1. (a) A sample path of the process is obtained when the outcome of all the random variables \( b_i \) are observed. An example is shown below for \(-2T \leq t \leq 3T\).

\[ b_{-2} = -1 \quad b_{-1} = 1 \quad b_0 = 1 \quad b_1 = -1 \quad b_2 = -1 \]

(b)

\[
m_X(t) = E\left[ \sum_{i=-\infty}^{\infty} b_i p_T(t-iT) \right] = \sum_{i=-\infty}^{\infty} E[b_i] p_T(t-iT)
\]

But since \( E[b_i] = 0 \) for all \( i \), we have \( m_X(t) = 0 \).

(c)

\[
R_X(.5T, .7T) = E[X(.5T)X(.7T)]
\]

But \( X(.5T) = b_0 \) and \( X(.7T) = b_0 \). (See the sample function above.) Therefore,

\[
R_X(.5T, .7T) = E[b_0^2] = 1
\]

\[
R_X(.9T, 1.1T) = E[X(.9T)X(1.1T)]
\]

But \( X(.9T) = b_0 \) and \( X(1.1T) = b_1 \). (See the sample function above.) Therefore,

\[
R_X(.9T, 1.1T) = E[b_0b_1] = E[b_0]E[b_1] = 0
\]

(d) From part c, the process is not WSS.

2. (a)

\[
E[Z(t)] = E[X] \cos(2\pi t) + E[Y] \sin(2\pi t) = 0
\]

\[
R_{ZZ}(t, t+\tau) = E\left\{ [X \cos(2\pi t) + Y \sin(2\pi t)] [X \cos(2\pi (t+\tau) + Y \sin(2\pi (t+\tau))] \right\}
\]

\[
= E[X^2] \cos(2\pi t) \cos(2\pi (t+\tau)) + E[XY] \cos(2\pi t) \sin(2\pi (t+\tau))
\]

\[
+ E[XY] \sin(2\pi t) \cos(2\pi (t+\tau)) + E[Y^2] \sin(2\pi t) \sin(2\pi (t+\tau))
\]

\[
= \cos(2\pi \tau)
\]

The above shows that the process is WSS.
(b) Let \( n \) and times \( t_1, t_2, \cdots, t_n \) be arbitrary. Then the vector
\[
[Z_{t_1}, Z_{t_2}, \cdots, Z_{t_n}]^T = [\cos(2\pi t_1), \cos(2\pi t_1), \cdots, \cos(2\pi t_1)]^T X \\
+ [\sin(2\pi t_1), \sin(2\pi t_1), \cdots, \sin(2\pi t_1)]^T Y
\]
is a linear combination of \((X, Y)\) and thus it is a Gaussian random vector. This implies that the process \( \{Z(t)\} \) is Gaussian.

(c) The covariance matrix is obtained from the autocorrelation function and the mean function.
\[
\Lambda = \begin{bmatrix}
1 & \cos\left(\frac{2\pi}{4}\right) & \cos\left(\frac{2\pi}{2}\right) \\
\cos\left(\frac{2\pi}{4}\right) & 1 & \cos\left(\frac{2\pi}{4}\right) \\
\cos\left(\frac{2\pi}{2}\right) & \cos\left(\frac{2\pi}{4}\right) & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 1 & 0
\end{bmatrix}
\]
Since \( \det(\Lambda) \neq 0 \), the PDF does exist.

3. (a) \( E[Y_k] = E[X^k - \frac{1}{k+1}] = 0 \)
because \( E[X^k] = \frac{1}{k+1} \).

\[
R_{YY}(k, l) = E[Y_k Y_l] = E\left[(X^k - \frac{1}{k+1})(X^l - \frac{1}{l+1})\right]
\]
\[
= E[X^{k+l}] - \frac{1}{(k+1)(l+1)} - \frac{1}{(k+1)(l+1)} + \frac{1}{(k+1)(l+1)}
\]
\[
= \frac{1}{k+l+1} - \frac{1}{(k+1)(l+1)}
\]
\[
= \frac{k l}{(k+l+1)(k+1)(l+1)}
\]
Since this is not a function of \( k - l \) only, the process is not WSS.

(b) \( F_{Y_k}(y) = P(Y_k \leq y) = P(X^k \leq y + \frac{1}{k+1}) \)
For \( 0 \leq y + \frac{1}{k+1} < 1 \),
\[
F_{Y_k}(y) = P[X \leq (y + \frac{1}{k+1})^{1/k}] = [y + \frac{1}{k+1}]^{1/k}
\]
Therefore,
\[
F_{Y_k}(y) = \begin{cases}
0 & y < -\frac{1}{k+1} \\
[y + \frac{1}{k+1}]^{1/k} & -\frac{1}{k+1} \leq y < 1 - \frac{1}{k+1} \\
1 & 1 - \frac{1}{k+1} \leq y
\end{cases}
\]
4. (a) $X_t$ and $Y_t$ are independent and have zero mean. Now

$$\text{var}(X_t) = E[X^2_t] = \int_{-\infty}^{\infty} S_{XX}(f) \, df = \int_{-5}^{5} df = 10$$

and

$$\text{var}(Y_t) = E[Y^2_t] = R_{YY}(0) = 1$$

Thus

$$f_{X_tY_t}(x, y) = \frac{1}{2\pi \sqrt{10}} e^{-\frac{x^2 + y^2}{20}}$$

(b) Since the input processes are independent, so are the output processes. Now

$$E[V_{s_1}] = m_X \int h_1(x) \, dx = 0$$

and

$$E[W_{s_2}] = m_Y \int h_2(x) \, dx = 0$$

Also

$$\text{var}(V_{s_1}) = E[V^2_{s_1}] = \int_{-\infty}^{\infty} S_{XX}(f)|H_1(f)|^2 \, df = \int_{-1}^{1} a_1^2 \, df = 2a_1^2$$

$$\text{var}(W_{s_1}) = E[W^2_{s_1}] = \int_{-\infty}^{\infty} S_{YY}(f)|H_2(f)|^2 \, df$$

Now

$$S_{YY}(f) = \frac{1}{1 + \pi^2 f^2}$$

Thus

$$\text{var}(W_{s_1}) = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{a_2^2}{1 + \pi^2 f^2} \, df = \sigma^2$$

where $\sigma^2$ is the value of the integral. Then

$$F_{V_{s_1}W_{s_2}}(x, y) = \frac{1}{2\pi a_1 \sigma \sqrt{2}} e^{-\frac{x^2}{4a_1^2} + \frac{y^2}{2\sigma^2}}$$

5. (a) Since $E[X_t] = 0$, we get $E[Y_t] = E[Y_t] = 0$. Now

$$\text{var}(Y_t) = E[Y^2_t] = R_{YY}(0) = \int_{-\infty}^{\infty} S_{YY}(f) \, df$$

But

$$S_{YY}(f) = S_{XX}(f)|H_1(f)|^2 = \begin{cases} 4f^4 & |f| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\text{var}(Y_t) = \int_{-1}^{1} 4f^4 \, df = \frac{8}{5}$$
(b) Similarly
\[ S_{ZZ}(f) = S_{XX}(f)|H_2(f)|^2 = \begin{cases} 
  f^2 & \text{if } 1 \leq |f| \leq 2 \\
  0 & \text{otherwise}.
\end{cases} \]

and
\[ \text{var}(Z_t) = 2 \int_1^2 f^2 \, df = \frac{14}{3} \]

(c) \[ \text{Cov}(Y_{t_1}, Z_{t_2}) = E[Y_{t_1}Z_{t_2}] = R_{YZ}(t_1, t_2) \]

We know that \{Y_t\} and \{Z_t\} are jointly WSS. Thus let \( \tau = t_1 - t_2 \). Then
\[ \text{Cov}(Y_{t_1}, Z_{t_2}) = R_{YZ}(\tau) = \mathcal{F}^{-1}\{S_{YZ}(f)\} \]

But
\[ S_{YZ}(f) = S_{XX}(f)H_1(f)H_2(-f) = 0 \]

Thus \( R_{YZ}(\tau) = 0 \) and \( \text{Cov}(Y_{t_1}, Z_{t_2}) = 0 \).

6. \[ R_Y(\tau) = E[Y(t)Y(t+\tau)] \]
\[ = E[X(t)\cos(2\pi f_1 t + \Theta)X(t+\tau)\cos(2\pi f_1 (t+\tau) + \Theta)] \]
\[ = E[X(t)X(t+\tau)]E[\cos(2\pi f_1 t + \Theta)\cos(2\pi f_1 (t+\tau) + \Theta)] \]
\[ = R_X(\tau)\{1/2\cos(2\pi f_1 \tau) + E[\cos(2\pi f_1 (2t+\tau) + 2\Theta)]\} \]
\[ = 1/2R_X(\tau)\cos(2\pi f_1 \tau) \]

Therefore, using the convolution
\[ S_Y(f) = \frac{1}{4} S_X(f) \ast [\delta(f-f_1) + \delta(f+f_1)] \]
\[ = \frac{1}{4} [S_X(f-f_1) + S_X(f+f_1)] \]
\[ = \frac{\sigma^2}{8f_0} [\text{rect}(\frac{f-f_1}{2f_0}) + \text{rect}(\frac{f+f_1}{2f_0})] \]

where
\[ \text{rect}(x) = \begin{cases} 
  1 & -1/2 \leq x \leq 1/2 \\
  0 & \text{otherwise}.
\end{cases} \]

7. By inspection we see that
\[ h(t) = \begin{cases} 
  \frac{1}{T} & -T \leq t \leq T \\
  0 & \text{otherwise}
\end{cases} \]

and therefore,
\[ H(f) = \frac{\sin(2\pi f T)}{2\pi f T} \]

and
\[ S_Y(f) = S_X(f) |H(f)|^2 \]
8. (a) Three sample paths are plotted below for $t \in [0, 1]$ for the three outcomes $Y = 0$, $Y = -1$ and $Y = -2$.

(b) $X_t$ takes values in the set 
\[ \{1, e^{-t}, e^{-2t}, e^{-3t}, \ldots \} \]
Therefore $X_t$ is a discrete random variable. It has a probability mass function given by
\[ P(X_t = e^{-kt}) = P(Y = -k) = (1 - p)p^k \quad \text{for} \quad k = 0, 1, 2, \ldots \]

(c) \[ E[X_t] = \sum_{k=0}^{\infty} (1 - p)p^k e^{-kt} = \frac{1 - p}{1 - pe^{-t}} \]

9. We first need to show that the process is wide sense stationary. For this we compute the mean and autocorrelation function of the process.

(a) \[ E[X(t)] = E[a \cos(2\pi Z t + \Theta)] = a E_Z \{ E_\Theta[\cos(2\pi Z t + \Theta)|Z] \} \]

Now for any $\alpha$, 
\[ E_\Theta[\cos(2\pi Z t + \Theta)|Z = \alpha] = E_\Theta[\cos(2\pi \alpha t + \Theta)|Z = \alpha] = E_\Theta[\cos(2\pi \alpha t + \Theta)] = 0 \]
where the equality before last follows from the independence of $Z$ and $\Theta$. It then follows that $E[X(t)] = 0$. 

5
Again by conditioning on \( Z = \alpha \) we get
\[
E[X(t)X(t+\tau)|Z=\alpha] = \frac{a^2}{2} \int_{-\infty}^{\infty} \cos(2\pi \alpha t) f_Z(\alpha) \, d\alpha
\] (1)

Parts (a) and (b) above show that \( \{X(t)\} \) is WSS. Therefore the Fourier transform of \( R_X(\tau) \) is the power spectral density. Now since \( f_Z(\alpha) \) is an even function, equation (1) above shows that \( R_X(\tau) \) and \( f_Z(\alpha) \) form a Fourier transform pair. This shows that
\[
S_X(f) = \frac{a^2}{2} f_Z(f).
\]
To make sure find the inverse Fourier transform of \( \frac{a^2}{2} f_Z(f) \).

\[
\mathcal{F}^{-1}\left\{ \frac{a^2}{2} f_Z(f) \right\} = \frac{a^2}{2} \int_{-\infty}^{\infty} \cos(2\pi f \tau) f_Z(f) \, df
\]

But the last integral is zero since the integrand is an odd function. Thus
\[
\mathcal{F}^{-1}\left\{ \frac{a^2}{2} f_Z(f) \right\} = \frac{a^2}{2} \int_{-\infty}^{\infty} \cos(2\pi f \tau) f_Z(f) \, df = R_X(\tau)
\]
Thus \( \frac{a^2}{2} f_Z(f) \) must be the power spectral density.