Homology

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Topological Classification of Surfaces

Theorem (Classification Theorem for Surfaces)

Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a g-torus, or a g-crosscaps.
The sphere and tori are orientable surfaces, whereas crosscaps are unorientable.

Any closed surface S is the connected sum

$$S = S_1 \# S_2 \# \cdots S_g,$$

- If S is genus-g and orientable, then S_i is a torus.
- If S is genus g and non-orientable, then S_i is a projective plane.

Simplical Complex

Definition (Simplex)

Suppose $k+1$ points in the general positions in $\mathbb{R}^n$, $v_0, v_1, \cdots, v_k$, the *standard simplex* $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \cdots, v_k$ the *vertices* of the simplex σ.

Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a *facet* of σ.

Simplicial Complex

Definition (Simplicial Complex)

A *simplicial complex* Σ is a union of simplices, such that

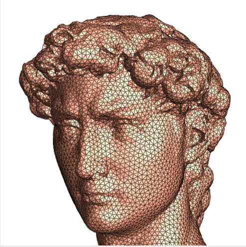
1. If a simplex σ belongs to K, then all its facets also belong to Σ.
2. If $\sigma_1, \sigma_2 \subset K$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then the intersection of σ_1 and σ_2 is also a common facet.

Triangular Mesh

Definition (Triangular Mesh)

A triangular mesh is a simplicial complex Σ:

1. Each face is counter clock wisely oriented with respect to the normal of the surface.
2. Each edge has two opposite half edges.

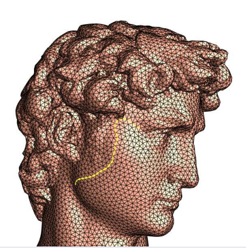


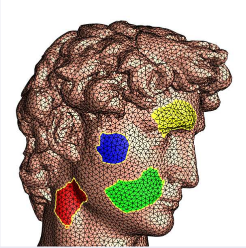
Chain Space

Definition (Chain Space)

A k chain is a linear combination of all k simplicies in Σ, $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathcal{Z}$. The n dimensional *chain space* is a linear space formed by all the n chains, we denote n dimensional chain space as $C_n(\Sigma)$.

- A curve on the mesh Σ is a 1-chain;
- A surface patch on Σ is a 2-chain.





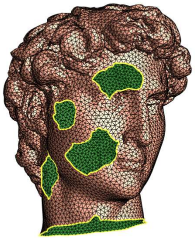
Boundary Operator

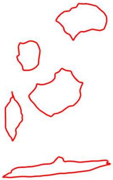
Definition (Boundary Operator)

The n-th dimensional boundary operator $\partial_n : C_n \to C_{n-1}$, is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \cdots v_n] = \sum_i (-1)^i [v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n].$$

The boundary operator extracts the boundary of the chain.



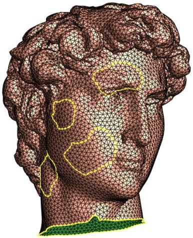


Closed chains

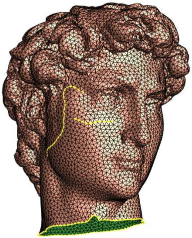
Definition (Closed chain)

A k-chain $\gamma \in C_k(\Sigma)$, if $\partial_k \gamma = 0$, then σ is closed.

A closed 1-chain is a loop. A non-closed 1-chain has boundary vertices.



closed 1-chain

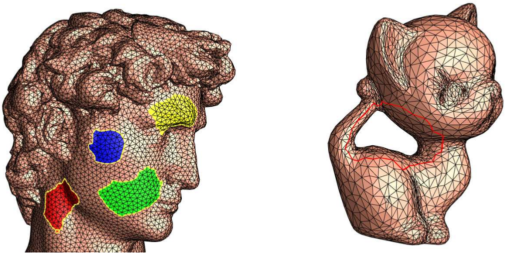


open 1-chain

Exact chains

Definition (Exact k-chain)

A k-chain $\gamma \in C_k(\Sigma)$ is exact, if there exists a (k+1)-chain σ, such that $\gamma = \partial_{k+1}\sigma$.



exact 1-chain   closed 1-chain

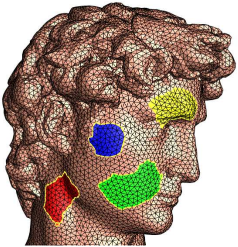
Boundary of Boundary

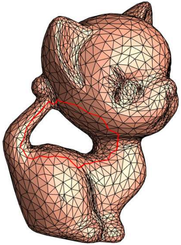
Theorem (Boundary of Boundary)

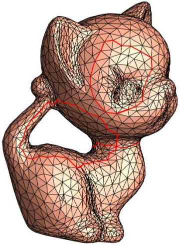
The boundary of a boundary is empty.

$$\partial_{k-1} \circ \partial_k \equiv \emptyset$$

Namely, exact chains are closed. But the reverse is not true.





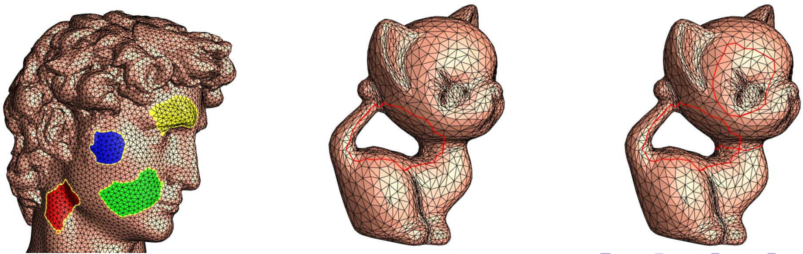


Homology

The difference between the closed chains and the exact chains indicates the topology of the surface.

Any closed 1-chain on genus zero surface is exact.

On tori, some closed 1-chains are not exact.



Homology Group

Closed k-chains form the kernel space of the boundary operator ∂_k.
Exact k-chains form the image space of ∂_{k+1}.

Definition (Homology Group)

The k dimensional homology group $H_k(\Sigma, \mathbb{Z})$ is the quotient space of $ker\partial_k$ and $img\partial_{k+1}$.

$$H_k(\Sigma, \mathbb{Z}) = \frac{ker\partial_k}{img\partial_{k+1}}.$$

Two k-chains γ_1, γ_2 are homologous, if they bound a (k+1)-chain σ,

$$\gamma_1 - \gamma_2 = \partial_{k+1}\sigma$$

Homology Group

Computation

1. The chain space C_1 is a linear space, the oriented edges are the basis. The chain space C_2 is also a linear space, the oriented face are the basis.
2. The boundary operators are linear operators, they can be represented as matrices.

$$\partial_2 = ([f_i, e_j]),$$

$[f_i, e_j]$ is zero if e_j is not on the boundary of f_i; $+1$ if e_j is on the boundary of f_i with consistent orientation; -1 if e_j is on the boundary of f_i with opposite orientation.
3. The basis of $H_1(\Sigma, \mathbb{Z})$ is formed by the eigenvectors of zero eigen values of the matrix

$$\Delta = \partial_2 \circ \partial_2^T + \partial_1^T \circ \partial_1.$$

Homology Group

Algebraic Method

The eigen vectors of Δ can be computed using the Smith norm of integer matrices. It is general for all dimensional complexes, but impractical.

Combinatorial Method

Combinatorial method is efficient and simple. The key is to find a cut graph.

Canonical Homology Basis

Definition (Canonical Homology Basis)

A homology basis $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$ is *canonical*, if

1. a_i and b_i intersect at the same point p.
2. a_i and a_j, b_i and b_j only touch at p.

The surface can be sliced along a set of canonical basis and form a simply connected patch, the fundamental domain. The fundamental domain is with the boundary

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Canonical Homology Basis

Definition (Canonical Homology Basis)

For a genus g closed surface M, there exist canonical bases for $\pi_1(M, p_0)$.

Especially, through any point $p \in M$, we can find a set of canonical basis for $\pi_1(M)$, the surface can be sliced open along them and form a canonical fundamental polygon:

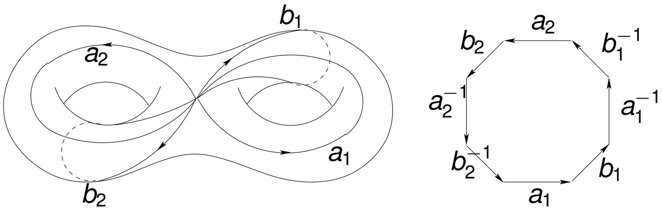
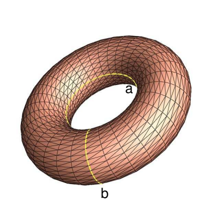
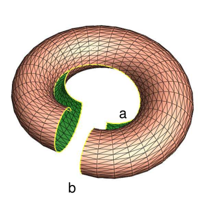


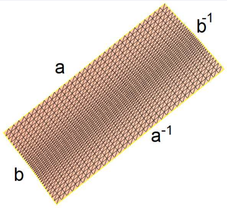
Figure: canonical basis of fundamental group $\pi_1(M, p_0)$.

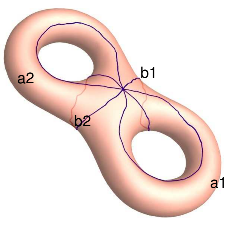
Canonical Homology Basis

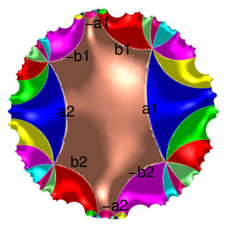
Canonical Homology Basis and Fundamental Domain







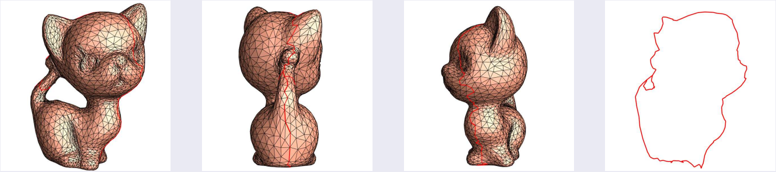




Cut Graph

Definition (cut graph)

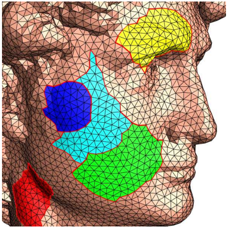
A cut graph G of a mesh Σ is a graph formed by non-oriented edges of Σ, such that Σ/G is a topological disk.



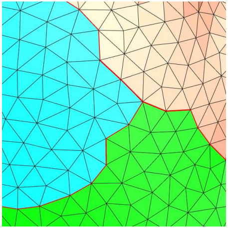
Corner and Wedge

Definition (Wedge)

On a face f, the corner with vertex v is denoted as (f, v). Given a vertex v, the corners are ordered counter-clockwisely. A maximal sequence of adjacent corners without sharp edges form a wedge.



wedges



3 wedges around a vertex

Fundamental Domain

Algorithm for Fundamental Domain

Input : A mesh Σ and a cut graph G.
Output : A fundamental domain $\tilde{\Sigma}$.

1. Label the edges on G as sharp edges.
2. Compute the wedges of Σ formed by the sharp edges.
3. Construct an empty $\bar{\Sigma}$.
4. For each wedge w, insert a vertex v, the vertex position is same as that of the vertex in of the wedge.
5. For each face $f = [v_0, v_1, v_2]$ on Σ, insert a face $\tilde{f} = [w_0, w_1, w_2]$ in $\tilde{\Sigma}$, such that the corner on f at v_i belongs to wedge w_i, $(v_i, f) \in w_i$.

Cut Graph

Algorithm: Cut Graph

Input : A triangular Mesh Σ.
Output: A cut graph G

1. Compute the dual mesh $\bar{\Sigma}$, each edge $e \in \Sigma$ has a unique dual edge $\bar{e} \in \bar{\Sigma}$.
2. Compute a spanning tree $\bar{T}$ of $\bar{\Sigma}$.
3. The cut graph is the union of all edges whose dual are not in $\bar{T}$.

$$G = \{e \in \Sigma | \bar{e} \notin \bar{T}\}.$$

Loop Basis for the Cut Graph

Theorem (Homology Basis)

Suppose Σ is a closed mesh, G is a cut graph of Σ, then the basis of loops of G (assigned with an orientation) is also a homology basis of Σ.

Algorithm: Loop Basis for the Cut Graph

Input : A graph G.
Output: A basis of loops on G.

1. Compute a spanning tree T of G.
2. $G/T = \{e_1, e_2, \cdots, e_n\}$.
3. $e_i \cup T$ has a unique loop, denoted as γ_i.
4. $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ form a basis for all loops of G.

Homotopic Maps

Definition (Homotopy of maps)

Two continuous maps $f_1, f_2 : S \to M$ between manifolds S and M are homotopic, if there exists a continuous map

$$F : S \times [0,1] \to M$$

with

$$\begin{aligned} F|_{S\times 0} &= f_1, \\ F|_{S\times 1} &= f_2. \end{aligned}$$

we write $f_1 \sim f_2$.

Homotopic Loops

Intuition: Two closed curves on a surface are homotopic to each other, if they can deform to each other without leaving the surface.

Definition (Homotopy of Curves)

Let $\gamma_i : S^1 \to \Sigma, i = 1, 2$ be closed curves on Σ, we say two curves are homotopic if the maps γ_1 and γ_2 are homotopic. we write $\gamma_1 \sim \gamma_2$.

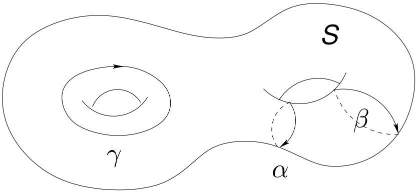


Figure: α is homotopic to β, not homotopic to γ.

Product of Curves

Definition (Product)

Let $\gamma_1, \gamma_2 : [0,1] \to M$ be curves with

$$\gamma_1(1) = \gamma_2(0),$$

the product of $\gamma_1\gamma_2 := \gamma$ is defined by

$$\gamma(t) := \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

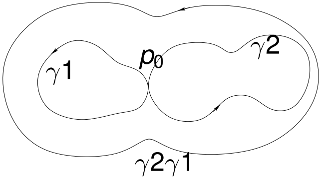


Figure: product of two closed curves.

Fundamental Group

Definition (Fundamental Group)

For any $p_0 \in M$, the fundamental group $\pi_1(M, p_0)$ is the group of homotopy classes of paths $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p_0$, i.e. closed paths with p_0 as initial and terminal point.

- $\pi_1(M, p_0)$ is a group w.r.t. the operation of multiplication of homotopy classes. The identity element is the class of the constant path $\gamma_0 \equiv p_0$.
- For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.
- If $f : M \to N$ be a continuous map, and $q_0 := f(p_0)$, then f induces a homomorphism $f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$ of fundamental groups.

Homology vs. Homotopy Groups

Abelianization

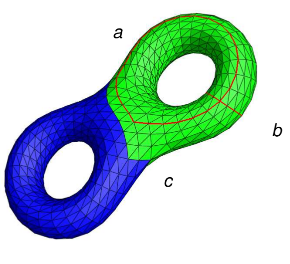
The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

$$H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)],$$

where $[\pi_1(\Sigma), \pi_1(\Sigma)]$ is the commutator of π_1, $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$.

- Fundamental group encodes more information than homology group, but is more difficult to compute.

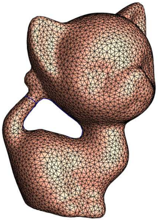
Homotopy Group vs. Homology Group

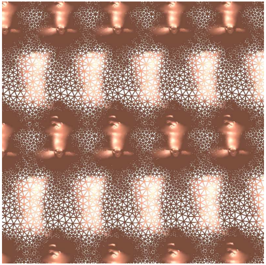


- c separate the surface to 2 handles.
- c is homotopic to $aba^{-1}b^{-1}$
- c is homologous to zero.

This shows the homotopy group is non-abelian, homotopy group encodes more information than homology group.

Universal Covering Space and Deck Transformation





Universal Cover

A pair $(\bar{\Sigma}, \pi)$ is a universal cover of a surface Σ, if

- Surface $\bar{\Sigma}$ is simply connected.
- Projection $\pi : \bar{\Sigma} \to \Sigma$ is a local homeomorphism.

Deck Transformation

A transformation $\phi : \bar{\Sigma} \to \bar{\Sigma}$ is a deck transformation, if

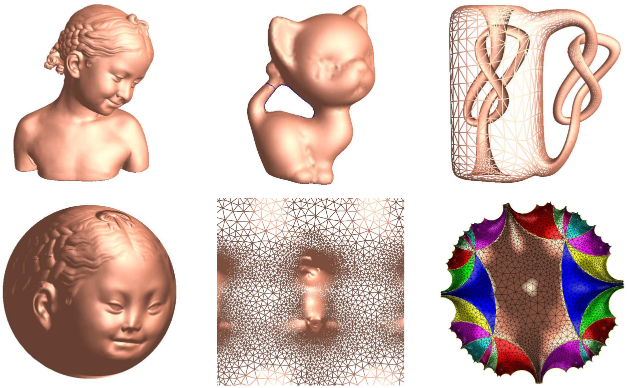
$$\pi = \pi \circ \phi.$$

- A deck transformation maps one fundamental domain to another.

Universal Covering Space (UCS)

Theorem (Universal Covering Space)

The universal covering spaces of closed surfaces are sphere (genus zero), plane (genus one) and disk (high genus).



Spherical Euclidean Hyperbolic

UCS and Fundamental Group

Intuition: A closed curve on the surface is "lifted" to a path in its UCS.

Theorem

Suppose $p \in \Sigma$, $(\bar{\Sigma}, \pi)$ is the universal cover of Σ,

$$\pi^{-1}(p) = \{\bar{p}_0, \bar{p}_1, \bar{p}_2, \cdots\},$$

a curve $\bar{\gamma}_i$ connecting $\bar{p}_0$ and $\bar{p}_i$, a curve $\bar{\gamma}_j$ connecting $\bar{p}_0$ and $\bar{p}_j$, $\pi(\bar{\gamma}_i)$ is homotopic to $\pi(\bar{\gamma}_j)$ if and only if i equals to j.

- Therefore, there is a one to one map between the fundamental group of Σ and $\pi^{-1}(p)$.
- A deck transformation maps $\bar{p}_0$ to $\bar{p}_i$. Therefore, the fundamental group is isomorphic to the deck transformation group.

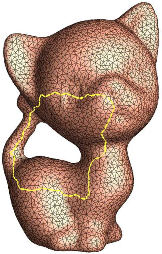
Universal Covering Space

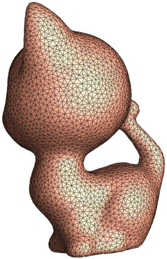
- Any topological non-trivial loop $\subset \Sigma$ is lifted to a path on its UCS $\bar{\Sigma}$. shortest loops $\to$ shortest paths
- The homotopy group of Σ can be traversed by: (1) connecting $\bar{p}_0$ to different $\bar{p}_k$ on $\bar{\Sigma}$; then (2) projecting to Σ.
- The number of fundamental domains on a UCS grows exponentially fast for high genus surfaces.

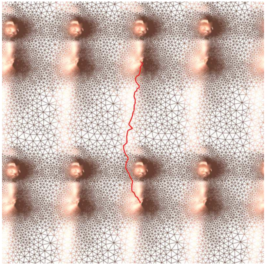
Loops and Lifting in UCS

Loop Lifting

Any nontrivial closed loop γ on Σ is lifted to an open curve $\bar{\gamma}$ on $\bar{\Sigma}$. The homotopy class of γ is determined by the starting and ending points of $\bar{\gamma}$.



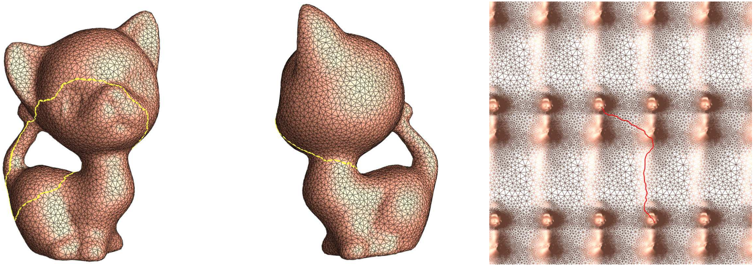




Shortest Loop in UCS

Shortest Loop

A shortest loop on surface Σ is lifted to a shortest path on the universal cover $\bar{\Sigma}$.



Computing UCS

Algorithm Universal Cover

Input : A mesh Σ.
Output: A finite portion of the universal cover $\bar{\Sigma}$.

1. Compute a cut graph G of Σ. We call a vertex on G with valence greater than 2 a knot. The knots divide G to segments, assign an orientation to each segment, labeled as $\{s_1, s_2, \cdots, s_n\}$.
2. Slice Σ along G to get a fundamental domain $\tilde{\Sigma}$, the boundary is composed of $s_k^{\pm}$.
3. Initialize $\bar{\Sigma} \leftarrow \tilde{\Sigma}$, keep $\partial\bar{\Sigma}$ using $s_k^{\pm}$'s.
4. Glue a copy of $\tilde{\Sigma}$ to current $\bar{\Sigma}$ along only one segment $s_k \in \partial\bar{\Sigma}$ with $s_k^{-1} \in \partial\tilde{\Sigma}$, $\bar{\Sigma} \leftarrow \bar{\Sigma} \cup_{s_k} \tilde{\Sigma}$.
5. Update $\partial\bar{\Sigma}$, if $s_i^{\pm}$ are adjacent on $\partial\bar{\Sigma}$, glue along s_i. Repeat this until no adjacent $s_i^{\pm}$ exists on the boundary $\partial\bar{\Sigma}$.
6. Repeat gluing the copies of $\tilde{\Sigma}$ until $\bar{\Sigma}$ is large enough.