1. Let $S$ consist of the integers $\{1, 2, 3, \ldots, 14, 15\}$ and let $A_2, A_3, A_5, A_7$ denote the subsets of $S$ consisting of integers that are multiples of $2, 3, 5,$ and $7,$ respectively.

(a) Express each of the following sets in terms of $A_2, A_3, A_5, A_7$ and their complements.

(i) $\{1, 3, 5, 7, 9, 11, 13\}$

(ii) $\{6, 12\}$

(iii) $7$

(iv) $\{5, 10, 14\}$

(v) $\{5, 7, 14\}$

(vi) $\{1, 11, 13\}$

(vii) $\{1, 2, 4, 8, 11, 13\}$

(viii) $\{x : x \in S, x \geq 14\}$

(b) Find a subset of $S$ that cannot be expressed in terms of $A_2, A_3, A_5, A_7$ and their complements (the answer is not unique).

2. Three campus organizations $A, B,$ and $C$ have 57, 49 and 43 members respectively. $A$ and $B$ have 13 members in common, $A$ and $C$ have 7 members in common, $B$ and $C$ have 4 members in common and 1 person belongs to all three. Find the number of people that are members of

(a) $A$ but not $B,$ nor of $C.$

(b) $B$ but not of $A,$ nor of $C.$

(c) $C$ but not of $A,$ nor of $B.$

3. An experiment consists of tossing a coin until a head appears. The outcome is the number of tosses required. Thus $S = \{1, 2, 3, \ldots\}.$ Suppose that each outcome is an event and that $P(n) = 2^{-n}$ (i.e., the probability of the event that $n$ tosses are required is $2^{-n}$). Find $P(A)$ where

(a) $A = \{i : i \text{ is a multiple of } 4\} = \{4, 8, 12, \ldots\}$

(b) $A = \{i : i \leq N\}$

4. A logic circuit has two inputs and one output. Let $A_i$ denote the event that input $i$ is a logical "1" $(i = 0, 1)$ and let $B$ denote the event that the output is a logical "1". Suppose that $P(A_0) = P(A_1) = 0.5$, $P(A_0 \cap A_1) = 0.25$, $P(B) = 0.5$, $P(B \cap A_0) = 0.35$, $P(B \cap A_1) = 0.3$ and $P(B \cap A_0 \cap A_1) = 0.2$.

(a) Find $P(B^c \cap A_0^c \cap A_1^c)$ and $P(B \cap A_0^c \cap A_1^c)$.

(b) The circuit is supposed to function as an OR gate. Find the probability that the output is incorrect, i.e., different from the OR function. (An OR gate’s output is zero if both inputs are zero. Otherwise the output is one.)

(c) The circuit is supposed to function as an AND gate. Find the probability that the output is incorrect, i.e., different from the AND function. (An AND gate’s output is one if both inputs are one. Otherwise the output is zero.)
5. A and B are two events. Let

\[ C = \{ x : x \in A \implies x \in B \} \]

Find the event C in terms of A, B and their complements.

6. Prove that preimage preserves set theoretic operations, i.e., let \( f : S_1 \to S_2 \) be a function. Let \( F \subset S_2 \) and \( G \subset S_2 \) and let \( E_1, E_2, ..., E_n \) be a partition of \( S_2 \). Prove that

(a) \( f^{-1}(F^c) = [f^{-1}(F)]^c \).
(b) \( f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G) \).
(c) \( f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G) \).
(d) \( f^{-1}(E_1), f^{-1}(E_2), ..., f^{-1}(E_n) \) forms a partition of \( S_1 \).

7. The circuit shown below represents a telephone communication link. Switches \( \alpha_i, i = 1, \ldots, 6 \) are open or closed and operate independently. The probability that a switch is closed is \( p \). Let \( A_i \) represent the event that switch \( i \) is closed.

(a) In terms of the \( A_i \)'s write the event that there exists at least one closed path from 1 to 2.
(b) Compute the probability of there being at least one closed path from 1 to 2.