1. (a) We know \((Y_1, Y_2)\) takes on values in the set \(\{(0, 0), (0, 1), (1, 0), (1, 1)\}\). Now given that \(X = x\), \(Y_1\) and \(Y_2\) are conditionally independent. Thus
\[
P[(Y_1, Y_2) = (0, 0)|X = x] = P(Y_1 = 0|X = x)P(Y_2 = 0|X = x) = (1 - x)^2
\]
\[
P[(Y_1, Y_2) = (0, 1)|X = x] = P(Y_1 = 0|X = x)P(Y_2 = 1|X = x) = x(1 - x)
\]
\[
P[(Y_1, Y_2) = (1, 0)|X = x] = P(Y_1 = 1|X = x)P(Y_2 = 0|X = x) = x(1 - x)
\]
\[
P[(Y_1, Y_2) = (1, 1)|X = x] = P(Y_1 = 1|X = x)P(Y_2 = 1|X = x) = x^2
\]
Now
\[
P[(Y_1, Y_2) = (\alpha, \beta)] = \int_0^1 P[(Y_1, Y_2) = (0, 0)|X = x] f_X(x) \, dx
\]
Using this formula we get
\[
P[(Y_1, Y_2) = (0, 0)] = \frac{1}{3},
\]
\[
P[(Y_1, Y_2) = (0, 1)] = P[(Y_1, Y_2) = (1, 0)] = \frac{1}{6},
\]
\[
P[(Y_1, Y_2) = (1, 1)] = \frac{1}{3}
\]
(b) We need to show that
\[
P[(Y_1, Y_2) = (\alpha, \beta)] = P[Y_1 = \alpha]P[Y_2 = \beta]
\]
for all \(\alpha\) and \(\beta\). From the above we can calculate that
\[
P[Y_1 = 0] = P[Y_2 = 0] = \frac{1}{2}
\]
Thus
\[
\frac{1}{3} = P[(Y_1, Y_2) = (0, 0)] \neq P[Y_1 = 0]P[Y_2 = 0] = \frac{1}{4}
\]
Thus the random variables are dependent.

2. (a)
\[
\text{var}(Y) = E[|Y - E(Y)|^2] = E[\sum_{i=1}^n X_i - E(Y)]^2
\]
Now since \(E(Y) = \sum_{i=1}^n E(X_i)\), we can write
\[
\text{var}(Y) = E[\sum_{i=1}^n X_i - \sum_{i=1}^n E(X_i)]^2 = E\left(\sum_{i=1}^n [X_i - E(X_i)]\right)^2
\]
\[
= E \left\{ \sum_{i=1}^n \sum_{j=1}^n [X_i - E(X_i)][X_j - E(X_j)] \right\}
\]
\[
= \sum_{i=1}^n \sum_{j=1}^n E[X_i - E(X_i)][X_j - E(X_j)]
\]
\[
= \sum_{i=1}^n \sum_{j=1}^n \text{cov}(X_i, X_j)
\]
(b) When $X_i$ and $X_j$ are uncorrelated for all $i \neq j$,

$$\text{cov}(X_i, X_j) = \begin{cases} \text{var}(X_i) & i = j \\ 0 & i \neq j \end{cases}$$

Thus

$$\text{var}(Y) = \sum_{i=1}^{n} \text{var}(X_i)$$

3. (a) We have

$$E[e^X] = \int_{1}^{\infty} e^x \frac{1}{x^2} \, dx = +\infty$$

This implies that $E[e^X]$ is defined but it is $+\infty$.

(b)

$$E[\sin(X)] = \int_{1}^{\infty} \frac{\sin(x)}{x^2} \, dx$$

This is well defined and finite because

$$E[|\sin(X)|] = \int_{1}^{\infty} \frac{|\sin(x)|}{x^2} \, dx \leq \int_{1}^{\infty} \frac{1}{x^2} \, dx < \infty$$

4. (a) We have

$$E[X_k] = E[\cos(kU)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(k\alpha) \, d\alpha = 0,$$

and

$$\text{var}(X_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(k\alpha) \, d\alpha = \frac{1}{2}$$

$$E[X_iX_j] = E[\cos(iU) \cos(jU)] = E[\frac{1}{2} \cos((i+j)U) + \frac{1}{2} \cos((i-j)U)] = 0$$

Now

$$ES_n = \sum_{k=1}^{n} EX_k = 0$$

and because $X_k$’s are uncorrelated,

$$\text{var}(S_n) = \text{var}(\sum_{k=1}^{n} X_k) = \sum_{k=1}^{n} \text{var}(X_k) = \frac{n}{2}.$$ 

(b) Now using the Chebychev inequality, for any $\epsilon > 0$,

$$P(|\frac{S_n}{n}| \geq \epsilon) \leq \frac{\text{var}(\frac{S_n}{n})}{\epsilon^2} = \frac{1}{n^2\epsilon^2} \text{var}(S_n) = \frac{n}{2n^2\epsilon} = \frac{1}{2n\epsilon}$$

Therefore for any $\epsilon > 0$,

$$P(|\frac{S_n}{n}| \geq \epsilon) \leq \frac{1}{2n\epsilon} \quad \text{for all } n,$$

which implies that $\lim_{n \to \infty} P(|\frac{S_n}{n}| \geq \epsilon) = 0$. 

2
5. Let $X$ denote the time until the miner reaches safety and let $Y$ denote the door he initially chooses. Then by the law of iterated expectations,


$$= 1/3E[X|Y = 1] + 1/3E[X|Y = 2] + 1/3E[X|Y = 3]$$

But

$$E[X|Y = 1] = 2, \ E[X|Y = 2] = 3 + E[X], \text{ and } E[X|Y = 3] = 5 + EX.$$ 

Note: If he chooses door number 2, then he will be back in the tunnel after 3 hours. Therefore given that $Y = 2$, the average number of hours it will take him to reach freedom is $3 + EX$, and similarly for $Y = 3$. Thus

$$EX = \frac{2}{3} + \frac{1}{3}(EX + 3) + \frac{1}{3}(EX + 5)$$

which gives $EX = 10$.

6.

$$E[Y|X = x] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy$$

$$E[Y] = E[E[Y|X]] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf_{Y|X}(y|x) f_X(x) \, dydx$$

$$= \int_{-\infty}^{\infty} f_X(x) \, dx \left[ \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} \, dy \right]$$

$$= \int_{-\infty}^{\infty} xf_X(x) \, dx = EX = m.$$ 

7. $f_X(\alpha) = \frac{1}{\pi(1+\alpha^2)}$ for $-\infty < \alpha < \infty$. We have

$$EX = \int_{-\infty}^{\infty} \alpha f_X(\alpha) \, d\alpha$$

$$= \int_{-\infty}^{0} \alpha \frac{1}{\pi(1+\alpha^2)} d\alpha + \int_{0}^{\infty} \alpha \frac{1}{\pi(1+\alpha^2)} d\alpha$$

$$= -\infty + \infty$$

Therefore $EX$ is undefined.

8. **On the matter of King, Queen and the Pauper**

The sample space should be the collection of all ordered pairs the first component of which is the person to be freed and the second component of which is the person the hangman points to. Thus

$$S = \{(p, k), (p, q), (k, q), (q, k)\}$$
Now we have

The event that the pauper lives \(A = \{(p, k), (p, q)\}\).
The event that the Queen lives \(Q = \{(q, k)\}\).
The event that the King lives \(K = \{(k, q)\}\).
The event that the hangman points to the queen \(B = \{(p, q), (k, q)\}\).

By the first lottery the probability of each one of these events is 1/3. However, \(P((p, k))\) and \(P((p, q))\) is not known, although it is know that their sum is 1/3. Assume \(P((p, k)) = r/3\) and thus \(P((p, q)) = (1 - r)/3\) for some \(0 \leq r \leq 1\). Then the probability that the pauper lives given that the hangman points to the queen is given by

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P((p, q))}{P((p, q)) + P((k, q))} = \frac{r/3}{r/3 + 1/3} = \frac{r}{r + 1}
\]

For \(r = 1/2\), we get \(P(A|B) = 1/3\) which is the same as the unconditional probability of pauper’s freedom. This implies that for \(r = 1/2\), the paupers assessment is incorrect.

Note that

\[
P(\text{pauper lives|queen dies}) = \frac{P(\text{queen dies|pauper lives})P(\text{pauper lives})}{P(\text{queen dies})} = \frac{1 \times 1/3}{2/3} = 1/2.
\]

But this is not what we had wanted.

9. **Game show**

Without loss of generality we can assume that you have selected box 1. The sample space is then given as the collection of all ordered pairs the first component of which is the box with the treasure and the second component of which is the box opened by the game show host. Thus

\[
S = \{(1, 2), (1, 3), (2, 3), (3, 2)\}
\]

Now we have

The event that you have the box with treasure \(A = \{(1, 2), (1, 3)\}\).
The event that the host opens box 2, \(O_2 = \{(1, 2), (3, 2)\}\).
The event that the host opens box 3, \(O_3 = \{(1, 3), (2, 3)\}\).

Now since your first selection is random, \(P(A) = 1/3\). Also \(P[\{(2, 3)\}] = P[\{(3, 2)\}] = 1/3\).

Now suppose that in the event that boxes 2 and 3 are empty the host opens one at random, i.e., \(P(O_2|A) = P(O_3|A) = 1/2\). Then

\[
P(O_2|A) = \frac{P(O_2 \cap A)}{P(A)} = \frac{P[\{(1, 2)\}]}{P(A)} = 1/2
\]

Thus

\[
P(\{(1, 2)\}) = P(\{(1, 3)\}) = 1/6
\]
Then

\[ P(A|O_2) = P(A|O_3) = \frac{1}{3} \]

Thus whether the host opens box 2 or box 3, your chance of winning (if you stay with your selection of box 1) is 1/3. On the other hand if you exchange your box, your probability of winning is \( P(A^c|O_2) \) or \( P(A^c|O_3) \) which is 2/3. Therefore, the right strategy is to exchange your box.