Delaunay Triangulation and Voronoi Diagram

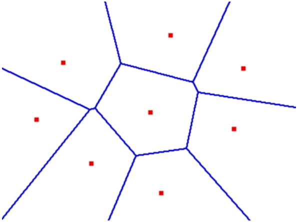
Xin Shane Li

ECE, CCT, CSC
Louisiana State University

April, 2010

Voronoi Diagrams

Given a finite set of points in the plane, assign to each point a region of influence, such that the regions decompose the plane...



Definition (Voronoi Regions)

Let $S \subset \mathbb{R}^2$ be a set of n points and define the **Voronoi region** of $p \in S$ as the set of points $x \in \mathbb{R}^2$ that are at least as close to p as to any other point in S; i.e.:

$$V_p = \{x \in \mathbb{R}^2 | ||x - p|| \leq ||x - q||, \forall q \in S\}.$$

- named after the Georges Voronoi

Voronoi Diagrams (cont.)

Define the half-plane of points at least as close to p as to q:

$$H_{pq} = \{x \in \mathbb{R}^2 | ||x - p|| \leq ||x - q||\}.$$

The **Voronoi region** of p is the intersection of H_{pq}, for all $q \in S - \{p\}$.

- V_p: convex polygonal region ($< n$ edges), possibly unbounded.
- $\forall x \in \mathbb{R}^2$ has at least 1 nearest point in S, so it lies in at least 1 Voronoi region $\rightarrow$ the Voronoi regions cover the entire plane.
- Two Voronoi regions lie on opposite sides of the perpendicular bisector separating the two generating points (they do not share interior points, except on the bisector boundary).

Definition (Voronoi Diagrams)

The Voronoi regions together with their shared edges and vertices form the **Voronoi Diagram** of S.

Delaunay Triangulation

Delaunay Triangulation (Graph)

- A dual diagram : if we draw a straight edge connecting points $p, q \in S$ if and only if their Voronoi regions intersect along a common line segment.
- In general, these edges (called Delaunay edges) decompose the convex hull of S into triangular regions, called **Delaunay triangles**.

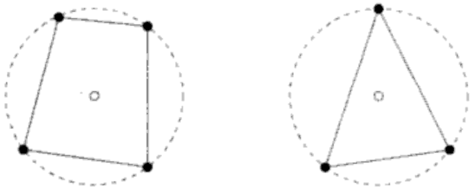
- no 2 Delaunay edges cross each other (proved later);
- now we can use Euler equation ($\chi = n_f - n_e + n_v$):
  1. a planar graph with $n \geq 3$ vertices has $\leq 3n - 6$ edges and $\leq 2n - 4$ faces.
  2. there is a bijection between the Voronoi edges and the Delaunay edges: $\rightarrow$ Voronoi edges : $\leq 3n - 6$, Voronoi vertices : $\leq 2n - 4$

- named after the Boris Delaunay

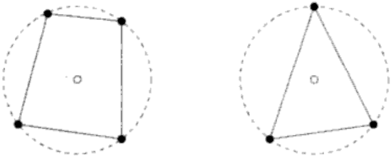
Degenerate Delaunay Triangle

Degeneracy

- if four or more Voronoi regions meet at a common point u;
- all four sites have the same distance from u (probabilistically, the chance is 0) $\rightarrow$ an arbitrarily small perturbation suffices to remove the degeneracy and to reduce it to the general case;
- we discuss general cases first...



Circumcircle Claim



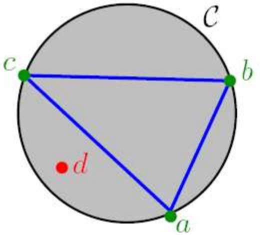
- For a Delaunay triangle $[abc]$, consider the circumcircle U (passing through a, b, c, centered at $u = V_a \cap V_b \cap V_c$;
- $r_U = ||u - a|| = ||u - b|| = ||u - c||$;
- U is called **empty** if it encloses no point of S.

Definition (Circumcircle Claim)

Let $S \subset \mathbb{R}^2$ be finite and in general position, and let $a, b, c \in S$ be three points. Then $[abc]$ is a Delaunay triangle if and only if the circumcircle of $[abc]$ is empty.

In Circle Test

Given a triangle $[a, b, c]$, and a fourth point d:



$$\mathrm{inCircle}(a, b, c, d) < 0$$

$$\mathrm{inCircle}(a, b, c, d) = \det \begin{pmatrix} 1 & a_x & a_y & {a_x}^2 + {a_y}^2 \\ 1 & b_x & b_y & {b_x}^2 + {b_y}^2 \\ 1 & c_x & c_y & {c_x}^2 + {c_y}^2 \\ 1 & d_x & d_y & {d_x}^2 + {d_y}^2 \end{pmatrix}$$

Why? (Answered later in the slides.)

Triangulations of Planar Point Sets

A triangulation of P is a maximal planar subdivision whose vertex $\in P$.

Theorem

Let P be a set of n points in the plane, not all collinear, and let k denote the number of points in P that lie on the boundary of the convex hull of P. Then any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.

Triangulation and its Angle-vector

- Let $\mathcal{T}$ be a triangulation of P, and suppose it has m triangles. Consider the $3m$ angles of the triangles of $\mathcal{T}$, sorted by increasing value. Let $\alpha_1, \alpha_2, \ldots, \alpha_{3m}$ be the resulting sequence of angles ($\alpha_i \leq \alpha_j$ for $i < j$).
- $A(\mathcal{T}) := (\alpha_1, \ldots, \alpha_{3m}) \rightarrow$ *angle-vector* of $\mathcal{T}$.
- We say the angle-vector of $\mathcal{T}$ is larger than the angle-vector of $\mathcal{T}'$ if $A(\mathcal{T})$ is lexicographically larger than $A(\mathcal{T}')$ (formal definition in next page).

Thales's Theorem

Definition

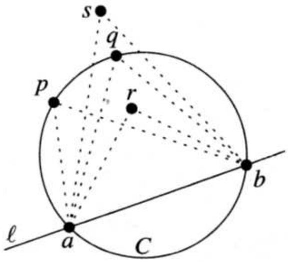
We denote $A(\mathcal{T}) > A(\mathcal{T}')$ if $\exists 1 \leq i \leq 3m$ such that

$$\alpha_j = \alpha'_j, \text{ for all } j < i, \text{ and } \alpha_i > \alpha'_i.$$

Theorem (Thales's Theorem)

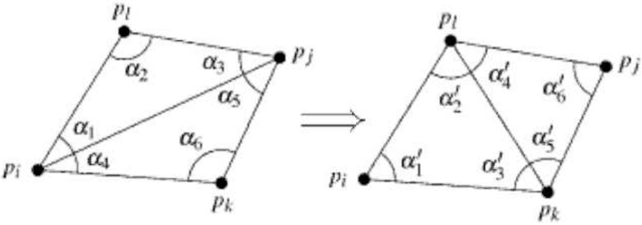
Let C be a circle, l a line intersecting C in points a and b, and p, q, r, s being points lying on the same side of l. Suppose that p and q lie on C, that r lies inside C, and that s lies outside C. Then

$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb.$$



Edge Flip

- Suppose we have a triangulation $\mathcal{T}$ of P.
- For an edge $[p_i, p_j]$ not on the boundary, it is incident to two triangles $[p_i, p_j, p_k]$ and $[p_j, p_i, p_l]$.
- On the quadrilateral $[p_i, p_k, p_j, p_l]$, removing the edge $[p_i, p_j]$, and inserting the edge $[p_k, p_l]$ we get a new triangulation $\mathcal{T}'$ of P.
- This removing+inserting operation is called **edge flip**.
- The only difference in the angle-vector of $\mathcal{T}$ and $\mathcal{T}'$ are the six angles:



Edge Flip and Angle Vector

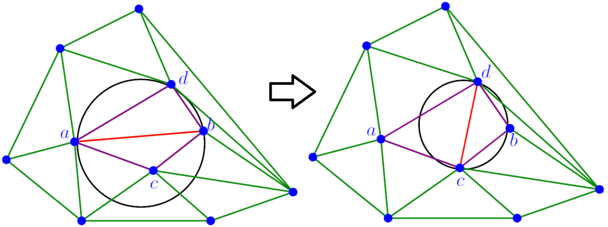
Definition (Illegal Edge)

We call edge $[p_i, p_j]$ an illegal edge if we can locally increase the smallest angle by flipping that edge, i.e.:

$$\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i .$$

Lemma

Let $\mathcal{T}$ be a triangulation with an illegal edge e. Let $\mathcal{T}'$ be the triangulation obtained from $\mathcal{T}$ by flipping e. Then $A(\mathcal{T}') > A(\mathcal{T})$.



Illegal vs Legal Edges

Lemma (One Illegal Edge Each Quadrangle)

Let edge $[p_i, p_j]$ be incident to triangles $[p_i, p_j, p_k]$ and $[p_j, p_i, p_l]$, and let C_{ijk} be the circle through p_i, p_j, p_k :

1. *The edge $[p_i, p_j]$ is **illegal** if and only if the point p_l lies in the interior of C_{ijk}.*
2. *Furthermore, if the points p_i, p_k, p_j, p_l do not lie on a common circle, then **exactly one between** $[p_i, p_j]$ **and** $[p_k, p_l]$ **is an illegal edge**.*

To prove it, we can show:

1. if p_l in C_{ijk}, edge flip increase $A(\mathcal{T})$;
2. either neither in circumcircles, or both in circumcircles
   (this proves 2, together with that if edge flip increases $A(\mathcal{T})$, it must be flipping an illegal edge to legal);

□

Illegal vs Legal Edges

Lemma (One Illegal Edge Each Quadrangle)

Let edge $[p_i, p_j]$ be incident to triangles $[p_i, p_j, p_k]$ and $[p_j, p_i, p_l]$, and let C_{ijk} be the circle through p_i, p_j, p_k :

1. *The edge $[p_i, p_j]$ is **illegal** if and only if the point p_l lies in the interior of C_{ijk}.*
2. *Furthermore, if the points p_i, p_k, p_j, p_l do not lie on a common circle, then **exactly one between** $[p_i, p_j]$ **and** $[p_k, p_l]$ **is an illegal edge**.*

To prove it, we can show:

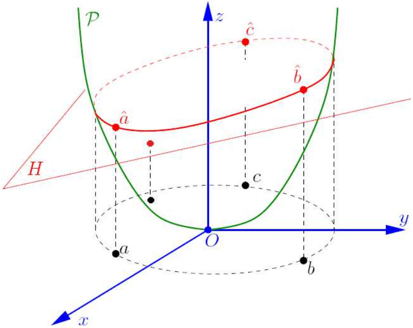
1. if p_l in C_{ijk}, edge flip increase $A(\mathcal{T})$;
2. either neither in circumcircles, or both in circumcircles (this proves 2, together with that if edge flip increases $A(\mathcal{T})$, it must be flipping an illegal edge to legal);

□

Proof of the “One-illegal edge Lemma”

Lifted Circle Claim (LCC)

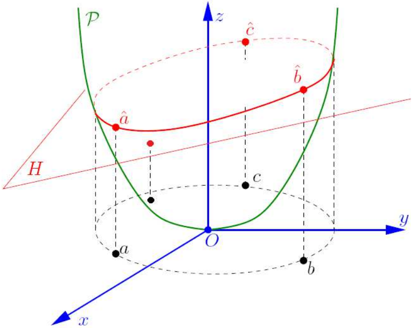
- Let a, b, c, d be points on the $x - y$ plane, and $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be the vertical projections of them onto the paraboloid $z = x^2 + y^2$.
- LCC: Point d lies inside C_{abc} if and only if point $\hat{d}$ lies vertically below the plane passing through $\hat{a}, \hat{b}, \hat{c}$.
- consider the **flip** using a tetrahedron in 3D space



Proof of the “One-illegal edge Lemma”

Lifted Circle Claim (LCC)

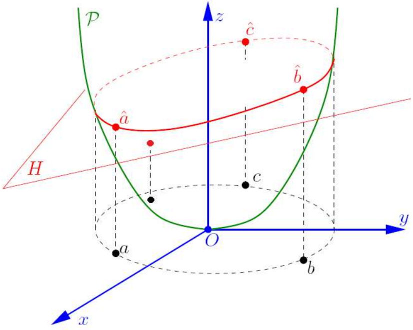
- Let a, b, c, d be points on the $x - y$ plane, and $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be the vertical projections of them onto the paraboloid $z = x^2 + y^2$.
- LCC: Point d lies inside C_{abc} if and only if point $\hat{d}$ lies vertically below the plane passing through $\hat{a}, \hat{b}, \hat{c}$.
- consider the **flip** using a tetrahedron in 3D space



Proof of the “One-illegal edge Lemma”

Lifted Circle Claim (LCC)

- Let a, b, c, d be points on the $x - y$ plane, and $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be the vertical projections of them onto the paraboloid $z = x^2 + y^2$.
- LCC: Point d lies inside C_{abc} if and only if point $\hat{d}$ lies vertically below the plane passing through $\hat{a}, \hat{b}, \hat{c}$.
- consider the **flip** using a tetrahedron in 3D space



Angle-optimal Triangulation

Angle-optimal and Legal Triangulations

- An angle-optimal triangulation is one that has the largest $A(\mathcal{T})$.
- A legal triangulation is one that no illegal edge exists.
- legal triangulation $\Leftrightarrow$ angle-optimal triangulation
- A straightforward algorithm to compute it (see next page):
  - repeat the process of searching and flipping illegal edges until all edges are locally legal.
  - The algorithm terminates because (1) the angle-vector keeps increasing iteratively and (2) there is a finite number of different triangulations.
  - Another interpolation of the proof: each time we flip and edge, the lifted triangulation gets lower.
  - $O(n^2)$ complexity, since there are $O(n^2)$ edges.

An Algorithm for Computing Delaunay Triangulation

Algorithm *SlowDelaunay*(P)
Input: a set P of n points in $\mathbb{R}^2$
Output: $\mathcal{DT}(P)$

compute a triangulation $\mathcal{T}$ of P
initialize a stack containing all the edges of $\mathcal{T}$
while stack is non–empty
    do pop ab from stack and unmark it
        if ab is illegal then
            do flip ab to cd
                for $xy \in \{ac, cb, bd, da\}$
                    do if xy is not marked
                        then mark xy and push it on stack
return $\mathcal{T}$

Delaunay Triangulation

Definitions

- Consider the dual graph $\mathcal{G}$ of the Voronoi diagram $\mathcal{V}(P)$: nodes correspond to sites, arcs between two nodes if the corresponding cells share an edge.
- The straight-line embedding of $\mathcal{G}$ is called *Delaunay graph* of P, or $\mathcal{DG}(P)$.

Theorem (Embedding of Delaunay Graph)

The Delaunay graph of a planar point set is a plane graph (i.e. no two edges in the embedding cross).

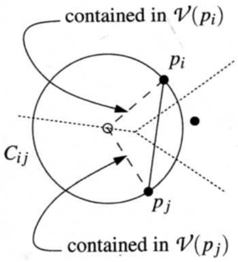
Proof of the Embedding of Delaunay Graph

Theorem (Embedding of Delaunay Graph)

$\mathcal{DG}(P)$ is a plane graph (i.e. no two edges in the embedding cross).

Proof.

(1) following Voronoi diagram property: the edge $[p_i, p_j]$ is in $\mathcal{DG}(P)$ iff exists a circle C_{ij} with p_i, p_j on its boundary and no other sites in it (the center o_{ij} of such a disc lies on the common edge of $V(p_i)$ and $V(p_j)$).
(2) define triangle $t_{ij} := [o_{ij}, p_i, p_j]$, edges $[o_{ij}, p_i] \subset V(p_i)$, $[o_{ij}, p_j] \subset V(p_j)$.
(3) suppose $[p_k, p_l]$ is another edge of $\mathcal{DG}(P)$ and it intersects $[p_i, p_j]$:
˜(3.1) then both p_k and p_l must lie outside C_{ij} (by (1)), therefore, outside t_{ij}.
˜(3.2) from (3.1) $\rightarrow$ $[p_k, p_l]$ must intersect either $[o_{ij}, p_i]$ or $[o_{ij}, p_j]$, and also $[p_i, p_j]$ must intersect either $[o_{kl}, p_k]$ or $[o_{kl}, p_l]$ $\Rightarrow$ there exists intersections among $[o_{..}, p_.]$ which is impossible. □



Delaunay as the Dual of $V(P)$

The Delaunay graph of P is an embedding of the dual graph of the Voronoi diagram.

Theorem (Circumcircle and Supporting Circle Claims)

Let P be a set of points in the plane.

(i) *Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the $\mathcal{DG}(P)$ if and only if the circle through p_i, p_j, p_r contains no point of P in its interior.*

(ii) *Two points $p_i, p_j \in P$ form an edge of $\mathcal{DG}(P)$ if and only if there is a closed disc C that contains p_i and p_j on its boundary and does not contain any other point of P.*

Delaunay and Legal Triangulation

Theorem

Let P be a set of points in the plane, and let $\mathcal{T}$ be a triangulation of P. Then $\mathcal{T}$ is a Delaunay triangulation of P if and only if the circumcircle of any triangle of $\mathcal{T}$ does not contain a point of P in its interior.

Theorem (Delaunay triangulation ⇔ Legal triangulation)

Let P be a set of points in the plane. A triangulation $\mathcal{T}$ of P is legal if and only if $\mathcal{T}$ is a Delaunay triangulation of P.

Proof of Delaunay triangulation ⇔ Legal triangulation.

- ⇐: Delaunay is legal (directly following the definition).
- ⇒: To prove legal triangulation is a Delaunay triangulation:
  Exercise: (Hint: show the contradiction that every edge is legal, but there exists a triangle Δ_{ijk}, its circumcircle C_{ijk} includes another point p_l can't happen.)

□

Delaunay and Angle-optimal Triangulation

Angle-optimal and Delaunay

- Any angle-optimal triangulation must be legal $\rightarrow$ angle-optimal triangulation is a Delaunay triangulation.
- When P is in general position, there is only one legal triangulation, which is then the only angle-optimal triangulation, or the unique Delaunay triangulation.
- When P degenerates, any triangulation of the $\mathcal{DG}(P)$ is legal, and is Delaunay triangulation.
  Not all these Delaunay triangulations are angle-optimal, but the minimum angle is the same (by Thales's theorem).

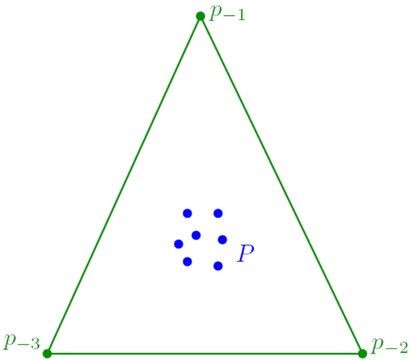
Theorem

Let P be a set of points in the plane. Any angle-optimal triangulation of P is a Delaunay triangulation of P. Furthermore, any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

Computing Delaunay Triangulation (Algorithm-2)

A Randomized Incremental Algorithm

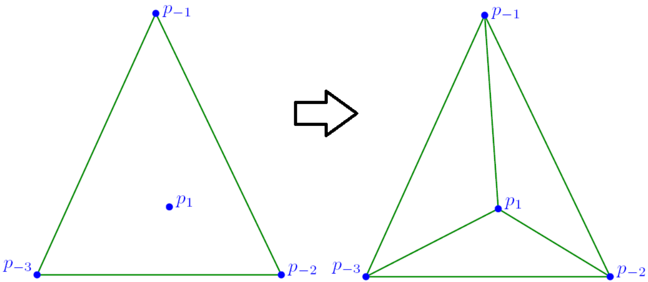
- Let $(p_1, p_2, \ldots, p_n)$ be a random permutation of P.
- Let $[p_{-3}, p_{-2}, p_{-1}]$ be a large triangle containing P.
- Denote point set $P_i = \{p_{-3}, p_{-2}, p_{-1}, p_1, \ldots, p_i\}$.



Algorithm Overview

A Randomized Incremental Algorithm

1. Start from p_1, then do the step for $p_2, \ldots, p_n$;
2. Every step starts from $\mathcal{DT}(P_{i-1})$, then insert p_i and split a triangle into 3
   1. Point Location
   2. Perform edge flips until no illegal edge remains only need to flip around p_i, on average, this step takes constant time
3. Done with $\mathcal{DT}(P_i)$, do $i = i + 1$ and goto Step 2, until $i = n$.



Algorithm Overview (cont.)

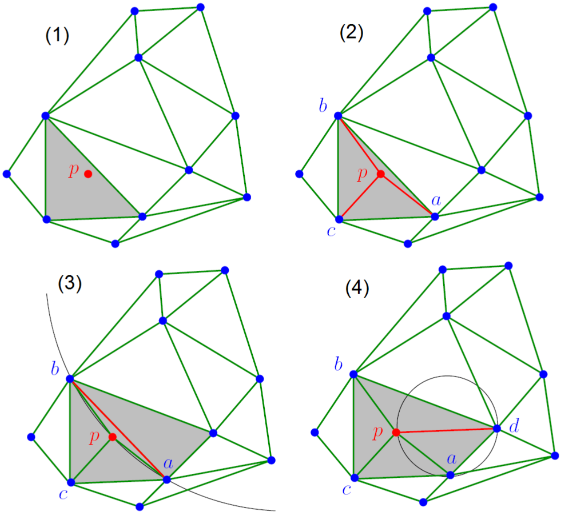
On Each Step

(1) Insert p_i, do point location;

(2) Split triangle *abc* that contains it into *abp*, *bcp* and *cap*;

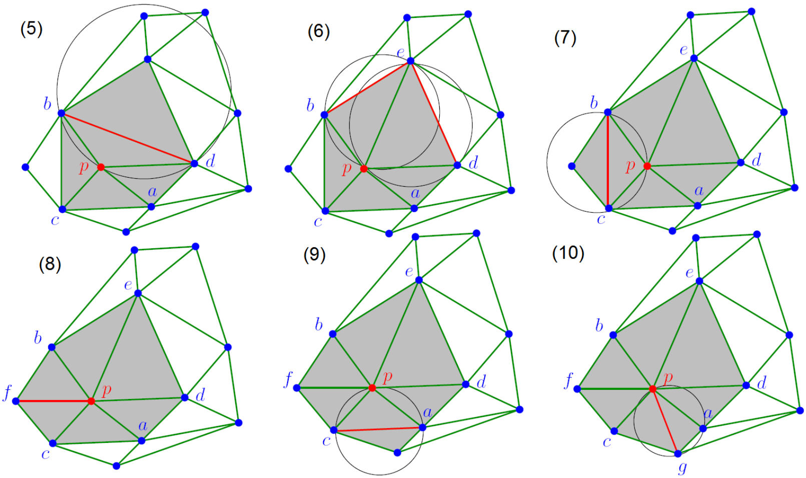
(3) For each illegal edge (e.g. *ab*), flip it;

(4) For each legal edge (e.g. *ad*), keep it.



Algorithm Overview (cont.)

Consider triangles in ccw order around *p* and flip illegal edges. Checking only *p*'s one-ring triangles are enough!



Algorithm Efficiency

- Why flipping edges of triangles that contain p is sufficient?
  - because edges between two triangles that do not contain p was locally Delaunay before the insertion, and they are still Delaunay;
  - local Delaunay implies global Delaunay;
  - therefore we only need to pay attention to incident triangles.
- Expected time $O(n \log n)$.
  - In Circle test $O(1)$.
  - Point Location $O(\log n)$ on average.
  - With a randomization preprocessing, this is over even the worst case input.
  - Deterministic $O(n \log n)$ algorithm exists, but harder and less practical.
- Knowing the Delaunay triangulation of P, we can find the Voronoi diagram of P in $O(n)$ time.