

EE 3140 HOMEWORK #3 SOLUTIONS

$$\begin{aligned} \textcircled{1}(a) & P[X \leq 6, |Y| < 2, Z^2 > 2] \\ &= P[X \leq 6] P[|Y| < 2] P[Z^2 > 2] \\ &= [F_X(6^-)] [F_Y(2^-) - F_Y(-2)] [1 - F_Z(\sqrt{2}) + F_Z(-\sqrt{2})] \end{aligned}$$

$$\begin{aligned} (b) & P[X < 5, Y=0, Z \leq 1] \\ &= P[X < 5] P[Y=0] P[Z \leq 1] = F_X(5^-) [F_Y(0) - F_Y(0^-)] F_Z(1) \end{aligned}$$

$$\begin{aligned} (c) & P[\min(X, Y, Z) > 0] \\ &= P[X > 0, Y > 0, Z > 0] = [1 - F_X(0)] [1 - F_Y(0)] [1 - F_Z(0)] \end{aligned}$$

$$\textcircled{2}(a) \text{ All outcomes are equiprobable since } P[X_1=i, X_2=j] = P[X_1=i] P[X_2=j] = \left(\frac{1}{6}\right)^2$$

$$[1 \leq i \leq 6; 1 \leq j \leq 6]$$

$$\begin{aligned} (b) & P[X=i, Y=j] = 0 & i > j \\ &= \frac{1}{36} & i = j \\ &= \frac{2}{36} & i < j \end{aligned}$$

$$(c) P[Y=6] = 5 \cdot \frac{2}{36} + \frac{1}{36} = \frac{11}{36}$$

$$P[Y=5] = 4 \cdot \frac{2}{36} + \frac{1}{36} = \frac{9}{36}$$

$$P[Y=4] = \frac{7}{36}$$

$$P[Y=3] = \frac{5}{36}$$

$$P[Y=2] = \frac{3}{36}$$

$$P[Y=1] = \frac{1}{36}$$

$$\text{Similarly } P[X=i] = P[Y=7-i] \quad i=1 \dots 6$$

$$\textcircled{3}(i) P[X=-1] = \frac{1}{3}$$

$$P[X=0] = 0$$

$$P[X=1] = \frac{2}{3}$$

$$P[Y=-1] = \frac{1}{3}$$

$$P[Y=0] = \frac{1}{3}$$

$$P[Y=1] = \frac{1}{3}$$

$$(ii) P[X=-1] = \frac{1}{2}$$

$$P[X=0] = \frac{1}{4}$$

$$P[X=1] = \frac{1}{4}$$

$$P[Y=-1] = \frac{1}{4}$$

$$P[Y=0] = \frac{1}{2}$$

$$P[Y=1] = \frac{1}{4}$$

$$(iii) P[X=-1] = \frac{1}{3}$$

$$P[X=0] = \frac{1}{3}$$

$$P[X=1] = \frac{1}{3}$$

$$P[Y=-1] = 0$$

$$P[Y=0] = \frac{1}{3}$$

$$P[Y=1] = \frac{2}{3}$$

	(i)	(ii)	(iii)
(b) $P\{A\}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{2}{3}$
$P\{B\}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$P\{C\}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{2}{3}$

④ (a) For $x > 0, y > 0$

$$F_{XY}(x, y) = \int_0^x \int_0^y 2e^{-x}e^{-2y} dx dy$$

$$= 2e^{-x}e^{-2y}$$

$$(b) P\{X > Y\} = \int_0^\infty \int_0^x 2e^{-x}e^{-2y} dy dx = 1$$

$$(c) f_X(x) = \int_0^\infty 2e^{-x}e^{-2y} dy = 2e^{-x}$$

$$f_Y(y) = \int_0^\infty 2e^{-x}e^{-2y} dx = 2e^{-2y}$$

$$(5) (a) 1 = K \int_0^2 \int_0^2 (x+y) dx dy = K \int_0^2 \left[\frac{x^2}{2} + xy \right]_0^2 dy$$

$$= K \int_0^2 [2 + 2y] dy = K [2y + y^2]_0^2 = 8K \Rightarrow K = \frac{1}{8}$$

	$\frac{x(x+2)}{8}$	1
2	$\frac{xy(x+y)}{16}$	$\frac{y(y+2)}{8}$
0		2

$$(c) F_X(x) = \lim_{y \rightarrow \infty} F_{XY}(x, y) = F_{XY}(x, 2)$$

$$f_X(x) = \frac{d}{dx} [F_X(x)] = \frac{d}{dx} \left[\frac{x^2}{8} + \frac{2x}{8} \right] = \frac{x}{4} + \frac{1}{4}$$

$$\text{Similarly } f_Y(y) = \frac{y}{4} + \frac{1}{4}$$

$$(6) (a) K \cdot \pi = 1 \Rightarrow K = \frac{1}{\pi}$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi} \quad -1 < x < 1$$

$$\text{Similarly } f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi} \quad -1 < y < 1$$

$$(7) \frac{1}{2} \cdot K \cdot 1 \cdot 1 = 1 \Rightarrow K = 2$$

$$f_X(x) = \int_0^{1-x} 2 dy = 2(1-x); \quad 0 < x < 1$$

$$f_Y(y) = 2(1-y); \quad 0 < y < 1$$

$$(8) (a) P\{X+Y \leq 5\} = \int_0^5 \int_0^{5-x} 2e^{-x} e^{-2y} dy dx = 1 - 2e^{-5} + e^{-10}$$

$$(b) P\{X < Y\} = \int_0^\infty \int_x^\infty 2e^{-x} e^{-2y} dy dx = \int_0^\infty e^{-x} e^{-2x} dx = \frac{1}{3}$$

$$(c) P\{X-Y \leq 8\} = \int_0^\infty \int_0^{8+y} 2e^{-x} e^{-2y} dx dy = 1 - \frac{2}{3}e^{-8}$$

$$(d) P\{X^2 < Y\} = \int_0^\infty \int_{x^2}^\infty 2e^{-x} e^{-2y} dy dx = 0.438$$

$$(9) f_X(x) = \int_0^\infty 2e^{-x} e^{-xy} dy = x e^{-x} \left(-\frac{e^{-xy}}{x} \right)_0^\infty = e^{-x}$$

$$f_Y(y) = \int_0^\infty 2e^{-2(1+y)t} dt = \frac{e^{-2(1+y)} [(1+y)t - 1]}{(1+y)^2} \Big|_0^\infty = \frac{1}{(1+y)^2}$$

⑩ Dependent in all 3 cases

⑪ Independent

⑫ Dependent

⑬ (a) Since the first n trials are independent of the remaining m trials

$$(b) p(N) = \binom{n}{N} p^N (1-p)^{n-N}$$

$$p(M) = \binom{m}{M} p^M (1-p)^{m-M}$$

$$p(N, M) = p(N) p(M)$$

$$(c) p(K) = \binom{n+m}{K} p^K (1-p)^{n+m-K}$$

$$\begin{array}{l} \textcircled{14} \quad P\{Y=-1|X=-1\} = \\ P\{Y=0|X=-1\} = \\ P\{Y=1|X=-1\} = \end{array} \begin{array}{c|c|c} \text{(i)} & \text{(ii)} & \text{(iii)} \\ \hline \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{array}$$

$$\textcircled{15} \quad f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}$$

$$\begin{aligned} \textcircled{16} (a) \quad 1 &= K \int_0^1 \int_0^1 \int_0^1 (x+y+z) dx dy dz = K \int_0^1 \int_0^1 \left(\frac{1}{2} + y + z\right) dy dz \\ &= K \int_0^1 \left[\left(\frac{1}{2} + z\right) + \frac{1}{2}\right] dz = K \left[1 + \frac{1}{2}\right] \Rightarrow K = \frac{2}{3} \end{aligned}$$

$$(b) f_{XY}(x,y) = \frac{2}{3} \int_0^1 (x+y+z) dz = \frac{2}{3} \left[x+y+\frac{1}{2}\right]$$

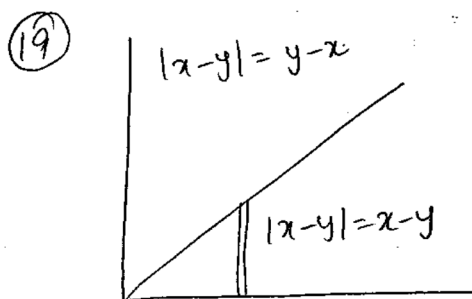
$$f_Z(z|x,y) = \frac{x+y+z}{x+y+\frac{1}{2}}$$

$$\begin{aligned} (17) \quad f_{XYZ}(x, y, z) &= f_z(z|x, y) f_{XY}(x, y) \\ &= f_z(z|x, y) f_Y(y|x) f_X(x) \end{aligned}$$

$$(18) (a) \quad E[(X+Y)^2] = E(X^2) + 2E(XY) + E(Y^2)$$

$$\begin{aligned} (b) \quad \text{var}[X+Y] &= E[(X+Y)^2] - [E(X+Y)]^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - [E(Y)]^2 - 2E(X)E(Y) \end{aligned}$$

(c) If $E(XY) = E(X)E(Y)$ that is if X and Y are uncorrelated.



$$\begin{aligned} E[|X-Y|] &= \int_0^\infty \int_0^\infty |x-y| e^{-(x+y)} dx dy \\ &= 2 \int_0^\infty \int_0^\infty (x-y) e^{-x} e^{-y} dy dx \\ &= 2 \int_0^\infty e^{-x} \left[x(1-e^{-x}) - \int_0^x y e^{-y} dy \right] dx \\ &= 2 \int_0^\infty (x e^{-x} + e^{-2x} - e^{-x}) dx \\ &= 2 \left[1 + \frac{1}{2} - 1 \right] = 1 \end{aligned}$$

$$\begin{aligned} (20) \quad E[X^2 Y] &= E(X^2) E(Y) \\ &= (1)(1) = 1 \end{aligned}$$