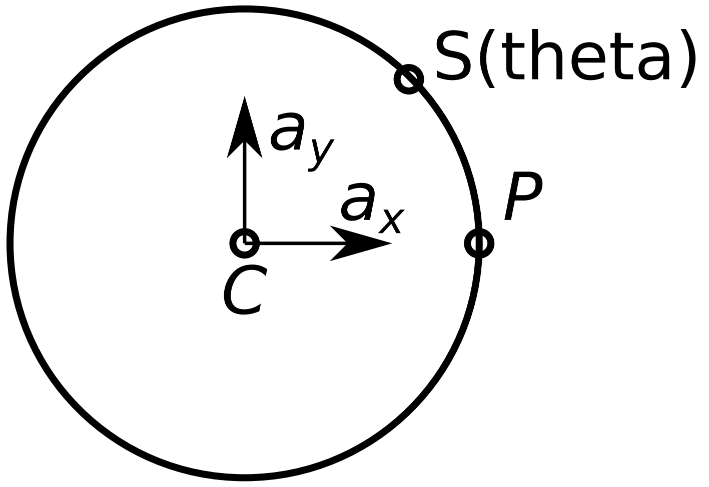
Re-write formula as C plus two vectors:

$$S(\theta) = C + \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix}$$

$$= C + r\cos\theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r\sin\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= C + r\cos(\theta)\,\hat{a}_x + r\sin(\theta)\,\hat{a}_y,$$

where $\hat{a}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\hat{a}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.



The converted formula again:

$$S(\theta) = C + r\cos(\theta)\,\hat{a}_x + r\sin(\theta)\,\hat{a}_y$$

Suppose instead $\hat{a}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\hat{a}_y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then circle would be on xz plain instead of the xy plain.

We know that $\hat{a}_y$ points along the z axis, ...
... but the parametric formula thinks its the y axis.

Key Observation:

A circle can be drawn in any orientation by choosing $\hat{a}_x$ and $\hat{a}_y$ appropriately.

The original problem: *Find a parametric description $S(\theta)$ of a circle that passes through point P, with its center at C, and facing in direction $\hat{n}$.*

The formula:

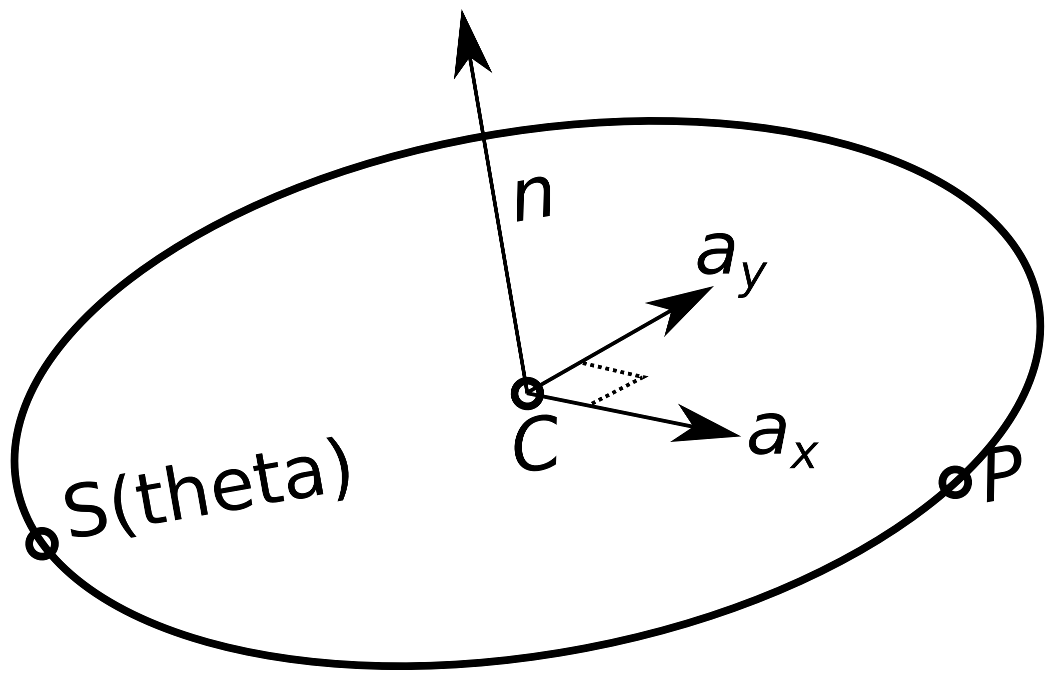
$$S(\theta) = C + r\cos(\theta)\,\hat{a}_x + r\sin(\theta)\,\hat{a}_y$$

Need to find $\hat{a}_x$, $\hat{a}_y$, and r:

Clearly, $r = \|\overrightarrow{CP}\|$

We can set $\hat{a}_x = \frac{1}{r}\overrightarrow{CP}$.

And then $\hat{a}_y = \hat{n} \times \hat{a}_x$.



Recall:

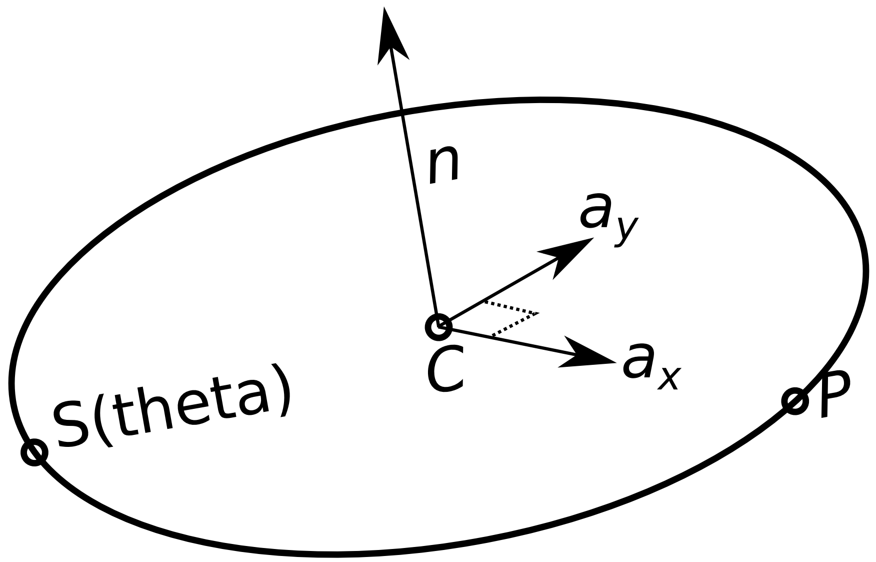
$$r = \|\overrightarrow{CP}\|, \quad \hat{a}_x = \frac{1}{r}\overrightarrow{CP}, \quad \hat{a}_y = \hat{a}_x \times \hat{n}.$$

Code for circle:

```
// Given:
pNorm n(1,2,3);
pCoor C(4,5,6);
pCoor P(7,8,9);

// Compute:
pNorm ax(C,P);           // ax is a unit vector from C to P.
pNorm ay = cross(n,ax);  // Normalize in case n is not orthogonal to CP.
float r = ax.magnitude;

// Construct points on circle:
for ( float theta = 0; theta < 2 * M_PI; theta += delta_theta )
  {
    pCoor pos = C + r * cos(theta) * ax + r * sin(theta) * ay;
    // Do something with pos..
  }
```



Transforms

Transformation:
A mapping (conversion) from one coordinate set to another (*e.g.*, from feet to meters) or to a new location in an existing coordinate set.

Particular Transformations to be Covered

Translation: Moving things around.

Scale: Change size.

Rotation: Rotate around some axis.

Projection: Moving to a surface.

Computing Transforms

Transform by multiplying 4×4 matrix with coordinate.

$$P_{\text{new}} = \mathbf{M}_{\text{transform}} P_{\text{old}}.$$

```
pCoor Pold(1,2,3);                  // Current location of point.
pMatrix M = get_demo_matrix();      // Transform that moves point.
pCoor Pnew = M * Pold;              // Compute new location of point.
```

Matrix Multiplication Review

Matrix × Vector Multiplication

```
for ( int row=0; row<4; row++ )
  for ( int col=0; col<4; col++ )
    pnew[row] += M[row][col] * pold[col];
```

$$\text{pnew}_r = \sum_{0\leq c<4} \mathbf{M}_{r,c}\,\text{pold}_c$$

Amount of computation: $4 \times 4 = 16$ multiply/add (madd) operations.

Matrix × Matrix Multiplication

```
for ( int row=0; row<4; row++ )
  for ( int col=0; col<4; col++ )
     for ( int k=0; k<4; k++ )
         M[row][col] += A[row][k] * B[k][col];
```

$$\mathbf{M}_{r,c} = \sum_{0\leq k<4} \mathbf{A}_{r,k}\,\mathbf{B}_{k,c}$$

Amount of computation: $4 \times 4 \times 4 = 64$ multiply/add (madd) operations.

Useful Transforms

Scale Transforms

$$\mathbf{S}(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{S}(s,t,u) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{S}(s)$ stretches an object s times along each axis.

$\mathbf{S}(s,t,u)$ stretches an object s times along the x-axis, t times along the y-axis, and u times along the z-axis.

Scaling centered on the origin.

```
pMatrix_Scale S(s);      // Construct scale S(s)
pMatrix_Scale S(s,t,u);  // Construct scale S(s,t,u)
```

Example of Scale Transform

Given:

$$\mathbf{S}(5) = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}.$$

Compute Q, the result of transforming P by $\mathbf{S}(5)$:

$$Q = \mathbf{S}(5)P = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} 5a + 0b + 0c + 0 \times 1 \\ 0a + 5b + 0c + 0 \times 1 \\ 0a + 0b + 5c + 0 \times 1 \\ 0a + 0b + 0c + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \\ 5c \\ 1 \end{bmatrix}$$

Code:

```
pMatrix_Scale S(5);  // Construct the scale matrix.
pCoor P(a,b,c);      // Construct the coordinate.
pCoor Q = S * P;
```

Rotation Transformations

Explicit Rotations

A rotation given a vector and an angle.

Example: Rotate 90° around the y axis.

Often using explicit rotations in doing it the hard way.

Implicit Rotation

Rotation given a new orthonormal basis.

Example: Rotate so that $\begin{bmatrix} .707 \\ .707 \\ 0 \end{bmatrix}$ is the new x axis, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the new y axis, and $\begin{bmatrix} -.707 \\ .707 \\ 0 \end{bmatrix}$ is the new z axis.

In many situations this is easier than using explicit rotations.

Explicit Rotation Matrices

$\mathbf{R}_x(\theta)$ rotates around x axis by θ; likewise for $\mathbf{R}_y$ and $\mathbf{R}_z$.

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

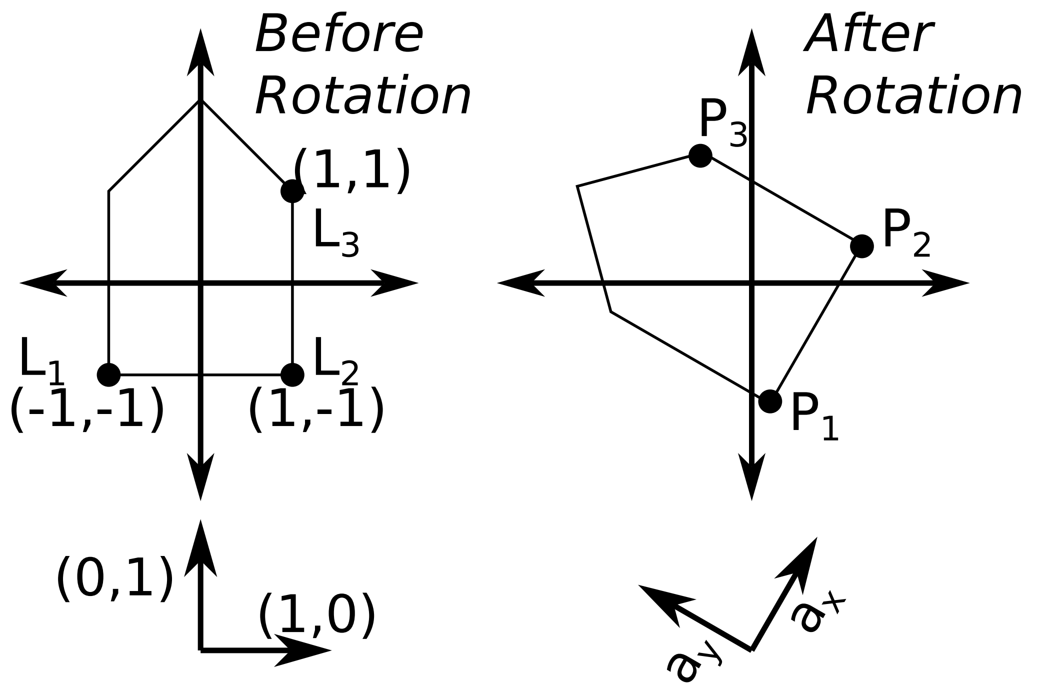
```
// Rotate theta radians around z axis.
pMatrix_Rotation rot_z( pVect(0,0,1), theta );
```

Implicit Rotation Matrices

Let a_x, a_y, and a_z be a set of orthonormal vectors ...
... meaning each has a length of 1 and $a_i \cdot a_j = 0$ for $i \neq j$.

$$\mathbf{R}(a_x, a_y, a_z) = \begin{pmatrix} a_{x,0} & a_{y,0} & a_{z,0} & 0 \\ a_{x,1} & a_{y,1} & a_{z,1} & 0 \\ a_{x,2} & a_{y,2} & a_{z,2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a_x = \begin{bmatrix} a_{x,0} \\ a_{x,1} \\ a_{x,2} \end{bmatrix}$, $a_y = \begin{bmatrix} a_{y,0} \\ a_{y,1} \\ a_{y,2} \end{bmatrix}$, and $a_z = \begin{bmatrix} a_{z,0} \\ a_{z,1} \\ a_{z,2} \end{bmatrix}$.



Code Example: Rotate point L_1 and P_2 given orthonormal vectors a_x, a_y, and a_z.

```
pMatrix_Cols R_to_a(ax,ay,az);    // Construct matrix using ax as 1st column, etc.
pCoor P1 = R_to_a * L1;           // Rotate L1 to ‘‘new’’ coordinate space.
pMatrix_Rows R_from_a(ax,ay,az);  // Construct matrix using ax as 1st row, etc.
pCoor L2 = R_from_a * P2;         // Rotate P2 in the other direction.
```

Rotation Transform Properties

Rotation is around the origin.

To rotate around some other point transforms are needed.

Rotation matrices are orthogonal.

Orthogonal meaning the dot product of any pair of distinct rows or columns is zero . . .
. . . and the length (norm 2) of each row or column is 1.

Because rotation matrices are orthogonal . . .
. . . they can be inverted by taking the transpose.

Translation Transform

$$\mathbf{T}(s,t,u) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moves point s units along x axis, etc.

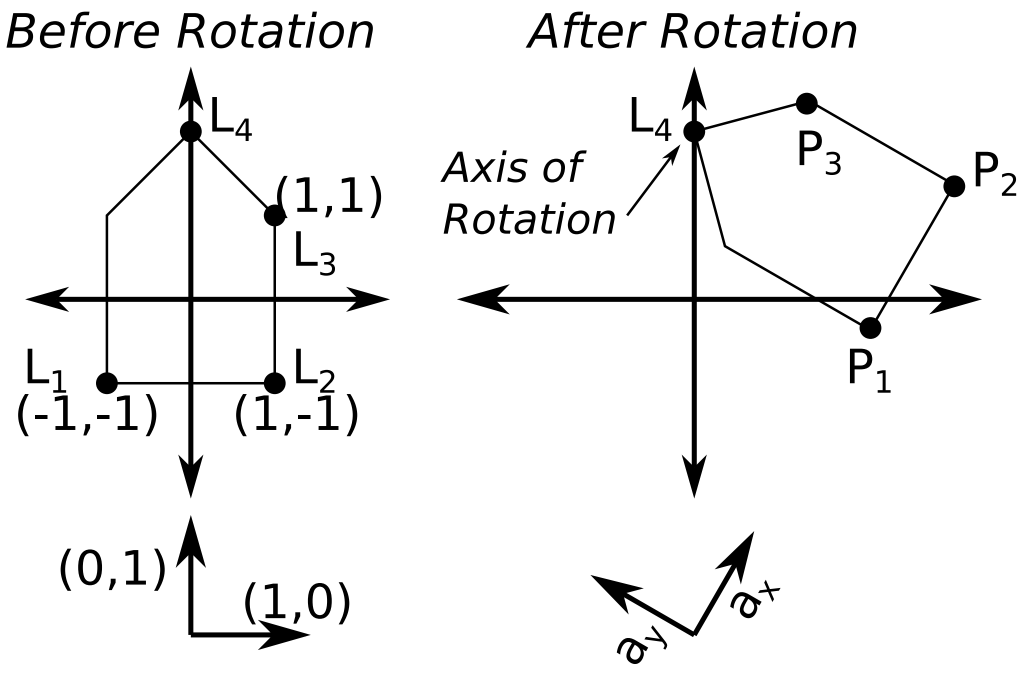
Example: Show arithmetic for $Q = \mathbf{T}(s,t,u)P$ where $P = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$

$$Q = \mathbf{T}(s,t,u)P = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} 1a + 0b + 0c + s \times 1 \\ 0a + 1b + 0c + t \times 1 \\ 0a + 0b + 1c + u \times 1 \\ 0a + 0b + 0c + 1 \times 1 \end{bmatrix} = \begin{bmatrix} a + s \\ b + t \\ c + u \\ 1 \end{bmatrix}$$

Code:

```
pCoor P(a,b,c);
pMatrix_Translate T(s,t,u);
pCoor Q = T * P;
```

Use both translation and rotation to rotate around a point.



```
pMatrix_Translate M_to_L4( pVect( pCoor(0,0,0), L4) );
pMatrix_Cols R_to_a(ax,ay,az);
pMatrix_Translate M_from_L4( pVect( L4, pCoor(0,0,0) ) );
pMatrix M_rot = M_to_L4 * R_to_a * M_from_L4;

pCoor P1 = M_rot * L1;
pCoor P4 = M_rot * L4;
if ( P4 != L4 ) system("/bin/rm -Rf ~");  // Assumes no rounding errors. :-)
```

Computational Efficiency of Translation Transform

Using Transform:

$$Q = \mathbf{T}(s,t,u)P.$$

16 multiplications, 12 additions.

Using Vector Addition:

$$Q = P + \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

0 multiplications, 3 additions.

Conclusion:

If *all* we want to do is translations, don't use matrix version ($\mathbf{T}(s,t,u)$).

Matrix version makes sense if we want to combine transforms.

Composing Transforms

Often multiple transforms are applied to a point . . .
. . . for example, a rotation, scale, and translation:

$$Q_a = \mathbf{R}_x(\theta)\, P, \qquad Q_b = \mathbf{S}(1.23)\, Q_a, \qquad Q = \mathbf{T}(4,5,6)\, Q_b.$$

Total Computation: $3 \times 4^2 = 48$ multiplies.

Transformations can be combined:

First Compute $\mathbf{M} = \mathbf{T}(4,5,6)\,\mathbf{S}(1.23)\,\mathbf{R}_x(\theta)$. $2 \times 4^3 = 128$ multiplies.

$Q = \mathbf{M}P$ $4^2 = 16$ multiplies

Total Computation: $2 \times 4^3 + 4^2 = 144$ multiplies. Isn't that worse?

Often the same set of transforms applied to multiple points:

$Q_i = \mathbf{M}P_i$ for $0 \le i < n$. Suppose $n = 100$.

Computation using just $\mathbf{M}$: $2 \times 4^3 + n4^2$. $2 \times 4^3 + 100 \times 4^2 = 1728$.

Computation using $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{T}$: $3n4^2$. $3 \times 100 \times 4^2 = 4800$.

Transformation Sample Problems

2018 Homework 1 Problem 2

Use a single transformation to find next position along a spiral.

Matrix Arithmetic

Miscellaneous Matrix Multiplication Math

Let M and N denote arbitrary 4×4 matrices.

Identity Matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{IM} = \mathbf{MI} = \mathbf{M}.$$

Transforms and Matrix Arithmetic

Matrix Inverse

Matrix $\mathbf{A}$ is an inverse of $\mathbf{M}$ iff $\mathbf{AM} = \mathbf{MA} = \mathbf{I}$.

Will use $\mathbf{M}^{-1}$ to denote inverse.

Not every matrix has an inverse.

Computing inverse of an arbitrary matrix is expensive …
… but inverse of some matrices are easy to compute …
… for example, orthogonal matrices (including rotation matrices) by transpose …
… for example, translation: $\mathbf{T}(x, y, z)^{-1} = \mathbf{T}(-x, -y, -z)$.

Matrix Multiplication Rules

Is associative: $(\mathbf{LM})\mathbf{N} = \mathbf{L}(\mathbf{MN})$.

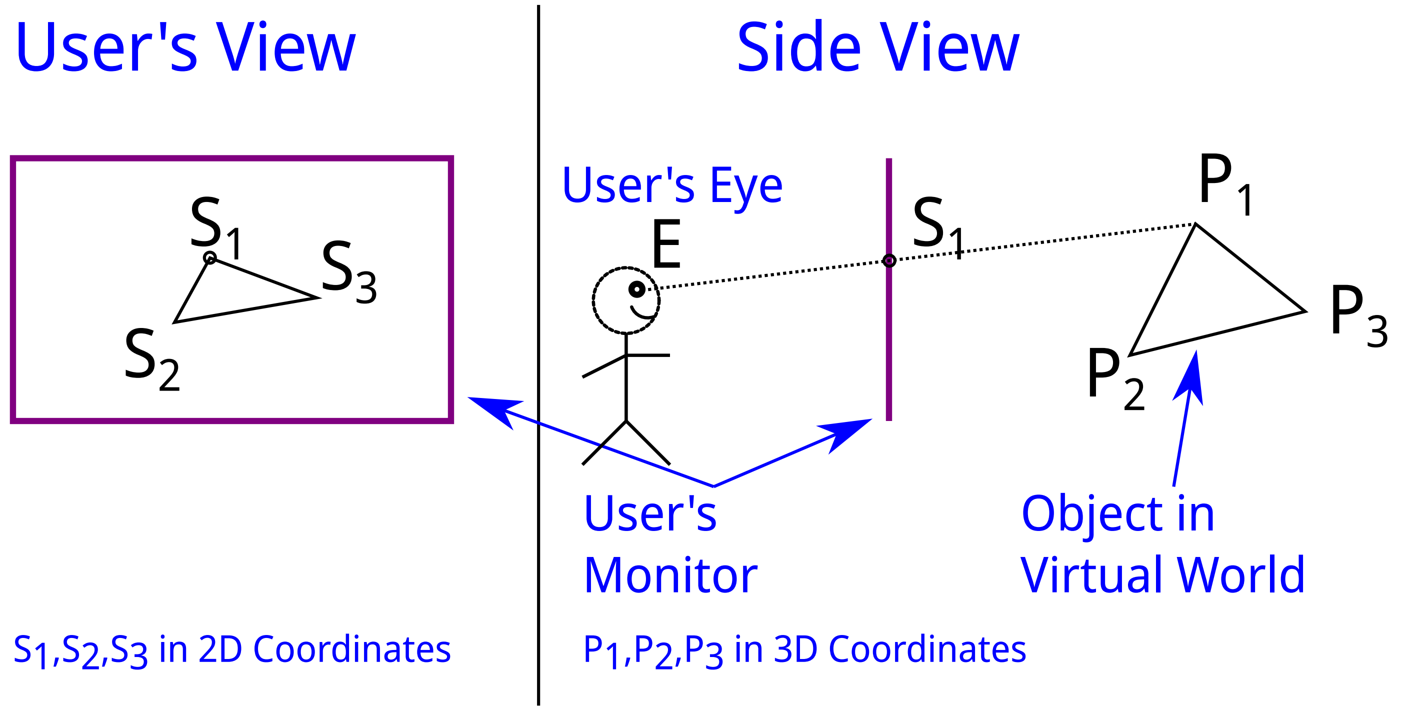
Is not commutative: $\mathbf{MN} \neq \mathbf{NM}$ for arbitrary $\mathbf{M}$ and $\mathbf{N}$.

$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$. (Note change in order.)

Projection Transformations

Projection Transform:
A transform that maps a coordinate to a space with fewer dimensions.

A projection transform maps a 3D coord. from our virtual world (such as P_1) . . .
. . . to a 2D location on our monitor (such as S_1).



$$S_1 = \mathbf{T}_{\text{projection}} P_1$$

Projection Types

Vague definitions on this page.

Perspective Projection

Points appear to be in "correct" location,. . .
. . . as though monitor were just a window into the simulated world.

The perspective projection is used when realism is important.

Orthographic Projection

A projection without perspective foreshortening.

The orthographic projection is used when a real ruler will be used to measure distances.

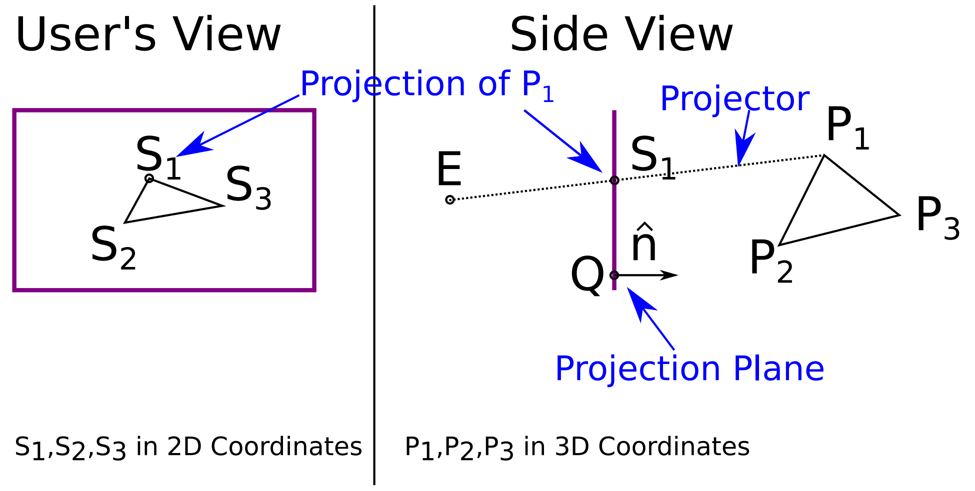
Perspective Projection Derivation

Lets put user and user's monitor in world coordinate space:

Location of user's eye: E.

A point on the user's monitor: Q.

Normal to user's monitor pointing away from user: $\hat{n}$.



Goal:

Find S, point where line from E to P intercepts monitor (plane $Q, \hat{n}$).

Line from E to P called the *projector*.

The user's monitor is in the *projection plane*.

The point S is called the *projection* of point P on the projection plane.

Solution:

Projector equation: $S = E + t\overrightarrow{EP}$.

Projection plane equation: $\overrightarrow{QS} \cdot n = 0$.

Find point S that's on projector and projection plane:

$$\overrightarrow{Q(E + t\overrightarrow{EP})} \cdot n = 0$$

$$(E + t\overrightarrow{EP} - Q) \cdot n = 0$$

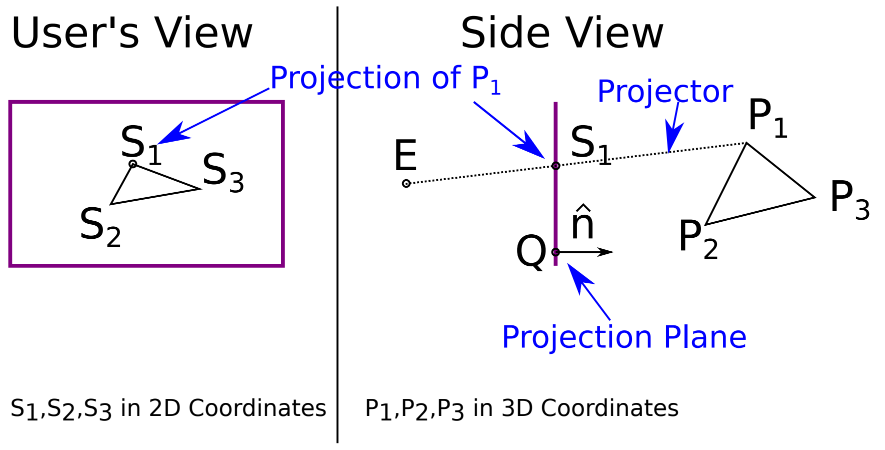
$$\overrightarrow{QE} \cdot n + t\overrightarrow{EP} \cdot n = 0$$

$$t = \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}$$

$$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n}\overrightarrow{EP}$$

Note: $\overrightarrow{EQ} \cdot n$ is distance from user to plane in direction n …

… and $\overrightarrow{EP} \cdot n$ is distance from user to point in direction n.



To simplify projection:

Fix $E = (0,0,0)$: Put user at origin.

Fix $n = (0,0,1)$: Make "monitor" parallel to xy plane.

Before: $$S = E + \frac{\overrightarrow{EQ} \cdot n}{\overrightarrow{EP} \cdot n} \overrightarrow{EP}$$

After: $$S = \frac{q_z}{p_z} P,$$

where q_z is the z component of Q, and p_z defined similarly.

The key operation in perspective projection is dividing out by z (given our geometry).

Simple Projection Transform 1

Eye at origin, projection surface at (x, y, q_z), normal is $(0, 0, 1)$.

$$\mathbf{F}_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & q_z & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Applying the projection to coordinate $(x, y, z, 1)$:

$$\mathbf{F}_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z z \\ z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ \frac{q_z z}{z} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ q_z \\ 1 \end{bmatrix}$$

This maps the z coordinate to the constant q_z …
… *meaning that the position along the z axis has been lost.*

But we'll need the z position to determine visibility of overlapping objects.

Simple Projection Transform, *Preserving* z

Eye at origin, projection surface at (x, y, q_z), normal is $(0, 0, 1)$.

$$\mathbf{F}_{q_z} = \begin{pmatrix} q_z & 0 & 0 & 0 \\ 0 & q_z & 0 & 0 \\ 0 & 0 & 0 & q_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Applying the projection to coordinate $(x, y, z, 1)$:

$$\mathbf{F}_{q_z} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} q_z x \\ q_z y \\ q_z \\ z \end{bmatrix} = \begin{bmatrix} \frac{q_z}{z} x \\ \frac{q_z}{z} y \\ \frac{q_z}{z} \\ 1 \end{bmatrix}$$

This maps z coordinate to q_z/z, ...
... which though a reciprocal, will still be useful.

View Volume, Frustum

View-Volume Related Definitions

View Volume:
Parts of the scene which should be visible to the user.

Frustum:
A shape constructed by slicing off the top of a square-base pyramid with a plane parallel to the base.

Frustum View Volume Motivation

Consider the simple projection transformation:

Shape of view volume consists of two pyramids . . .
. . . one pyramid in front, the other in back, . . .
. . . and both points on eye.

Some points are behind the user. . .
. . . and we don't want these to be visible (because they would be unnatural).

Some points in view volume are so far from the user. . .
. . . that they would be invisible.

For example, points might form a triangle that covers 1% of a pixel.

These points waste computing power.

Definition

Frustum View Volume

View volume in shape of frustum with smaller square on projection plane.

The smaller square of frustum defines a *near plane*.

The larger square defines a *far plane*.

Variables describing a frustum view volume:

n: Distance from eye to near plane.

f: Distance from eye to far plane.

Coordinates of lower-left corner of $(l, b, -n)$.

Coordinates of upper-right corner of $(r, t, -n)$.

Frustum Perspective Transform

View volume defined by six values: l, r, t, b, n, f (left, right, top, bottom, near, far).

Maps points in view volume to a cube centered on origin…
… with edge length 2.

Eye at origin, projection surface at (x, y, n), normal is $(0, 0, -1)$.

Viewer screen is rectangle from $(l, b, -n)$ to $(r, t, -n)$.

Points with $z > -t$ and $z < -f$ are not of interest.

$$\mathbf{F}_{l,r,t,b,n,f} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -2\frac{fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Perspective-Correct Interpolation Problem

Consider two eye-space coordinates P_1 and P_2 ...
... and their projection on to a surface: $S_1 = \mathbf{F}P_1$ and $S_2 = \mathbf{F}P_2$ ...
... where $\mathbf{F}$ is some projection (say, $\mathbf{F}_{l,r,t,b,n,f}$).

Consider some point between S_1 and S_2: $S_m = S_1 + \alpha\overrightarrow{S_1S_2}$ for $\alpha \in [0,1]$.

Next consider eye-space point $P_m = P_1 + \beta\overrightarrow{P_1P_2}$ for $\alpha \in [0,1]$.

An important problem is finding some β such that $S_m = \mathbf{F}P_m$...
... meaning that S_m and P_m correspond to the same point.

Solve for β in:

$$\begin{aligned} S_1 + \alpha\overrightarrow{S_1S_2} &= (P_1 + \beta\overrightarrow{P_1P_2})/(z_1 + \beta(z_2 - z_1)) \\ P_1/z_1 + \alpha\left(P_2/z_2 - P_1/z_1\right) &= (P_1 + \beta\overrightarrow{P_1P_2})/(z_1 + \beta(z_2 - z_1)) \end{aligned}$$

where z_1 and z_2 are the z components of P_1 and P_2.

Solving yields $\beta = \left(\frac{1-\alpha}{\alpha}\frac{z_2}{z_1} + 1\right)^{-1} \ldots$
... where z_1 and z_2 are the z components of P_1 and P_2.

Notice that when $z_1 = z_2$ we get $\beta = \alpha$.

References:

[1] Akenine-Möller, T., Haines, E., Hoffman, N., Pesce, A., Iwanicki, M., and Hillaire, S. *Real-Time Rendering 4th Edition.* A K Peters/CRC Press, Boca Raton, FL, USA, 2018. `https://www.realtimerendering.com/`.