1. $X$ is a random variable with pdf $f_X(x)$. Let $B = \{a \leq X \leq b\}$ and let $Y = g(X)$ for a given function $g$. Write an expression for $E[Y|B]$.

2. $X_1, X_2$ and $X_3$ are independent identically distributed Gaussian random variables each with mean zero and variance 1. Let

$$U = X_1 + 2X_2$$
$$V = 2X_1 - X_2 + X_3 + 3$$

Find the joint density function of $U$ and $V$.

3. Let $X_1, X_2$ and $X_3$ be three independent Gaussian random variables with $E[X_1] = E[X_2] = E[X_3] = 0$ and $\text{var}(X_1) = 1$, $\text{var}(X_2) = 2$, $\text{var}(X_3) = 1$. Let

$$Y_1 = X_1 + X_2 - X_3 + 1$$
$$Y_2 = X_1 - X_2 - 1$$

Find the joint density function of $Y_1$ and $Y_2$.

4. Let $X$ be a Cauchy random variable with density function $f_X(x) = \frac{1}{\pi(1+x^2)}$.

(a) Show that the characteristic function of $X$ is given by $\phi_X(v) = e^{-|v|}$.

(b) Let $X_1, X_2, X_3, ...$ be a sequence of independent and identically distributed random variables all of which are identical to $X$. Let $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Find the density function and the characteristic function of $Y_n$. How do you explain the behavior of $Y_n$ in light of the weak law of large numbers?

5. Uncorrelatedness of two r.v.’s $X$ and $Y$ does not imply independence of $X$ and $Y$. Verify this statement for the two examples given below by checking for (i) independence and (ii) uncorrelatedness.

(a) $f_X(x)$ is symmetrical about the origin and $Y = X^2$.

(b) $f_{XY}(x, y) = \begin{cases} 4|x|y & 0 \leq y \leq |x| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

6. Let $X$ and $Y$ be two independent, identically distributed random variables with probabilities $P(X = k) = P(Y = k) = (1 - p)p^k$ for $k = 0, 1, 2, \cdots$, where $0 < p < 1$. Let $Z = X - Y$.

(a) Calculate the characteristic function of $X$ in closed form (not in infinite sum).

(b) Compute the characteristic function of $Z$. 

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(c) Find the mean and variance of \( X \) by using the characteristic function of \( X \).
(d) Find the mean and variance of \( Z \).

7. \( X_1, X_2, \ldots, X_n \) are \( n \) independent random variables and \( a_1, a_2, \ldots, a_n \) are real constants.
   Let \( Z = \sum_{i=1}^{n} a_i X_i \). Find \( \Phi_Z(w) \) and \( f_Z(z) \) if
   
   (a) \( X_i \) is a Gaussian random variable with mean \( m_i \) and variance \( \sigma_i^2 \) for \( i = 1, 2, \ldots, n \).
   (b) \( X_i \) is a Poisson random variable with mean \( \lambda_i \) for \( i = 1, 2, \ldots, n \) and \( a_i = 1 \) for all \( i \).

8. \( (X_1, X_2, X_3, \ldots) \) are independent random variables such that for every \( i \), \( X_i = 0 \) or 1 and
   \[ P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p \]
   Let
   \[ Z = \frac{1}{n} \sum_{k=1}^{n} X_k. \]
   
   (a) Find \( E[Z] \) and \( \text{var}(Z) \).
   (b) Prove that for any \( \epsilon > 0 \)
   \[ P(|Z - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2} \]
   What conclusion can you draw from the above inequality?
   (c) Let \( N \) be a Poisson random variable with mean \( \lambda \). Assume \( N \) is independent of \( X_1, X_2, X_3, \ldots \), and let \( Y = \sum_{k=0}^{N} X_k \). Find \( E[Y] \).