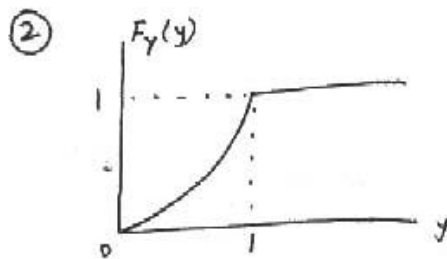


EE 3140 HOMEWORK 2 SOLUTIONS

① (a) $S_Y = \{Y: 0 \leq Y < 1\}$

(b) $\{Y \leq y\} = \{\text{point is inside circle of radius } y\}$

(c) $P[Y \leq y] = \frac{\pi y^2}{\pi \cdot 1^2} = y^2$



Y is a continuous random variable

③ $P[X > \frac{1}{3}] = 1 - P[X \leq \frac{1}{3}] = 1 - F_X(\frac{1}{3}) = 1 - 1 = 0$

$P[|X| \geq 1] = P[X = -1] = \frac{1}{3}$

$P[|X - \frac{1}{3}| < 1] = P[-\frac{2}{3} < X < \frac{4}{3}] = F_X(\frac{4}{3}) - F_X(-\frac{2}{3}) = 1 - \frac{2}{3} \cdot (\frac{1}{3})^2 = \frac{25}{27}$

$P[X < 0] = F_X(0) = 1$

④ (a) X is a mixed type of random variable

(b) $P(X < -\frac{1}{2}) = 0$

$P(X < 0) = 0$

$P(X \leq 0) = \frac{1}{2}$

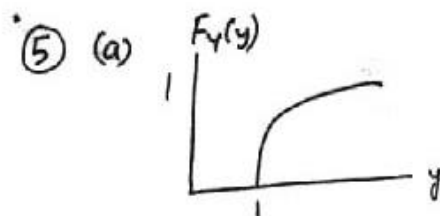
$P(\frac{1}{4} \leq X < 1) = F_X(1^-) - F_X(\frac{1}{4}) = \frac{3}{4} - \frac{9}{16} = \frac{3}{16}$

$P(\frac{1}{4} \leq X \leq 1) = F_X(1^+) - F_X(\frac{1}{4}) = 1 - \frac{9}{16} = \frac{7}{16}$

$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{5}{8} = \frac{3}{8}$

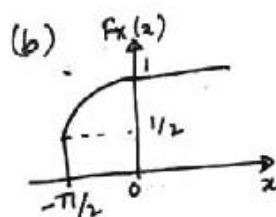
$P(X \geq 5) = 1 - P(X < 5) = 0$

$P(X < 5) = 1$



(b) $P[K < Y \leq K+1] = F_Y(K+1) - F_Y(K) = \left(\frac{1}{K}\right)^n - \left(\frac{1}{K+1}\right)^n$

⑥ (a) $F_X(0) = 1$ since it is continuous
 $\Rightarrow c[1 + \cos(0)] = 1 \Rightarrow c = \frac{1}{2}$



⑦ (a) We know $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$\therefore c \int_0^1 x(1-x) dx = c \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{c}{6} = 1 \Rightarrow c = 6$

(b) $P\left[\frac{3}{4} \leq X \leq 1\right] = 6 \int_{3/4}^1 x - x^2 dx = 6 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{3/4}^1 = \frac{5}{32}$

(c) $F_X(x) = 0$ for $x < 0$

$F_X(x) = 1$ for $x > 1$

$F_X(x) = \int_0^x f_X(x') dx' = 3x^2 - 2x^3$ for $0 \leq x \leq 1$

⑧ (a) $\int_{-1}^1 c(1-x^4) dx = 1 \Rightarrow c \left[x - \frac{x^5}{5} \right]_{-1}^1 = 1 \Rightarrow c = \frac{5}{8}$

(b) $F_X(x) = \int_{-1}^x f_X(x') dx'$ for $-1 \leq x \leq 1$
 $= \frac{1}{2} + \frac{5}{8}x - \frac{1}{8}x^5$

$F_X(x) = 0$; $x < -1$

$F_X(x) = 1$; $x > 1$

$$(c) P[|X| < \frac{1}{2}] = F_X(\frac{1}{2}) - F_X(-\frac{1}{2}) = \frac{79}{128}$$

$$(9) \text{ (a) Area under the curve} = 1$$

$$\Rightarrow \frac{1}{2} \cdot 2a \cdot c = 1 \Rightarrow c = \frac{1}{a}$$

$$\therefore f_X(x) = 0 \quad ; |x| > a$$

$$= \frac{1}{a} \left(1 - \frac{|x|}{a}\right) \quad ; |x| \leq a$$

$$(b) f_X(x) = 0 \quad x < -a$$

$$f_X(x) = \int_{-\infty}^x f_X(z) dz = \frac{1}{a} \int_{-a}^x \left(1 + \frac{z}{a}\right) dz = \frac{1}{2} + \frac{1}{a} \left[x + \frac{x^2}{2a}\right] \quad \text{for } -a \leq x \leq a$$

$$f_X(x) = \int_{-a}^0 f_X(z) dz + \int_0^x f_X(z) dz = \frac{1}{2} + \frac{1}{a} \int_0^x \left(1 - \frac{z}{a}\right) dz$$

$$= \frac{1}{2} + \frac{1}{a} \left[x - \frac{x^2}{2a}\right] \quad \text{for } 0 \leq x \leq a$$

$$f_X(x) = 1 \quad \text{for } x > a$$

$$(10) (a) P(X=0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-15} \quad [\lambda=15]$$

$$(b) P[X > 10] = 1 - P[X \leq 9] = 1 - \sum_{k=0}^9 \frac{e^{-15} (15)^k}{k!} = 0.8815$$

$$(11) c = \frac{1}{4} \therefore f_X(x) = \frac{1}{4} \quad ; |x| < 2$$

$$\left|\frac{dy}{dx}\right| = |2x| = |2\sqrt{y}|$$

$$f_Y(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} = \frac{1}{8\sqrt{y}} \quad ; 0 < y \leq 4 \quad \text{for negative } x$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{1}{8\sqrt{y}} \quad ; 0 < y \leq 4 \quad \text{for positive } x$$

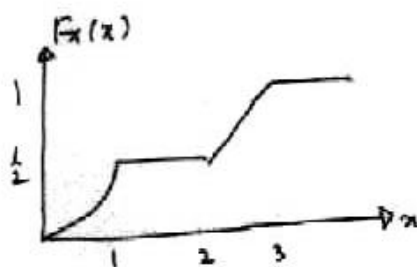
$$\therefore f_Y(y) = \frac{1}{8\sqrt{y}} + \frac{1}{8\sqrt{y}} = \frac{1}{4\sqrt{y}} \quad , 0 < y \leq 4$$

$$(12) \quad \frac{1}{2} \cdot 5 \cdot h = 1 \Rightarrow h = \frac{2}{5}$$

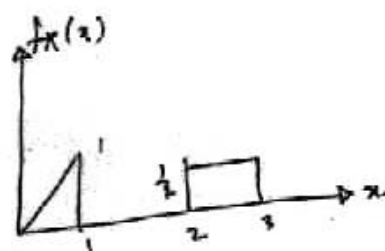
$$\frac{dy}{dx} = 3x^2 = 3y^{2/3}$$

$$f_y(y) = \frac{f_x(y^{1/3})}{3y^{2/3}} = \frac{2}{75y^{1/3}} + \frac{4}{75y^{2/3}} \quad -8 \leq y \leq 27$$

$$(13) \quad F_X(x) = \begin{cases} \frac{x^2}{2} & 0 \leq x \leq 1 \\ \frac{1}{2} & 1 \leq x \leq 2 \\ \frac{1}{2}(x-1) & 2 \leq x \leq 3 \\ 1 & 3 \leq x \end{cases}$$



$$f_X(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 0 & 1 \leq x \leq 2 \\ \frac{1}{2} & 2 \leq x \leq 3 \\ 0 & 3 \leq x \end{cases}$$



$$(14) \quad \frac{1}{2} \cdot 1 \cdot h + 1 \cdot h + \frac{1}{2} \cdot 1 \cdot h = 1 \Rightarrow h = \frac{1}{2}$$

$$E(X) = \int_0^1 \frac{1}{2} x \cdot x dx + \int_1^2 \frac{1}{2} \cdot x dx + \int_2^3 \left(-\frac{x}{2} + \frac{3}{2}\right) \cdot 2 dx = \frac{3}{2}$$

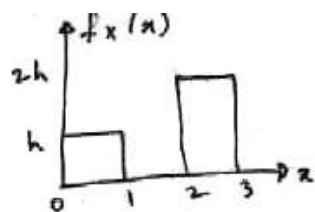
$$E(X^2) = \int_0^1 \frac{1}{2} x \cdot x^2 dx + \int_1^2 \frac{1}{2} x \cdot x^2 dx + \int_2^3 \left(-\frac{x}{2} + \frac{3}{2}\right) \cdot x^2 dx = \frac{8}{3}$$

$$\text{Var}[X] = \frac{8}{3} - \left(\frac{3}{2}\right)^2 = \frac{5}{12}$$

$$(15) \quad h \cdot 1 + 2h \cdot 1 = 1 \Rightarrow h = \frac{1}{3}$$

$$E(X) = \int_0^1 \frac{1}{3} \cdot x \, dx + \int_2^3 \frac{2}{3} \cdot x \, dx$$

$$= \frac{1}{3} \cdot \frac{x^2}{2} \Big|_0^1 + \frac{2}{3} \cdot \frac{x^2}{2} \Big|_2^3 = \frac{11}{6}$$



$$E(X^2) = \int_0^1 \frac{1}{3} \cdot x^2 \, dx + \int_2^3 \frac{2}{3} \cdot x^2 \, dx$$

$$= \frac{1}{3} \cdot \frac{x^3}{3} \Big|_0^1 + \frac{2}{3} \cdot \frac{x^3}{3} \Big|_2^3 = \frac{13}{3}$$

$$\text{Var}(X) = \frac{13}{3} - \left(\frac{11}{6}\right)^2 = \frac{35}{36}$$

$$(16) \quad E(X) = 2.583$$

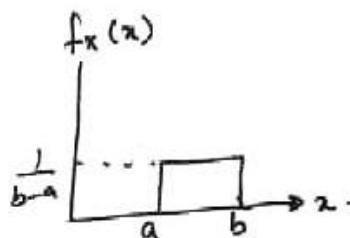
$$E(X^2) = 9.583$$

$$\text{Var}[X] = 1.7888 \quad 2.9097$$

$$(17) \quad E(X) = \int_a^b x \cdot \frac{1}{b-a} \, dx = \frac{a+b}{2}$$

$$E(X^2) = \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{b^2 + ab + a^2}{3}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$$



$$(18) \quad E(X) = \sum_{k=1}^n k P[X=k] = \sum_{k=1}^n \frac{k}{n} = \frac{1}{n} \sum_{k=1}^n k = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\sigma_X^2 = E(X^2) - E(X)^2 = \sum_{k=1}^n \frac{k^2}{n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2-1}{12}$$

$$\begin{aligned}
 (19) \quad E(X) &= \sum_{k=1}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \sum_{k=1}^{\infty} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
 &= np \cdot \sum_{k=1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
 &= np
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum_{k=1}^{\infty} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
 &= np \sum_{k=1}^{\infty} \frac{k(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
 &= np \sum_{k=1}^{\infty} \frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
 &= np \left[\sum_{k=1}^{\infty} \frac{(k-1)(n-1)!}{(k-1)!(n-k)!} p \cdot p^{k-2} (1-p)^{n-k} + \sum_{k=1}^{\infty} \frac{(1)(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \right] \\
 &= np \left[(n-1)p \sum_{k=1}^{\infty} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k} + \sum_{k=1}^{\infty} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \right] \\
 &= np[(n-1)p + 1] = n^2 p^2 - np^2 + np
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - [E(X)]^2 \\
 &= \underline{np(1-p)}
 \end{aligned}$$

$$\begin{aligned}
 (20) \quad E(X) &= \sum_{k=0}^{\infty} k \cdot \frac{\alpha^k e^{-\alpha}}{k!} \\
 &= e^{-\alpha} \left[\alpha + \frac{2\alpha^2}{2!} + \frac{3\alpha^3}{3!} + \dots \right] \\
 &= \alpha e^{-\alpha} \left[1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \dots \right] \\
 &= \alpha e^{-\alpha} e^{\alpha} = \alpha
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= \sum_{k=0}^{\infty} k^2 \cdot \frac{\alpha^k e^{-\alpha}}{k!} \\
 &= \alpha \sum_{k=0}^{\infty} \frac{k \alpha^{k-1}}{(k-1)!} e^{-\alpha} = \alpha \sum_{k=1}^{\infty} \frac{(k-1+1) \alpha^{k-1}}{(k-1)!} e^{-\alpha} \\
 &= \alpha \left[\underbrace{\sum_{k=1}^{\infty} \frac{(k-1) \alpha^{k-1}}{(k-1)!} e^{-\alpha}}_{\alpha} + \underbrace{\sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha}}_1 \right] \\
 &= \alpha [\alpha + 1] = \alpha^2 + \alpha
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - E(X)^2 \\
 &= \alpha^2 + \alpha - \alpha^2 = \alpha
 \end{aligned}$$

