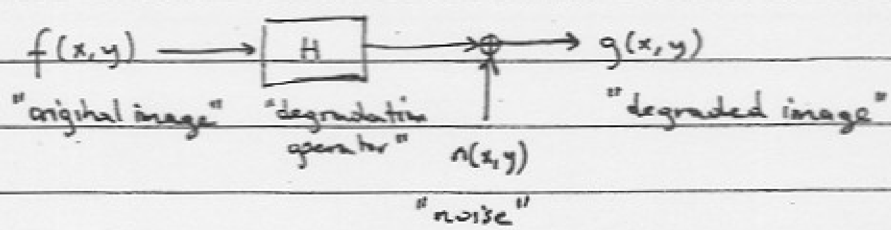
Image Restoration

Goal: To reconstruct or recover an image (that has been degraded) by using some <u>prior</u> knowledge of the degradation process.

Image Restoration	vs.	Image Enhancement
→ Model the degradation → Apply the inverse process based on a "goodness" criterion		→ Manipulate an image heuristically in order to take advantage of the psychophysical aspects of the human visual system (HVS).

1. <u>Degradation Model</u> : $g(x,y) = H[f(x,y)] + n(x,y)$



* Some definitions:

Additivity

$$H[f_1(x,y)+f_2(x,y)] = H[f_1(x,y)] + H[f_2(x,y)]$$

Homogeneity

$$H[k_1 f_1(x,y)] = k_1 H[f_1(x,y)]$$

Linearity

$$H[k_1 f_1(x,y)+k_2 f_2(x,y)] = k_1 H[f_1(x,y)] + k_2 H[f_2(x,y)]$$

Space-Invariance

$$H[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta) \quad \left(\text{where } g(x,y) = H[f(x,y)]\right.$$

* Degradation Model for Continuous Functions :

$f(x,y)$ can be expressed as follows: $f(x,y) = \iint f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)\,d\alpha\,d\beta$

$$\Rightarrow g(x,y) = H[f(x,y)] + n(x,y)$$

$$= H\left[\iint f(\alpha,\beta)\,\delta(x-\alpha,y-\beta)\,d\alpha\,d\beta\right] + n(x,y)$$

$$= \iint H[f(\alpha,\beta)\,\delta(x-\alpha,y-\beta)]\,d\alpha\,d\beta + n(x,y)$$

if H is linear

$$= \iint f(\alpha,\beta)\,\underbrace{H[\delta(x-\alpha,y-\beta)]}\,d\alpha\,d\beta + n(x,y)$$

Homogeneity

"impulse response of H"

"point spread function (PSF)"

$$\text{Let} \quad h(x,y;\alpha,\beta) \triangleq H[\delta(x-\alpha,y-\beta)]$$

$$\Rightarrow \boxed{g(x,y) = \iint f(\alpha,\beta)\,h(x,y;\alpha,\beta)\,d\alpha\,d\beta}$$

"Fredholm integral of the first kind"

If H is space-invariant, then $H[\delta(x-\alpha,y-\beta)] = h(x-\alpha,y-\beta)$

$$\Rightarrow \boxed{\begin{aligned} g(x,y) &= \iint f(\alpha,\beta)\,h(x-\alpha,y-\beta)\,d\alpha\,d\beta + n(x,y) \\ &= f(x,y) * h(x,y) + n(x,y) \end{aligned}}$$

* Discrete Formulation :

Let's start with the 1-D case and temporarily neglect the noise term

Suppose $f(x)$ and $h(x)$ are sampled uniformly to form arrays $f[n]$ and $h[n]$

$$f[n] = f[0], f[1], \dots, f[A-1]$$
$$h[n] = h[0], h[1], \dots, h[B-1]$$

Remember that the discrete convolution formulation is based on the assumption that the sampled functions are periodic, with a period N.

$$N \geq A+B-1$$

$f[n]$ and $h[n]$ are extended to avoid overlap.

Let $$f_e[n] = \begin{cases} f[n], & \text{for } 0 \leq n \leq A-1 \\ 0, & \text{for } A \leq n \leq N-1 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & \text{for } 0 \leq n \leq B-1 \\ 0, & \text{for } B \leq n \leq N-1 \end{cases}$$

$$f_e[n] * h_e[n] \overset{\text{DFT}}{\Longleftrightarrow} F_e[k] H_e[k] \qquad \text{(convolution property)}$$

Let $g_e[n]$ be the convolution of $f_e[n]$ and $h_e[n]$:

$$\boxed{g_e[n] \triangleq f_e[n] * h_e[n] = \sum_{m=0}^{N} f_e[m] h_e[n-m]}$$

($g_e[n]$ is also periodic with N)

In matrix notation

$$\begin{bmatrix} g_e[0] \\ g_e[1] \\ \vdots \\ g_e[N-1] \end{bmatrix}_{N\times 1} = \begin{bmatrix} h_e[0] & h_e[-1] & h_e[-2] & \cdots & h_e[-(N-1)] \\ h_e[1] & h_e[0] & h_e[-1] & \cdots & h_e[-(N-2)] \\ \vdots & \vdots & \vdots & & \vdots \\ h_e[N-1] & h_e[N-2] & h_e[N-3] & \cdots & h_e[0] \end{bmatrix}_{N\times N} \begin{bmatrix} f_e[0] \\ f_e[1] \\ \vdots \\ f_e[N-1] \end{bmatrix}_{N\times 1}$$

$$\underbrace{}_{\underline{g}} \qquad \underbrace{}_{H} \qquad \underbrace{}_{f}$$

$$\boxed{g = Hf}$$

Using the periodicity $h_e[n] = h_e[n+N]$

$$H = \begin{bmatrix} h_e[0] & h_e[N-1] & h_e[N-2] & \cdots & h_e[1] \\ h_e[1] & h_e[0] & h_e[N-1] & \cdots & h_e[2] \\ \vdots & \vdots & \vdots & & \vdots \\ h_e[N-1] & h_e[N-2] & h_e[N-3] & \cdots & h_e[0] \end{bmatrix}$$

$\Rightarrow$ The rows are related by a <u>circular shift</u> to the right.
(The right-most element in one row is equal to the left-most element in the row immediately below.)

$\Rightarrow$ H is a <u>circulant matrix</u>.

Example: Suppose $A = 4$, $B = 3$; and we choose $N = 6$

$$\begin{bmatrix} g_e[0] \\ g_e[1] \\ g_e[2] \\ g_e[3] \\ g_e[4] \\ g_e[5] \end{bmatrix} = \begin{bmatrix} h_e[0] & h_e[5] & h_e[4] & h_e[3] & h_e[2] & h_e[1] \\ h_e[1] & h_e[0] & h_e[5] & h_e[4] & h_e[3] & h_e[2] \\ h_e \vdots & & & & & \\ & & & & & \\ & & & & & \\ h_e[5] & h_e[4] & h_e[3] & h_e[2] & h_e[1] & h_e[0] \end{bmatrix} \begin{bmatrix} f_e[0] \\ f_e[1] \\ f_e[2] \\ f_e[3] \\ f_e[4] \\ f_e[5] \end{bmatrix}$$

$$= \begin{bmatrix} h[0] & 0 & 0 & 0 & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & 0 & h[2] \\ h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad g = Hf$$

when there is noise $\quad n \triangleq \begin{bmatrix} n[0] \\ n[1] \\ \vdots \\ n[N-1] \end{bmatrix}$

$$\boxed{g = Hf + n}$$

* Extension to 2-D signals

Let $f[m,n]$ and $h[m,n]$ have sizes $A \times B$ and $C \times D$, respectively.

The extended images of size $M \times N$ are

$$f_e[m,n] = \begin{cases} f[m,n], & 0 \leq m \leq A-1 \text{ and } 0 \leq n \leq B-1 \\ 0, & A \leq m \leq M-1 \text{ or } B \leq n \leq N-1 \end{cases}$$

$$h_e[m,n] = \begin{cases} h[m,n], & 0 \leq m \leq C-1 \text{ and } 0 \leq n \leq D-1 \\ 0, & C \leq m \leq M-1 \text{ or } D \leq n \leq N-1 \end{cases}$$

$$\Rightarrow g_e[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f_e[k,l]\, h_e[m-k, n-l]$$

* Vector-matrix form:

$$\underset{M \times N}{\begin{bmatrix} f_e[m,n] \end{bmatrix}} \xrightarrow{\text{stack}} \underset{MN \times 1}{\begin{bmatrix} f_e[0,0] \\ f_e[0,1] \\ \vdots \\ f_e[M-1,N-1] \end{bmatrix}}$$

$$\underset{MN \times 1}{\begin{bmatrix} g \end{bmatrix}} = \underset{MN \times MN}{\begin{bmatrix} H \end{bmatrix}} \underset{MN \times 1}{\begin{bmatrix} f \end{bmatrix}} + \underset{MN \times 1}{\begin{bmatrix} n \end{bmatrix}}$$

$$\boxed{g = Hf + n}$$

$\mathcal{H}$ is a block-circulant matrix

$$\mathcal{H} = \begin{bmatrix} [H_0] & [H_{M-1}] & \cdots & [H_1] \\ [H_1] & [H_0] & \cdots & [H_2] \\ \vdots & \vdots & & \vdots \\ [H_{M-1}] & [H_{M-2}] & \cdots & [H_0] \end{bmatrix}_{MN \times MN}$$

$[H_0] \rightarrow$ size $= N \times N$

$H_j \rightarrow$ corresponds to the j^{th} row

$\rightarrow$ has dimension $N \times N$

$\rightarrow$ is circulant matrix.

2. Diagonalization of Circulant and Block-Circulant Matrices:

Consider an $N \times N$ circulant matrix H:

$$H = \begin{bmatrix} h_e[0] & h_e[N-1] & h_e[N-2] & \cdots & h_e[1] \\ h_e[1] & h_e[0] & h_e[N-1] & \cdots & h_e[2] \\ h_e[2] & h_e[1] & h_e[0] & \cdots & h_e[3] \\ \vdots & \vdots & \vdots & & \\ h_e[N-1] & h_e[N-2] & h_e[N-3] & \cdots & h_e[0] \end{bmatrix}$$

Let

$$w[k] \triangleq \begin{bmatrix} 1 \\ e^{j\frac{2\pi}{N}k} \\ e^{j\frac{2\pi}{N}2k} \\ \vdots \\ e^{j\frac{2\pi}{N}(N-1)k} \end{bmatrix}, \quad k = 0, 1, \ldots, N-1$$

It can be shown that

$$\underset{\substack{\text{a scalar} \\ \text{"eigenvalue"}}}{H\, w[k] = \lambda_k}\; \underset{\text{"eigenvector of } H\text{"}}{w[k]}$$

where

$$\lambda_k = h_e[0] + h_e[N-1]e^{j\frac{2\pi}{N}k} + h_e[N-2]e^{j\frac{2\pi}{N}2k} + \cdots + h_e[1]e^{j\frac{2\pi}{N}(N-1)k}.$$

Putting $w[0], \ldots, w[N-1]$ into the columns of a matrix, we can get

$$H \underbrace{\begin{bmatrix} w[0] & w[1] & \cdots & w[N-1] \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} w[0] & w[1] & \cdots & w[N-1] \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} \lambda_0 & & & \underset{\sim}{0} \\ & \lambda_1 & & \\ & & \ddots & \\ \underset{\sim}{0} & & & \lambda_{N-1} \end{bmatrix}}_{D}$$

$$\left(W[k,\ell] = e^{j\frac{2\pi}{N}k\ell} \right)$$

↳ Diagonal matrix (All elements other than the diagonals are zero.)

$$\Rightarrow \boxed{HW = WD}$$

* Let W^{-1} be the inverse of W. $(WW^{-1} = W^{-1}W = I)$

It can be shown that $W^{-1}[k,\ell] = \frac{1}{N} e^{-j\frac{2\pi}{N}k\ell}$.

$$\Rightarrow \boxed{W^{-1} \text{ is the DFT matrix!}}$$

* Remember that $H_e[k] = \frac{1}{N} \sum_n h_e[n] e^{-j\frac{2\pi}{N}kn} \Longrightarrow \boxed{\lambda_k = N H_e[k]}$

* Effects of Diagonalization on the Degradation Model:

(Note that: $HW = WD \Rightarrow H = WDW^{-1}$ and $D = W^{-1}HW$)

$$g = Hf$$
$$= WDW^{-1}f$$

$$\Rightarrow W^{-1}g = DW^{-1}f$$

Remember that $W^{-1}[k,l] = \frac{1}{N} e^{-j\frac{2\pi}{N}kl}$

$$\Rightarrow \boxed{W^{-1}f \text{ is the DFT of } f\,!}$$

$$\left(W^{-1} = \frac{1}{N}\begin{bmatrix} 1 & e^{-j\frac{2\pi}{N}(0)1} & e^{-j\frac{2\pi}{N}(0)2} & \cdots & e^{-j\frac{2\pi}{N}(0)(N-1)} \\ 1 & e^{-j\frac{2\pi}{N}(1)(1)} & e^{-j\frac{2\pi}{N}(1)(2)} & \cdots & e^{-j\frac{2\pi}{N}(1)(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)(1)} & e^{-j\frac{2\pi}{N}(N-1)2} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix} \right)$$

Also remember that $D = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix}$

where $\lambda_k = h_e[0] + h_e[N-1]e^{j\frac{2\pi}{N}k} + h_e[N-2]e^{j\frac{2\pi}{N}2k} + \dots + h_e[1]e^{j\frac{2\pi}{N}(N-1)k}$

Note that: $e^{j\frac{2\pi}{N}(N-l)k} = e^{-j\frac{2\pi}{N}lk}$

$$\Rightarrow \lambda_k = h_e[0] + h_e[1]e^{-j\frac{2\pi}{N}k} + \dots + h_e[N-1]e^{-j\frac{2\pi}{N}(N-1)k}$$
$$= \underline{\underline{N H_e[k]}}$$

When there is noise in the process:

$$g = Hf + n$$

$$\Rightarrow \underbrace{W^{-1}g}_{\text{DFT of } g} = \underbrace{D}_{N\times N \text{ diagonal matrix}} \underbrace{W^{-1}f}_{\text{DFT of } f} + W^{-1}n$$

* DFTs can be computed using fast Fourier Transform algorithm. This provides computational simplification.

* 1-D result can be extended to 2-D signals:

$$W^{-1}g = D\,W^{-1}f + W^{-1}n$$

D: $\left(\begin{matrix} MN\times MN \\ \text{diagonal matrix} \end{matrix}\right)$ $\quad$ f: (MN×1 vector)

$$MN\begin{bmatrix} H[0,0] & & \\ & H[0,1] & \\ & & \ddots \end{bmatrix}$$

* [illegible]

$$W^{-1}f \rightarrow [F[0,0], F[0,1], \ldots, F[M-1,N-1]]^T$$

$$W^{-1}g \rightarrow [G[0,0], G[0,1], \ldots, G[0,N-1], G[1,0], G[1,1], \ldots, G[M-1,N-1]]^T$$

$$\Rightarrow \boxed{G[k,l] = \underbrace{MN\,H_0[k,l]}_{H[k,l]}\,F[k,l] + N[k,l]}$$

$$\Rightarrow \boxed{G[k,l] = H[k,l]\,F[k,l] + N[k,l]}$$

3. Algebraic Approach to Restoration :

Preliminaries :

Let $x = [x_1 \; x_2 \; \cdots \; x_N]^T$ be an $N \times 1$ vector

$$\Rightarrow \quad \frac{\partial}{\partial x} \triangleq \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_N} \end{bmatrix}$$

$$\frac{\partial}{\partial x}(x^T A) = A$$

$$\frac{\partial}{\partial x}(B^T x) = B$$

$$\frac{\partial}{\partial x}(x^T A x) = 2Ax$$

3.1 Unconstrained Restoration

$\boxed{g = Hf + n}$

$$\hat{f} = \arg\min_f \left\{ \| g - Hf \|^2 \right\}$$

$\Rightarrow$ Take the derivative of $\| g - Hf \|^2$ with respect to f, set it to zero, solve for f.

$$\frac{\partial}{\partial f} \| g - Hf \|^2 = \frac{\partial}{\partial f} (g - Hf)^T (g - Hf) = \frac{\partial}{\partial f} (g^T - f^T H^T)(g - Hf)$$

$$= -2H^T g + 2H^T H f = 0 \Rightarrow \boxed{\hat{f} = (H^T H)^{-1} H^T g}$$

When H is a square matrix and H^{-1} exists,

$$\hat{f} = (H^T H)^{-1} H^T g = H^{-1} H^{-T} H^T g = H^{-1} g \quad //$$

* DFT-domain formulation:

Remember $H = WDW^{-1}$ and $W^{-1} = \frac{1}{N} W^{*T}$

H is real $\Rightarrow H^* = H \Rightarrow W^* D^* W^{-*} = WDW^{-1}$

$$\Rightarrow H^T = W^{-T} D W^T \underset{H^T = H^{*T}}{=} W^{-*T} D^* W^{*T} = \frac{1}{N} W D^* N W^{-1} = W D^* W^{-1}$$

$$\begin{aligned}
\Rightarrow \hat{f} &= (H^T H)^{-1} H^T g = \left(W D^* W^{-1} W D W^{-1} \right)^{-1} W D^* W^{-1} g \\
&= \left(W D^* D W^{-1} \right)^{-1} W D^* W^{-1} g \\
&= W (D^* D)^{-1} \underbrace{W^{-1} W}_{I} D^* W^{-1} g
\end{aligned}$$

$$\Rightarrow \underbrace{W^{-1} \hat{f}}_{\substack{\hat{F} \\ \text{"DFT of } \hat{f}\text{"}}} = (D^* D)^{-1} D^* \underbrace{W^{-1} g}_{\substack{G \\ \text{"DFT of } g\text{"}}}$$

$$\Rightarrow \boxed{\hat{F} = (D^* D)^{-1} D^* G} \quad \text{where } D = \begin{bmatrix} H[0,0] & & \\ & \ddots & \\ & & H[M,N] \end{bmatrix}$$

$$\Rightarrow \boxed{\hat{F}[k,l] = \frac{H^*[k,l]}{|H[k,l]|^2} G[k,l]}$$

3.2. Constrained Restoration

Define a cost function $C(f) \triangleq \|g - Hf\|^2 + \gamma \|Qf\|^2$

a scalar

constrains the solution (Minimizing $\|g-Hf\|^2$ alone may yield a solution that suffers from large oscillating values. The term $\gamma\|Qf\|^2$ is used to prevent that.)

$$\hat{f} = \underset{f}{\operatorname{argmin}} \{ C(f) \}$$

$$\Rightarrow \frac{\partial C(f)}{\partial f} = -2H^T g + 2H^T H f + \gamma 2 Q^T Q f = 0$$

$$\Rightarrow \boxed{\hat{f} = (H^T H + \gamma Q^T Q)^{-1} H^T g}$$

Let $Q = W \begin{bmatrix} P[0,0] & & \\ & \ddots & \\ & & P[M,N] \end{bmatrix} W^{-1}$ be the diagonalization, where $P[k,l]$ is the DFT of a filter $p[m,n]$.

Similar to the unconstrained case, we can write the DFT-domain formulation:

$$\boxed{\hat{F}[k,l] = \frac{H^*[k,l]}{|H[k,l]|^2 + \gamma |P[k,l]|^2} G[k,l]}$$

4. Inverse Filtering

$$G[k,l] = H[k,l]\,F[k,l] + N[k,l]$$

Inverse filtering: $\hat{F}[k,l] = \dfrac{G[k,l]}{H[k,l]}$

When the noise term $N[k,l]$ is nonzero:

$$\hat{F}[k,l] = \frac{H[k,l]\,F[k,l] + N[k,l]}{H[k,l]} = F[k,l] + \frac{N[k,l]}{H[k,l]}$$

This term could dominate!

5. Wiener Filter

$$g[n] = h[n] * f[n] + \underset{\text{noise}}{\underset{\uparrow}{\eta[n]}} = \sum_{\ell} h[\ell] f[n-\ell] + \eta[n]$$

We are looking for a filter, $w[n]$, such that $E\{|w[n] * g[n] - f[n]|^2\}$ is minimum, where $E\{\cdot\}$ is the expectation operator.

$$\varepsilon \triangleq E\left\{\left|\sum_{\ell} w[\ell]\, g[n-\ell] - f[n]\right|^2\right\}$$

$$= E\left\{\left(\sum_{\ell} w[\ell]\, g[n-\ell] - f[n]\right)\left(\sum_{\ell} w[\ell]\, g[n-\ell] - f[n]\right)^*\right\}$$

$$= E\left\{\left(\sum_{\ell} w[\ell]\, g[n-\ell]\right)\left(\sum_{\ell} w^*[\ell]\, g^*[n-\ell]\right) - \sum_{\ell} w[\ell]\, g[n-\ell]\, f^*[n]\right.$$

$$\left. - f[n] \sum_{\ell} w^*[\ell]\, g^*[n-\ell] + f[n]\, f^*[n]\right\}$$

To find $w[k]$, $\dfrac{\partial \varepsilon}{\partial w^*[k]} = 0$

$$\Rightarrow E\left\{\left(\sum_{\ell} w[\ell]\, g[n-\ell]\right) g^*[n-k] - f[n]\, g^*[n-k]\right\} = 0$$

$$\Rightarrow \sum_{\ell} w[\ell]\, E\{g[n-\ell]\, g^*[n-k]\} = E\{f[n]\, g^*[n-k]\}$$

Define $R_{gg}[k-\ell] \triangleq E\{g[n-\ell]\, g^*[n-k]\}$ and

$$R_{fg}[k] \triangleq E\{f[n]\, g^*[n-k]\}$$

$$\Rightarrow \sum_{l} w[l]\, R_{gg}[k-l] = R_{fg}[k]$$

$$\Rightarrow \boxed{\underbrace{R_{gg}[k]}_{\text{"autocorrelation"}} * w[k] = \underbrace{R_{fg}[k]}_{\text{"cross-correlation"}}}$$

Let $S_{gg}(u)$ and $S_{fg}(u)$ be the Fourier Transform of $R_{gg}[k]$ and $R_{fg}[k]$, respectively.

Then

$$S_{gg}(u)\, W(u) = S_{fg}(u) \quad \Rightarrow \quad \boxed{W(u) = \frac{S_{fg}(u)}{S_{gg}(u)}}$$

They are called "power spectral density"

FT of $w[k]$

Assume $f[n]$ and $\eta[n]$ are uncorrelated. That is, $E\{f[n]\,\eta^*[n-k]\} = 0$

Using $g[n] = h[n] * f[n] + \eta[n]$, we can find that

$$R_{fg}[k] = h^*[-k] * R_{ff}[k]$$

Using the convolution property $\Rightarrow \boxed{S_{fg}(u) = H^*(u)\, S_{ff}(u)}$

$\left(\text{Note that } H(u) = \sum_{l} h[l]\, e^{-j\frac{2\pi}{N}ul} \Rightarrow H^*(u) = \sum_{l} h^*[-l]\, e^{-j\frac{2\pi}{N}ul} \right)$

Similarly, it can be shown that

$$R_{gg}[k] = E\{g[n]\,g^*[n-k]\} = h[k] * h^*[-k] * R_{ff}[k] + R_{\eta\eta}[k]$$

$$\Rightarrow \boxed{S_{gg}(u) = H(u)H^*(u)\,S_{ff}(u) + S_{\eta\eta}(u)}$$

As a result, the Wiener filter, $W(u)$, is

$$W(u) = \frac{S_{fg}(u)}{S_{gg}(u)} = \frac{H^*(u)\,S_{ff}(u)}{H(u)H^*(u)\,S_{ff}(u) + S_{\eta\eta}(u)}$$

$$= \frac{H^*(u)}{|H(u)|^2 + \dfrac{S_{\eta\eta}(u)}{S_{ff}(u)}}$$

As $\dfrac{S_{\eta\eta}(u)}{S_{ff}(u)} \to 0$, the Wiener filter becomes $W(u) \approx \dfrac{1}{H(u)}$, the inverse filter!