Basic Surface Topology - III

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Simplicial Complex

Planar Models

Surface Classification
Simplicial Complex - Simplex

• All smooth surfaces can be triangulated.
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- Refine the triangulation → the mesh becomes closer to the original smooth surface.
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**Definition (Simplex)**

Suppose $k + 1$ points $\{v_0, v_1, \ldots, v_k\}$ are in general positions in $\mathbb{R}^n$, $n \geq k + 1$, the standard simplex $[v_0, v_1, \ldots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \ldots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \ldots, v_k$ as the vertices of the simplex $\sigma$. Suppose $\tau \subset \sigma$ is also a simplex, then we say $\tau$ is a facet of $\sigma$. 
Orientation of a Simplex

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- **Definition (Orientation of a Simplex)**
  Suppose $k + 1$ points $\{v_0, v_1, \ldots, v_k\}$ are in the general positions in $\mathbb{R}^n, n \geq k + 1$. The orientation of a simplex $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}]$ is positive, if the permutation $(i_0, i_1, \ldots, i_k)$ differs from $(0, 1, \ldots, k)$ by an even number of two-element swaps; otherwise, the orientation is negative.
Simplicial Complex

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- **Definition (Simplicial Complex)**
  A simplicial complex $\Sigma$ is a union of simplices, such that:
  1. If a simplex $\sigma$ belongs to $\Sigma$, then all its facets also belong to $\Sigma$;
  2. If $\sigma_1, \sigma_2 \subset K, \sigma_1 \cap \sigma_2 \neq \emptyset$, then the intersection of $\sigma_1$ and $\sigma_2$ is also a common facet.
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- Triangular meshes are simplicial complexes (vertex, oriented edges, and oriented faces are 0-simplexes, 1-simplexes, and 2-simplexes respectively).
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**Definition (Planar Model)**

A planar model for a surface $S$ is a polygon in $\mathbb{E}^2$ with an identification on edges s.t. the resulting surface is $S$. We permit polygons with curved edges to allow the “2-sided polygon”.
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- Denote edges by lowercase letters, each has a direction. A curve path can be denoted as a sequence of letters. On each path if it traverses in the reversed direction along the edge $a$, we write an inverse $a^{-1}$.
## Definitions

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**Definition (Word)**

The sequence of letters for the boundary edges read ccw on a planar model is called the **word** for the planar model.
Examples

What are *words* for the following surfaces:

- Sphere :
- Projective plane :
- Torus :
- Klein bottle :
- 2-holed torus :
Examples

What are \textit{words} for the following surfaces:

- Sphere: $aa^{-1}$.
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If two words $W_1$ and $W_2$ both represent the same surface, then we say that the words are equivalent, and write $W_1 \sim W_2$. 
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- Projective plane: \( aa \).
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If two words \( W_1 \) and \( W_2 \) both represent the same surface, then we say that the words are equivalent, and write \( W_1 \sim W_2 \). A surface could have more than one words.
Theorems

Theorem

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- Every closed surface can be triangulated. A triangulation of a surface follows the coherent rule of the simplicial complex.

- Every closed connected surface has a planar model. (To show it: the surface can be triangulated, the triangulation mesh can be embedded onto the plane glued together.)

- Each letter in a word for a planar model for a closed surface appears exactly twice. If both instances of each letter have different exponents and the planar model is triangulated, the triangulated surface is orientable. If some letters appear twice with the same exponent and the planar model is triangulated, then the triangulated surface is nonorientable.
Classification Theorem

**Theorem**
Any closed surface is homeomorphic to a sphere with $g$ handles (i.e. $g$-holed torus) or sphere with $k$ crosscaps.

- This theorem simply indicates that any closed orientable surface can be classified by its genus.
- Its proof is tedious and not required (the following slides)...
Word Concatenation and Surface Connected Sum

Definition
The concatenation of words $W_1$ and $W_2$ is the word consisting of all the letters of $W_1$ (in order) followed by the letters of $W_2$ (in order), denoted as $W_1 W_2$.

e.g. $W_1 = aba^{-1} b^{-1}$, $W_2 = cdc^{-1} d^{-1}$, then $W_1 W_2 = aba^{-1} b^{-1} cdc^{-1} d^{-1}$. (Fig.)
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**Theorem**

*Let $W_1$ and $W_2$ be representative words for surfaces $S_1$ and $S_2$. A word for $S_1 \# S_2$ is $W_1 W_2$.***
Pairs of 1st and 2nd Kinds

1. A pair of $\cdots x \cdots x \cdots$ is called a **pair of the 1st kind**;
2. A pair of $\cdots x \cdots x^{-1} \cdots$ is called a **pair of the 2nd kind**.
Rules-1

- **Permutation Rule:** If $x_1x_2 \ldots x_n$ is a word for a surface $S$, then $x_kx_{k+1} \ldots x_nx_1 \ldots x_{k-1}$ is also a word for $S$. 
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- **Inverse Rule:** If \( W = x_1x_2 \ldots x_n \) is a word for a surface \( S \), then \( W^{-1} = x_n^{-1} \ldots x_1^{-1} \) is also a word for \( S \).
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- **Cancelation Rule:** If $Axx^{-1}B$ is a word for a surface $S$ and either $A$ or $B$ is nonempty, then $AB$ is also a word for $S$. 
**Rules-II**

- **Cylinder Cut-and-Paste Rule**: If $ABxCDx^{-1}$ is a word for a surface $S$, then $BAxDCx^{-1}$ is also a word for $S$. 

![Diagram showing the application of the Cylinder Cut-and-Paste Rule](image)
Rules-II

- **Cylinder Cut-and-Paste Rule**: If $ABxCDx^{-1}$ is a word for a surface $S$, then $BAxDCx^{-1}$ is also a word for $S$.

- **Möbius Strip Cut-and-Paste Rule**: If $AxBxC$ is a word for a surface $S$, then $AxxB^{-1}C$ is also a word for $S$. 

![](image)
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Classification Theorem and Proof

**Theorem**

*Any closed surface is homeomorphic to a sphere with g handles (i.e. g-holed torus) or sphere with k crosscaps.*

1. **Simplify**: \( W = Axy^{-1}zBxy^{-1}z \rightarrow W = AxBx. \)
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2. **Reduce to One Vertex**: If there are more than one vertices on the boundary of the planar model \( P \), keep removing them until there is only vertex on the boundary. (Think about \( T^2 \).)

**Proof**: suppose \( b = (u \rightsquigarrow v) \),

- if there are no other vertex \( u \) appears on the boundary \( \partial P \), then \( u \) connects \( b \) and \( b^{-1} \), and can be canceled .
- otherwise, find the corresponding edge and reduce the appearance of \( u \) on \( \partial P \) by one. (Fig.)
3. **Collect crosscaps**: For each edge $a$ that occurs twice in $W$ in the same exponent, we can use Möbius strip cut-and-paste rule to rearrange $W$ so that these edges appear consecutively in $W$. If all edges come in pairs with the same exponent, then $S$ is a sphere with $k$ crosscaps, where $k$ is the number of pairs of edges.
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4. **Collect handles**: Now $W = AxBx^{-1}$. Find the closest two corresponding blocks $x$, then

$$W = [Ax][B]y[C][x^{-1}D]y^{-1},$$

$$W = [B][Ax]y[x^{-1}D][C]y^{-1} = x[y][x^{-1}[DC][y^{-1}BA],$$

$$W = x[y][x^{-1}[y^{-1}BA][DC] = xyx^{-1}y^{-1}BADC.$$
Classification Theorem and Proof

\[ W = a_1 a_1 a_2 a_2 \cdots a_m a_m x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_n y_n x_n^{-1} y_n^{-1} \]

→ any surface is one with \( m \) Möbius strips and \( n \) handles.

5. **Combine crosscaps and handles**: If there are no crosscaps, then \( S \) is a \( g \)-holed torus. Otherwise, we can iteratively combine each handle with a crosscap to create three crosscaps, and finally get a sphere with \( k \) crosscaps.

**Proposition**

*The direct sum of a torus with a projective plane is homeomorphic to the connected sum of three projective planes.*
Classification Theorem and Proof

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*The direct sum of a torus with a projective plane is homeomorphic to the connected sum of three projective planes.*

(keep applying Möbius strip cut-and-paste rule:)

\[ W = aba^{-1} b^{-1} cc = a^{-1} b^{-1} cb^{-1} a^{-1} c, \]

\[ = abbc^{-1} ac = bbc^{-1} ac = bbc^{-1} c^{-1} aa. \]