Solutions Manual
ESSENTIALS OF ROBUST CONTROL

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January 9, 1998

Preface

This solutions manual contains two parts. The first part contains some intuitive derivations of H_2 and H_∞ control. The derivations given here are not strictly rigorous but I feel that they are helpful (at least to me) in understanding H_2 and H_∞ control theory. The second part contains the solutions to problems in the book. Most problems are solved in detail but not for all problems. It should also be noted that many problems do not have unique solutions so the manual should only be used as a reference. It is also possible that there are errors in the manual. I would appreciate very much of your comments and suggestions.

Kemin Zhou

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Part I

Understanding $H_2/LQG/H_\infty$ Control

Chapter 1

Understanding $\mathbf{H_2}$/LQG Control

We present a natural and intuitive approach to the H_2 /LQG control theory from the state feedback and state estimation point of view.

1.1 H_2 and LQG Problems

Consider the following dynamical system

$$\dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = 0 \tag{1.1}$$

$$z = C_1 x + D_{12} u \tag{1.2}$$

$$y = C_2 x + D_{21} w \tag{1.3}$$

We shall make the following assumptions:

(i) (A, B_2) is stabilizable and (C_2, A) is detectable;

(ii) $R_1 = D_{12}^* D_{12} > 0$ and $R_2 = D_{21} D_{21}^* > 0$;

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(iv) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has column column rank for all ω.

Let T_{zw} denote the transfer matrix from w to z.

H_2 **Control Problem:** find a control law

$$u = K(s)y$$

that stabilizes the closed-loop system and minimizes $\|T_{zw}\|_2$ where

$$\|T_{zw}\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left[T_{zw}^*(j\omega)T_{zw}(j\omega)\right] d\omega}$$

LQG Control Problem: Assume $w(t)$ is a zero mean, unit variance, white Gaussian noise

$$E\{w(t)\} = 0, \quad E\{w(t)w^*(\tau)\} = I\delta(t-\tau).$$

Find a control law

$$u = K(s)y$$

that stabilizes the closed-loop system and minimizes

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \|z\|^2\, dt\right\}$$

Define

$$A_x = A - B_2R_1^{-1}D_{12}^*C_1, \quad A_y = A - B_1D_{21}^*R_2^{-1}C_2,$$

Then assumptions (iii) and (iv) guarantee that the following algebraic Riccati equations have respectively stabilizing solutions $X_2 \geq 0$ and $Y_2 \geq 0$

$$X_2A_x + A_x^*X_2 - X_2B_2R_1^{-1}B_2^*X_2 + C_1^*(I - D_{12}R_1^{-1}D_{12}^*)C_1 = 0$$

$$Y_2A_y^* + A_yY_2 - Y_2C_2^*R_2^{-1}C_2Y_2 + B_1(I - D_{21}^*R_2^{-1}D_{21})B_1^* = 0$$

Define

$$F_2 := -R_1^{-1}(D_{12}^*C_1 + B_2^*X_2)$$

$$L_2 := -(B_1D_{21}^* + Y_2C_2^*)R_2^{-1}$$

It is well-known that the H_2 and LQG problems are equivalent and the optimal controller is given by

$$K_2(s) := \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

and

$$\min J = \min \|T_{zw}\|_2^2 = \operatorname{trace}(B_1^*X_2B_1) + \operatorname{trace}(F_2Y_2F_2^*R_1).$$

1.2 A New H_2 Formulation

In this section, we shall look at the H_2 problem from a time domain point of view, which will lead to a simple proof of the result. The following lemma gives a time domain characterization of the H_2 norm of a stable transfer matrix.

Lemma 1.1 *Consider a stable dynamical system below*

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$$

Let T_{zw} be the transfer matrix from w to z. Let $x(0) = 0$ and $w(t) = w_0\delta(t)$ with a random direction w_0 such that

$$E\{w_0\} = 0, \quad E\{w_0 w_0^*\} = I.$$

Then

$$\|T_{zw}\|_2^2 = E\left\{\int_0^\infty \|z(t)\|^2\, dt\right\}$$

Proof. Let the impulse response of the system be denoted by $g(t) = Ce^{At}B$. It is well-known by Parseval's Theorem that

$$\|T_{zw}\|_2^2 = \int_0^\infty \operatorname{trace}\left[g^*(t)g(t)\right] dt = \operatorname{trace}(B^*QB)$$

where

$$Q = \int_0^\infty e^{A^*t}C^*Ce^{At}dt \geq 0$$

is the solution of the following Lyapunov equation:

$$A^*Q + QA + C^*C = 0$$

Next, note that

$$z(t) = Ce^{At}Bw_0$$

$$\begin{aligned} E\left\{\int_0^\infty \|z(t)\|^2\, dt\right\} &= E\left\{\int_0^\infty w_0^*B^*e^{A^*t}C^*Ce^{At}Bw_0\, dt\right\} \\ &= \int_0^\infty \operatorname{trace}\left[B^*e^{A^*t}C^*Ce^{At}B\ E\{w_0w_0^*\}\right] dt \\ &= \int_0^\infty \operatorname{trace}\left[B^*e^{A^*t}C^*Ce^{At}B\right] dt = \operatorname{trace}\left(B^*QB\right) \end{aligned}$$

This completes the proof. □

In view of the above lemma, the H_2 control can be regarded as a problem of finding a controller $K(s)$ for the system described in equations (1.1)–(1.3) with $w = w_0\delta(t)$ and $x(0) = 0$ that minimizes

$$J_1 := E\left\{\int_0^\infty \|z\|^2\, dt\right\}$$

Now we are ready to consider the output feedback H_2 problem. We shall need the following fact.

Lemma 1.2 *Suppose $K(s)$ is strictly proper. Then*

$$x(0^+) = B_1 w_0.$$

Proof. Let $K(s)$ be described by

$$\dot{\xi} = \hat{A}\xi + \hat{B}y, \qquad u = \hat{C}\xi$$

Then the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ \hat{B}D_{21} \end{bmatrix} w =: \overline{A} \begin{bmatrix} x \\ \xi \end{bmatrix} + \overline{B}w$$

and

$$\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} = e^{\overline{A}t}\overline{B}w_0$$

which gives $x(0^+) = B_1 w_0$. $\square$

Suppose that there exists an output feedback controller such that the closed-loop system is stable. Then $x(\infty) = 0$. Note that

$$\begin{aligned}
J_1 &= E\left\{\int_0^\infty \|z(t)\|^2\, dt\right\} \\
&= E\left\{\int_0^\infty \left(\|z\|^2 + \frac{d}{dt}x^*(t)X_2x(t)\right) dt\right\} \\
&= E\left\{\int_0^\infty \left(\|z\|^2 + 2x^*X_2\dot{x}\right) dt\right\} \\
&= E\left\{\int_0^\infty \left(\|C_1x + D_{12}u\|^2 + 2x^*X_2(Ax + B_1w + B_2u)\right) dt\right\} \\
&= E\left\{\int_0^\infty ((u - F_2x)^*R_1(u - F_2x) + 2x^*(t)X_2B_1w(t))\, dt\right\} \\
&= E\left\{\int_0^\infty ((u - F_2x)^*R_1(u - F_2x) + 2x^*(t)X_2B_1w_0\delta(t))\, dt\right\} \\
&= E\left\{\int_0^\infty (u - F_2x)^*R_1(u - F_2x)dt\right\} + E\left\{x^*(0^+)X_2B_1w_0\right\}
\end{aligned}$$

$$
\begin{aligned}
&= E\left\{\int_0^\infty (u-F_2x)^* R_1 (u-F_2x)dt\right\} + E\left\{w_0^* B_1^* X_2 B_1 w_0\right\} \\
&= E\left\{\int_0^\infty (u-F_2x)^* R_1 (u-F_2x)dt\right\} + \operatorname{trace}\left(B_1^* X_2 B_1\right)
\end{aligned}
$$

Obviously, an optimal control law would be

$$u = F_2 x$$

if full state is available for feedback. Since full state is not available for feedback, we have to implement the control law using estimated state:

$$u = F_2 \hat{x} \tag{1.4}$$

where $\hat{x}$ is the estimate of x. A standard observer can be constructed from the system equations (1.1) and (1.3) as

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C_2\hat{x} - y) \tag{1.5}$$

where L is the observer gain to be determined such that $A + LC_2$ is stable and J_1 is minimized.

Let

$$e := x - \hat{x}$$

Then

$$
\begin{aligned}
\dot{e} &= (A + LC_2)e + (B_1 + LD_{21})w =: A_L e + B_L w \\
u - F_2 x &= -F_2 e
\end{aligned}
$$

$$e(t) = e^{A_L t} B_L w_0$$

and

$$
\begin{aligned}
J_2 &:= E\left\{\int_0^\infty (u-F_2x)^* R_1 (u-F_2x)dt\right\} \\
&= E\left\{\int_0^\infty e^* F_2^* R_1 F_2 e \; dt\right\} \\
&= E\left\{\int_0^\infty w_0^* B_L^* e^{A_L^* t} F_2^* R_1 F_2 e^{A_L t} B_L w_0 \; dt\right\} \\
&= \operatorname{trace}\left\{\int_0^\infty F_2^* R_1 F_2 e^{A_L t} B_L E\{w_0 w_0^*\} B_L^* e^{A_L^* t} \; dt\right\} \\
&= \operatorname{trace}\left\{F_2^* R_1 F_2 \int_0^\infty e^{A_L t} B_L B_L^* e^{A_L^* t} \; dt\right\} \\
&= \operatorname{trace}\left\{F_2^* R_1 F_2 Y\right\}
\end{aligned}
$$

where

$$Y = \int_0^\infty e^{(A+LC_2)t}(B_1 + LD_{21})(B_1 + LD_{21})^* e^{(A+LC_2)^* t} dt$$

or

$$(A + LC_2)Y + Y(A + LC_2)^* + (B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

Subtract the equation of Y_2 from the above equation to get

$$(A + LC_2)(Y - Y_2) + (Y - Y_2)(A + LC_2)^* + (L - L_2)R_2(L - L_2)^* = 0$$

It is then clear that $Y \geq Y_2$ and equality holds if $L = L_2$. Hence J_2 is minimized if $L = L_2$.

In summary, the optimal H_2 controller can be written as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 u + L_2(C_2\hat{x} - y) \\ u &= F_2\hat{x} \end{aligned}$$

i.e.,

$$K_2(s) := \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

and

$$\min \|T_{zw}\|_2 = \sqrt{\operatorname{trace}(B_1^* X_2 B_1) + \operatorname{trace}(F_2 Y_2 F_2^* R_1)}.$$

This is exactly the H_2 controller formula we are familiar with.

1.3 Traditional Stochastic LQG Formulation

The traditional stochastic LQG formulation assumes that $w(t)$ is a zero mean, unit variance, white Gaussian stochastic process. That is,

$$E\{w(t)\} = 0, \quad E\{w(t)w^*(\tau)\} = I\delta(t - \tau).$$

In this case, we have the following relationship.

Lemma 1.3 *Consider a stable dynamical system below*

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = 0 \\ z &= Cx \end{aligned}$$

Then

$$\|T_{zw}\|_2^2 = E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \|z(t)\|^2 \, dt\right\}$$

Proof. Note that

$$z(t) = \int_0^t Ce^{A(t-\tau)} Bw(\tau) d\tau$$

and

$$E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \|z(t)\|^2 \, dt\right\}$$

$$\begin{aligned}
&= E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t w^*(\tau)B^* e^{A^*(t-\tau)} C^* C e^{A(t-s)} Bw(s) \, d\tau \, ds \, dt\right\} \\
&= E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t \operatorname{trace}\left\{B^* e^{A^*(t-\tau)} C^* C e^{A(t-s)} Bw(s)w^*(\tau)\right\} \, d\tau \, ds \, dt\right\} \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t \operatorname{trace}\left\{B^* e^{A^*(t-\tau)} C^* C e^{A(t-s)} B \; E\left\{w(s)w^*(\tau)\right\}\right\} \, d\tau \, ds \, dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t \operatorname{trace}\left\{B^* e^{A^*(t-\tau)} C^* C e^{A(t-s)} B\delta(\tau - s)\right\} \, d\tau \, ds \, dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \operatorname{trace}\left\{B^* e^{A^*(t-s)} C^* C e^{A(t-s)} B\right\} \, ds \, dt \\
&= \operatorname{trace}\left\{B^* \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t e^{A^* s} C^* C e^{As} ds \, dt \; B\right\} \\
&= \operatorname{trace}(B^* Q B) = \|T_{zw}\|_2^2
\end{aligned}$$

□

Now consider the LQG control problem. Suppose that there exists an output feedback controller such that the closed-loop system is stable. Then $x(\infty) = 0$.

Lemma 1.4 *Suppose that $K(s)$ is a strictly proper stabilizing controller. Then*

$$E\left\{x(t)w^*(t)\right\} = B_1/2.$$

Proof. Let $K(s)$ be described by

$$\dot{\xi} = \hat{A}\xi + \hat{B}y, \qquad u = \hat{C}\xi$$

Then the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ \hat{B}D_{21} \end{bmatrix} w =: \overline{A} \begin{bmatrix} x \\ \xi \end{bmatrix} + \overline{B} w$$

Then

$$\begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} = \int_0^t e^{\overline{A}(t-\tau)} \overline{B} w(\tau) \, d\tau$$

Hence

$$
\begin{aligned}
E\begin{bmatrix} x(t)w^*(t) \\ \xi(t)w^*(t) \end{bmatrix} &= E\left\{\int_0^t e^{\overline{A}(t-\tau)}\overline{B}w(\tau)w^*(t)\, d\tau\right\} \\
&= \int_0^t e^{\overline{A}(t-\tau)}\overline{B}\ E\left\{w(\tau)w^*(t)\right\}\ d\tau \\
&= \int_0^t e^{\overline{A}(t-\tau)}\overline{B}\ \delta(t-\tau)\ d\tau \\
&= \int_0^t e^{\overline{A}\tau}\overline{B}\ \delta(\tau)\ d\tau = \overline{B}/2
\end{aligned}
$$

which gives $E\left\{x(t)w^*(t)\right\} = B_1/2$. □

Now note that

$$
\begin{aligned}
J &:= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left\|z(t)\right\|^2 dt\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\left\|z\right\|^2 + \frac{d}{dt}x^*(t)X_2x(t)\right) dt\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\left\|z\right\|^2 + 2x^*X_2\dot{x}\right) dt\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\left\|C_1x + D_{12}u\right\|^2 + 2x^*X_2(Ax + B_1w + B_2u)\right) dt\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T ((u - F_2x)^*R_1(u - F_2x) + 2x^*(t)X_2B_1w(t))\, dt\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T (u - F_2x)^*R_1(u - F_2x)dt\right\} \\
&\quad + E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \operatorname{trace}\left\{2X_2B_1\ E\left\{w(t)x^*(t)\right\}\right\}\right\} \\
&= E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T (u - F_2x)^*R_1(u - F_2x)dt\right\} + \operatorname{trace}\left(B_1^*X_2B_1\right)
\end{aligned}
$$

Again, an optimal control law would be

$$u = F_2x$$

if full state is available for feedback. Since full state is not available for feedback, we will have to implement the control law using estimated state:

$$u = F_2\hat{x} \tag{1.6}$$

where $\hat{x}$ is the estimate of x. A standard observer can be constructed from the system equations as

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C_2\hat{x} - y) \tag{1.7}$$

where L is the observer gain to be determined such that $A + LC_2$ is stable and J is minimized.

Let

$$e := x - \hat{x}$$

Then

$$\begin{aligned} \dot{e} &= (A + LC_2)e + (B_1 + LD_{21})w =: A_L e + B_L w \\ u - F_2 x &= -F_2 e \end{aligned}$$

$$e(t) = \int_0^t e^{A_L(t-\tau)} B_L w(\tau)\, d\tau$$

and

$$\begin{aligned} J_3 &:= E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T (u - F_2 x)^* R_1 (u - F_2 x) dt\right\} \\ &= E\left\{\int_0^\infty e^* F_2^* R_1 F_2 e\; dt\right\} \\ &= E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t w^*(t) B_L^* e^{A_L^*(t-\tau)} F_2^* R_1 F_2 e^{A_L(t-s)} B_L w(s)\; d\tau\; ds\; dt\right\} \\ &= \operatorname{trace}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t F_2^* R_1 F_2 e^{A_L(t-\tau)} B_L E\{w(s)w^*(\tau)\} B_L^* e^{A_L^*(t-s)}\; d\tau\; ds\; dt\right\} \\ &= \operatorname{trace}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t F_2^* R_1 F_2 e^{A_L(t-\tau)} B_L \delta(\tau - s) B_L^* e^{A_L^*(t-s)}\; d\tau\; ds\; dt\right\} \\ &= \operatorname{trace}\left\{F_2^* R_1 F_2 \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t e^{A_L t} B_L B_L^* e^{A_L^* t}\; ds\; dt\right\} \\ &= \operatorname{trace}\{F_2^* R_1 F_2 Y\} \end{aligned}$$

where

$$Y = \int_0^\infty e^{A_L t} B_L B_L^* e^{A_L^* t} dt$$

or

$$(A + LC_2)Y + Y(A + LC_2)^* + (B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

Subtract the equation of Y_2 from the above equation to get

$$(A + LC_2)(Y - Y_2) + (Y - Y_2)(A + LC_2)^* + (L - L_2)R_2(L - L_2)^* = 0$$

It is then clear that $Y \geq Y_2$ and equality holds if $L = L_2$. Hence J_3 is minimized if $L = L_2$.

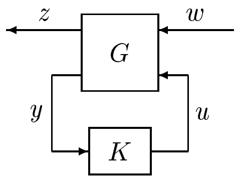
Chapter 2

Understanding H_∞ Control

We give an intuitive derivation of $\mathcal{H}_\infty$ controller.

2.1 Problem Formulation and Solutions

Consider again the standard system below:



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

The following assumptions are made:

(i) (A, B_1) is controllable and (C_1, A) is observable;

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable;

(iii) $D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Theorem 2.1 *There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:*

(i) There exists a stabilizing solution $X_\infty > 0$ *to*

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0. \tag{2.1}$$

(ii) There exists a stabilizing solution $Y_\infty > 0$ *to*

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0. \tag{2.2}$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, when these conditions hold, one central controller is

$$K_{\mathrm{sub}}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^* X_\infty, \qquad L_\infty := -Y_\infty C_2^*, \qquad Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

2.2 An Intuitive Derivation

Most existing derivations and proofs of the $\mathcal{H}_\infty$ control results given in Theorem 2.1 are mathematically quite complex. Some algebraic derivations (such as the one given in the book) are simple but they provide no insight to the theory for control engineers. Here we shall present an intuitive but nonrigorous derivation of the $\mathcal{H}_\infty$ results by using only some basic system theoretic concept such as state feedback and state estimation. In fact, we shall construct intuitively the output feedback $\mathcal{H}_\infty$ central controller by combining an $\mathcal{H}_\infty$ state feedback and an observer.

A key fact we shall use is the so-called bounded real lemma, which states that, for a system $z = G(s)w$ with state space realization $G(s) = C(sI - A)^{-1}B \in \mathcal{H}_\infty$, $\|G\|_\infty < \gamma$, which is essentially equivalent to

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt < 0, \quad \forall w \neq 0$$

if and only if there is an $X = X^* \geq 0$ such that

$$XA + A^*X + XBB^*X/\gamma^2 + C^*C = 0$$

and $A + BB^*X/\gamma^2$ is stable. Dually, there is a $Y = Y^* \geq 0$ such that

$$YA^* + AY + YC^*CY/\gamma^2 + BB^* = 0$$

and $A + YC^*C/\gamma^2$ is stable.

Note that the system has the following state space realization

$$\dot{x} = Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{12} u, \quad y = C_2 x + D_{21} w$$

We shall first consider state feedback $u = Fx$. Then the closed-loop system becomes

$$\dot{x} = (A + B_2 F)x + B_1 w, \qquad z = (C_1 + D_{12} F)x$$

By the bounded real lemma, $\|T_{zw}\|_\infty < \gamma$ implies that there exists an $X = X^* \geq 0$ such that

$$X(A + B_2 F) + (A + B_2 F)^* X + X B_1 B_1^* X/\gamma^2 + (C_1 + D_{12} F)^*(C_1 + D_{12} F) = 0$$

which is equivalent, by completing square with respect to F, to

$$XA + A^* X + X B_1 B_1^* X/\gamma^2 - X B_2 B_2^* X + C_1^* C_1 + (F + B_2^* X)^*(F + B_2^* X) = 0$$

Intuition suggests that we can take

$$F = -B_2^* X$$

which gives

$$XA + A^* X + X B_1 B_1^* X/\gamma^2 - X B_2 B_2^* X + C_1^* C_1 = 0$$

This is exactly the X_∞ Riccati equation under the preceding simplified conditions. Hence, we can take $F = F_\infty$ and $X = X_\infty$.

Next, suppose that there is an output feedback stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$. Then $x(\infty) = 0$ because the closed-loop system is stable. Consequently, we have

$$\begin{aligned}\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt &= \int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2 + \frac{d}{dt}(x^* X_\infty x)\right) dt \\ &= \int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2 + \dot{x}^* X_\infty x + x^* X_\infty \dot{x}\right) dt\end{aligned}$$

Substituting $\dot{x} = Ax + B_1 w + B_2 u$ and $z = C_1 x + D_{12} u$ into the above integral and using the X_∞ equation, and finally completing the squares with respect to u and w, we get

$$\int_0^\infty \left(\|z\|^2 - \gamma^2 \|w\|^2\right) dt = \int_0^\infty \left(\|v\|^2 - \gamma^2 \|r\|^2\right) dt$$

where $v = u + B_2^* X_\infty x = u - F_\infty x$ and $r = w - \gamma^{-2} B_1^* X_\infty x$. Substituting w into the system equations, we have the new system equations

$$\begin{aligned}\dot{x} &= (A + B_1 B_1^* X_\infty/\gamma^2)x + B_1 r + B_2 u \\ v &= -F_\infty x + u \\ y &= C_2 x + D_{21} r\end{aligned}$$

Hence the original $\mathcal{H}_\infty$ control problem is equivalent to finding a controller so that $\|T_{vr}\|_\infty < \gamma$ or

$$\int_0^\infty \left(\|u - F_\infty x\|^2 - \gamma^2 \|r\|^2\right) dt < 0$$

Obviously, this also suggests intuitively that the state feedback control can be $u = F_\infty x$ and a worst state feedback disturbance would be $w_* = \gamma^{-2} B_1^* X_\infty x$. Since full state is not available for feedback, we have to implement the control law using estimated state:

$$u = F_\infty \hat{x}$$

where $\hat{x}$ is the estimate of x. A standard observer can be constructed from the new system equations as

$$\dot{\hat{x}} = (A + B_1 B_1^* X_\infty/\gamma^2)\hat{x} + B_2 u + L(C_2\hat{x} - y)$$

where L is the observer gain to be determined. Let $e := x - \hat{x}$. Then

$$\begin{aligned} \dot{e} &= (A + B_1 B_1^* X_\infty/\gamma^2 + LC_2)e + (B_1 + LD_{21})r \\ v &= -F_\infty e \end{aligned}$$

Since it is assumed that $\|T_{vr}\|_\infty < \gamma$, it follows from the dual version of the bounded real lemma that there exists a $Y \geq 0$ such that

$$Y(A + B_1 B_1^* X_\infty/\gamma^2 + LC_2)^* + (A + B_1 B_1^* X_\infty/\gamma^2 + LC_2)Y + Y F_\infty^* F_\infty Y/\gamma^2$$
$$+ (B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

The above equation can be written as

$$(A + B_1 B_1^* X_\infty/\gamma^2)^* + (A + B_1 B_1^* X_\infty/\gamma^2)Y + Y F_\infty^* F_\infty Y/\gamma^2 + B_1 B_1^* - Y C_2^* C_2 Y$$
$$+ (L + Y C_2^*)(L + Y C_2^*)^* = 0$$

Again, intuition suggests that we can take

$$L = -Y C_2^*$$

which gives

$$Y(A + B_1 B_1^* X_\infty/\gamma^2)^* + (A + B_1 B_1^* X_\infty/\gamma^2)Y + Y F_\infty^* F_\infty Y/\gamma^2 - Y C_2^* C_2 Y + B_1 B_1^* = 0$$

It is easy to verify that

$$Y = Y_\infty (I - \gamma^{-2} X_\infty Y_\infty)^{-1}$$

where Y_∞ is as given in Theorem 2.1. Since $Y \geq 0$, we must have

$$\rho(X_\infty Y_\infty) < \gamma^2$$

Hence $L = Z_\infty L_\infty$ and the controller is give by

$$\begin{aligned}
\dot{\hat{x}} &= (A + B_1 B_1^* X_\infty/\gamma^2)\hat{x} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y) \\
u &= F_\infty \hat{x}
\end{aligned}$$

which is exactly the $\mathcal{H}_\infty$ central controller given in Theorem 2.1.

We can see that the $\mathcal{H}_\infty$ central controller can be obtained by connecting a state feedback with a state estimate under the worst state feedback disturbance.

Part II

Solutions Manual