Surface Parameterization

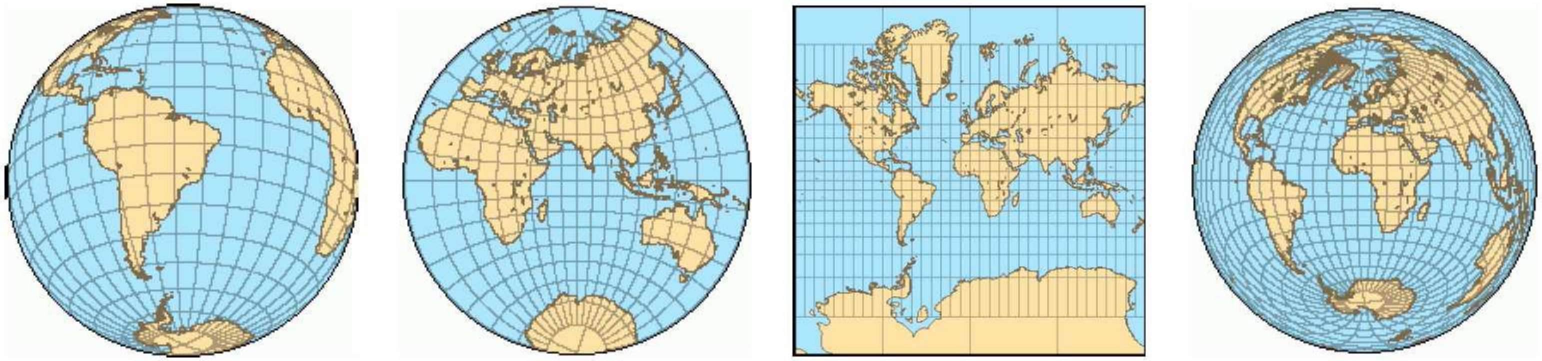
Problem Definition

- Recall the Texture Mapping that wrap an Image onto a mesh
  - A one-to-one map from geometry shape S to a texture image (2D domain) D
  - D here is a rectangular domain, e.g. $D = [0,1] \times [0,1]$
  - The mapping: a vector function $\vec{f} : S \rightarrow D \subset \mathbb{R}^2$,

    composed by two scalar function f_u and f_v.
  - ⇔Define a "u-v" coordinates over the surface S.
  - Infinite mapping ways, which one is good?

  - Intrinsic distortion is measured by 1^{st} fundamental forms
  - Ideal parameterization: isometry

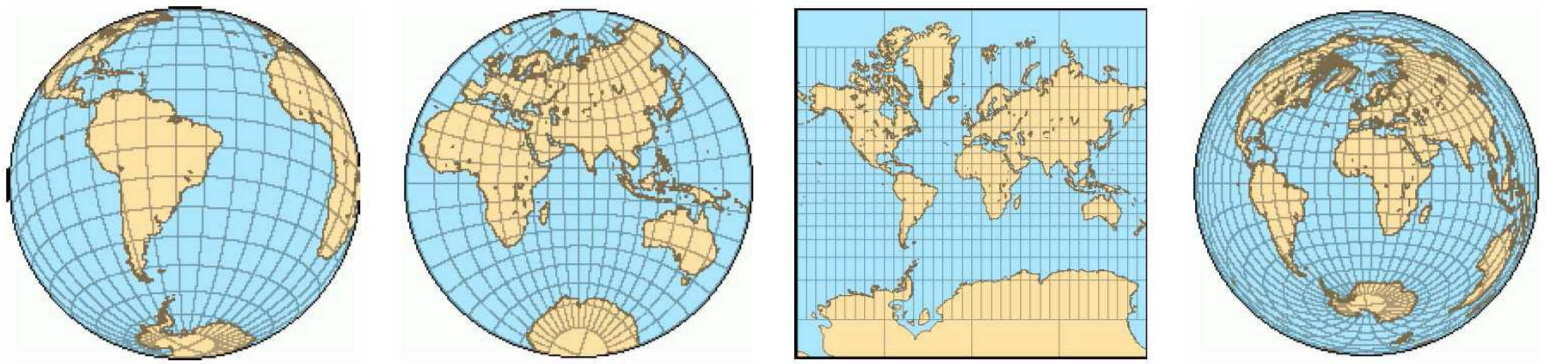
Historical Background



(a) Orthographic; (b) stereographic; (c) Mercator; and (d) Lambert

- Cartography
- Distortion: angles and areas distortion
  - Isometry: no distortion
  - Not all surfaces has the isometry to a planar region
  - Peeling oranges → can't be of no distortion
- Ptolemy was the first known to produce the data for creating a map showing the world (100-150AD)
  - [Geography] → project a sphere by longitude and latitude

Historical Background (cont.)



(a) Orthographic; (b) stereographic; (c) Mercator; and (d) Lambert

(a) The orthographic projection (Egyptians and Greeks, > 2000 years ago) → modifies both angles and areas

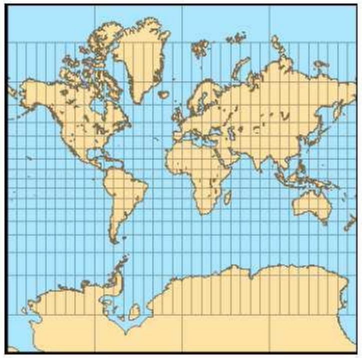
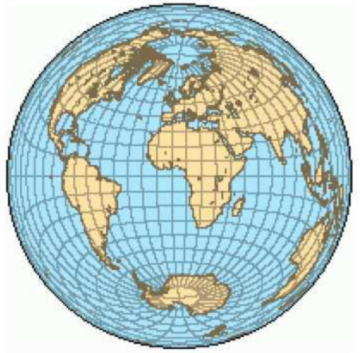
(b) Stereographic projection (Hipparchus, 190-120B.C.) → preserves angles, not areas

(c) Mercator projection (Mercator 1569) → preserves angles, not areas

(d) Lambert projection (Lambert 1772) → preserves areas, not angles

Good "UV" versus bad "UV"?

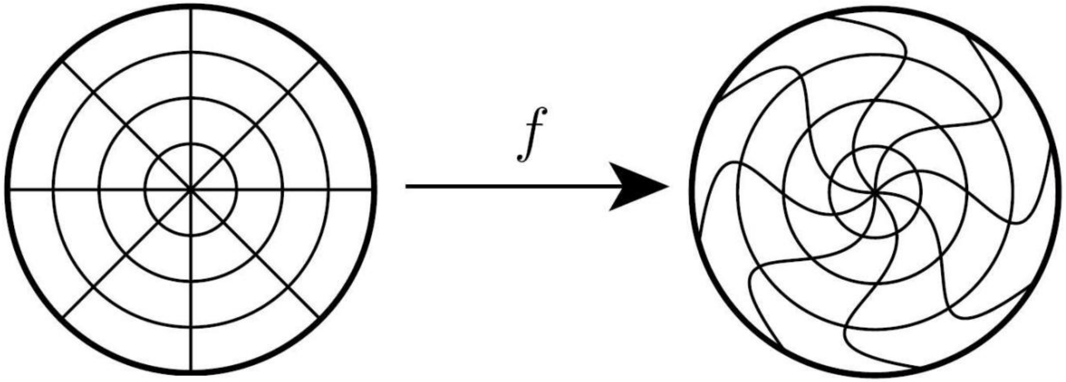


- What do we look for? What do we preserve?
- Should we map it onto a rectangle? Or a disk? Or something different? What do we choose?
- If the target shape is fixed (e.g. a rectangle, or a disk…), what is the best mapping then?
- Consider the easiest case:
  - Source: a genus-zero open surface (a topological disk)
  - Target: planar square

Mapping Criteria

- Angle Distortion: change of the local angles
  - Conformal mapping: no angle distortion (locally, a right angle → a right angle, or a circle → a circle), preserving the "shape"
- Area Distortion: change of the local area
  - Equiareal mapping: no area change
- Isometric Mapping: neither angles nor area distortion
- Isometric ⇔ conformal + equiareal
- Isometry exists between a given surface and a planar domain, only if this surface is "developable" (Guassian curvature=0 everywhere)
- Purely Equiareal Mapping is infinitely dimensional and not necessarily useful



Mapping Criteria

- Therefore:

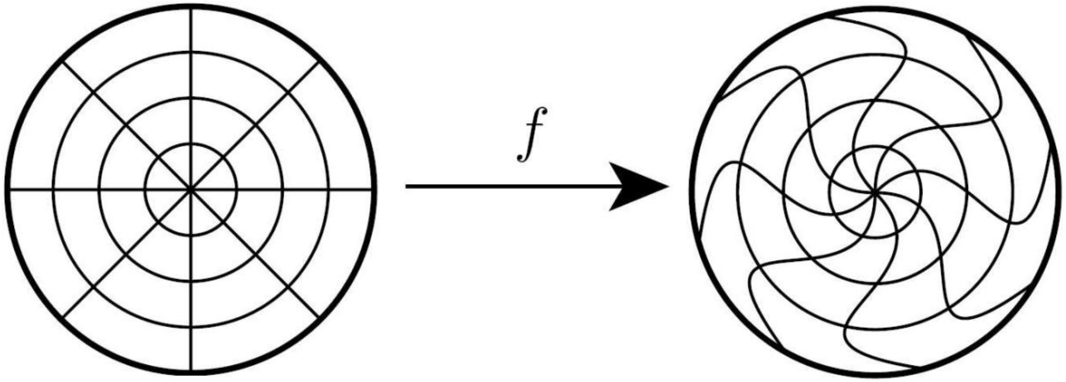
Given an arbitrary topological disk surface and a planar domain

- Isometric mapping rarely exists
- Conformal mapping always exists (Riemann Mapping Theorem)
- Infinitely many equiareal mapping, as a pure criterion, not easy to control and design

We will focus on:

A conformal mapping = an analytic function = two conjugate harmonic scalar fields (will be explained later)

A conformal map → harmonic



Harmonic Flattening of a Triangle Mesh

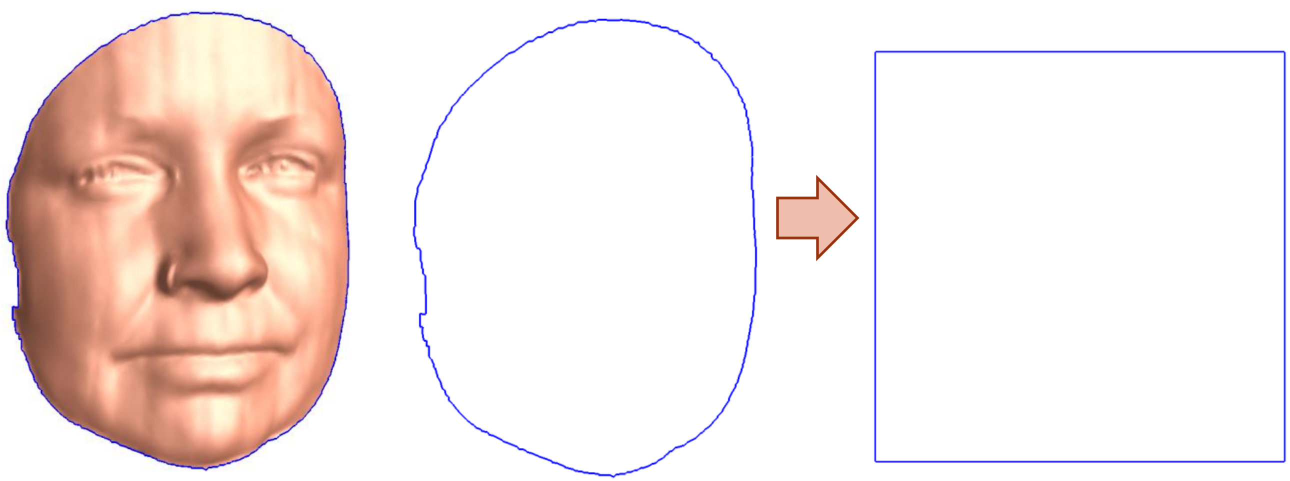
- Intuitively : considering that you are flattening a triangle mesh (deform it while preserving angles and make it flat)
  1) Pin vertices on the boundary loop on a planar rectangle boundary
  2) Move the interior vertices into the rectangle properly

<u>Algorithm Pipeline:</u>

computing two harmonic functions $f_u: (x,y,z) \rightarrow u$, and $f_v: (x,y,z) \rightarrow v$

1) For boundary vertices, map them to one of the following four segments

a) $u=0, 0<v<1$;
b) $0<u<1, v=0$;
c) $u=1, 0<v<1$;
d) $0<u<1, v=1$.



Flattening Triangle Mesh

- An intuitive way : considering that you are flattening a triangle mesh (deforming it and make it flat)
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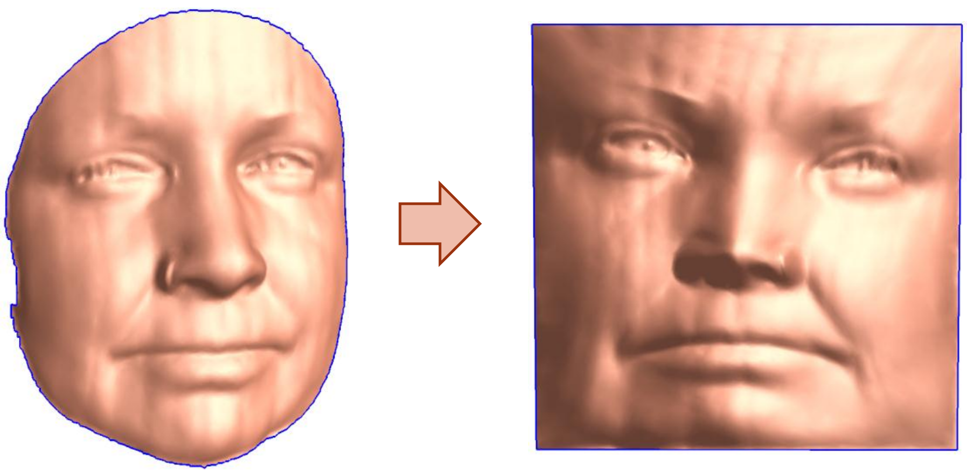
<u>Algorithm Pipeline:</u>

computing two harmonic functions $f_u: (x,y,z) \rightarrow u$, and $f_v: (x,y,z) \rightarrow v$

1) For each interior vertex, map it to $0<u<1, 0<v<1$

there should not be flip-over (roughly speaking, every vertex v_i should be mapped into the interior region of its one ring vertices v_j)



Flatten 3D Mesh by Harmonic Map

- Harmonic function: a smooth function that minimizes the magnitude of its gradient:

$$E(f) = \frac{1}{2}\int_S \| \nabla f \|^2 dx \qquad (1)$$

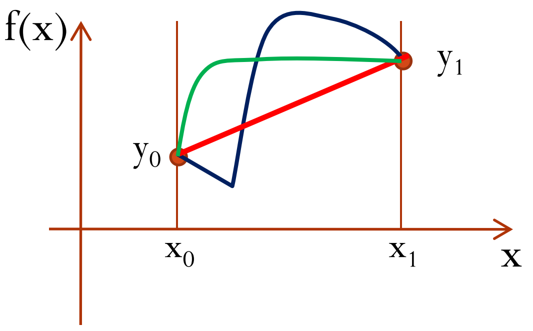
  → Called the harmonic energy, or Dirichlet energy

- A map composed of harmonic functions is called a harmonic map
  - It satisfies: $\vec{f} = (f_1, \ldots f_k)^T : S \rightarrow D; \qquad \Delta f_i(p) = 0, \forall p(x,y,z) \in S \quad (2)$
  - It is uniquely determined by the boundary condition

- Harmonic Function Examples:
  - 1D Curve:

  Given: $f(x_0)=y_0$, $f(x_1)=y_1$

  The harmonic function $f(x)$ is uniquely defined, and can be computed by minimizing E in (1)



Harmonic Function (1D)

- Harmonic Function Examples:
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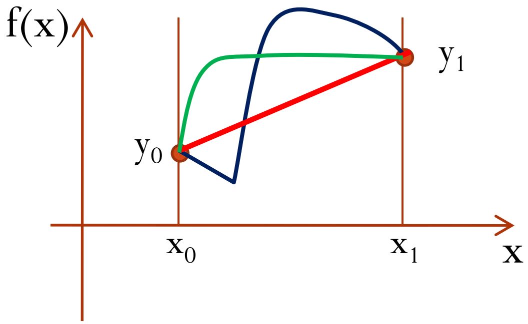


- Property of a harmonic function $f(x)$, (the red curve)
  - **Mean-value principle :**

$$f(x) = \frac{1}{2\varepsilon}\int_{|y-x|<\varepsilon} f(y)dy, \forall x, y \in S$$

  function value on a point is the average of values of it surrounding points

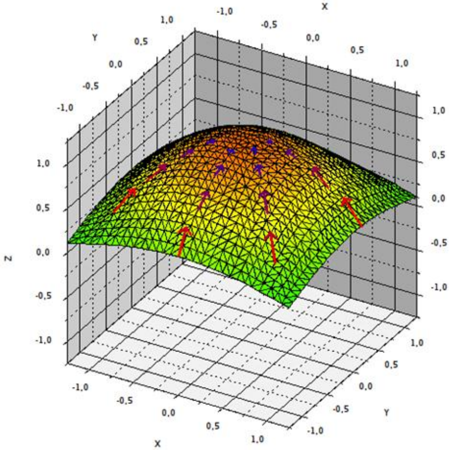
  → we can use this to numerically compute the function

  - **Maximal principle :**

  Maximal/minimal function values only exist on the boundary

Flatten 3D Mesh by Harmonic Map

- Flatten a 2D variable function f(u,v) , similarly minimize the harmonic energy $$E(\vec{f}) = \frac{1}{2}\int_{p\in S} \| \nabla \vec{f}(p) \|^2 dp$$
- It is equivalent to solving: $$\Delta \vec{f}(p) = \nabla \cdot \nabla \vec{f}(p) = 0, \forall p \in S$$
- If the boundary conditions:
  - (1) If $\vec{f}(p)|_{p\in\partial S} : \partial S \to const\, C$ ➔ $f(p)|_{p\in S} = C, \forall p \in S$
  - (2) If $\vec{f}(p)|_{p\in\partial S} : \partial S \to \partial D$ ➔ $f(u,v) \in D, \forall (u,v) \in S$



Flatten Mesh To Square

- Physical simulation:
  - Edges of the triangle mesh are springs (spring network)
  - Fix the boundary on the plane
  - Relax the interior of this network
  - Physical law being the only rule
  - Stabilized position → mapping for the interior vertices
- A mesh with n+b (interior: 1.. n, boundary: n+1…n+b) vertices:
  - The rest string length → 0
  - Potential energy → $(Ds^2)/2$, (D-constant, s-final string length)
  - Boundary vertices p_i → u_i (2d-vector u_i denotes its planar coordinates)
  - Minimize spring energy:

$$E = \frac{1}{2}\sum_{i=1}^{n+b}\sum_{j \in N_i} \frac{1}{2} D_{ij} \|\boldsymbol{u}_i - \boldsymbol{u}_j\|^2,$$

where $D_{ij} = D_{ji}$ is the spring constant of the spring between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$

Mesh Mapping (cont.)

- To find the minimized solution:

$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j \in N_i} D_{ij}(\boldsymbol{u}_i - \boldsymbol{u}_j) = 0 \quad \Rightarrow \quad \sum_{j \in N_i} D_{ij}\boldsymbol{u}_i = \sum_{j \in N_i} D_{ij}\boldsymbol{u}_j$$

(for any interior vertex i=1…n)

- Remove boundary points from the left to right hand side:

$$\boldsymbol{u}_i - \underline{\sum_{j \in N_i, j \le n}} \lambda_{ij}\boldsymbol{u}_j = \underline{\sum_{j \in N_i, j > n}} \lambda_{ij}\boldsymbol{u}_j, \qquad \lambda_{ij} = \frac{D_{ij}}{\sum_{j \in N_i} D_{ij}}$$

- Lead to two sparse linear systems (in two axis directions):

$$AU = \bar{U} \quad \text{and} \quad AV = \bar{V},$$

$$\bar{u}_i = \sum_{j \in N_i, j > n} \lambda_{ij} u_j \quad \text{and} \quad \bar{v}_i = \sum_{j \in N_i, j > n} \lambda_{ij} v_j \qquad (3)$$

$$A = (a_{ij})_{i,j=1,\dots,n} \; : \; a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\lambda_{ij} & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Boundary Mapping

- No fold-over → not direct projection
- Flatten a curve:
  a) Choosing the shape of the planar domain boundary
  b) Choosing the distribution of the points on the boundary

a) Boundary Shape: Usually rectangle, circle, etc.
   - Convex shape → bijectivity guarantees for many weights
   - Larger distortion when surface is highly concave
   - Choose square here

b) Distribution: Usually uniform length, chord length, …
   - Uniform distribution: works for well (uniformly) sampled data
   - Chord length: working well in most cases

(2) Interior Mapping – different weights

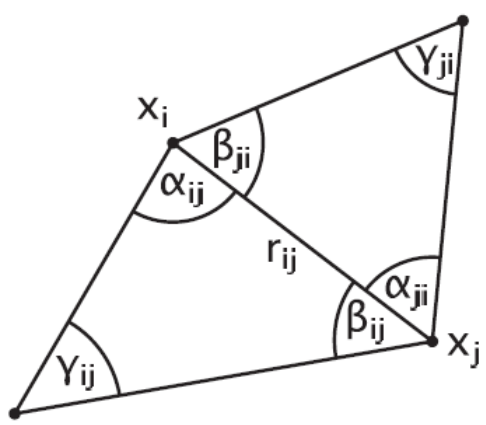
- Different D_{ij} :
  - Graph embedding:
  - Wachspress coordinates:
    - Earliest generalization of barycentric coordinates
    - Mainly used in finite element methods
  - Discrete Harmonic coordinates:
    - Standard piecewise linear approximation to Laplace equation
    - Minimizing deformation energy
  - Mean value coordinates:
    - Discretizing mean value theorem of harmonic function
    - Positive weights guaranteed, stable parameterization

$$D_{ij} = 1$$

$$D_{ij} = \frac{\cot \alpha_{ji} + \cot \beta_{ij}}{(r_{ij})^2}$$

$$D_{ij} = \cot \gamma_{ji} + \cot \gamma_{ij}$$

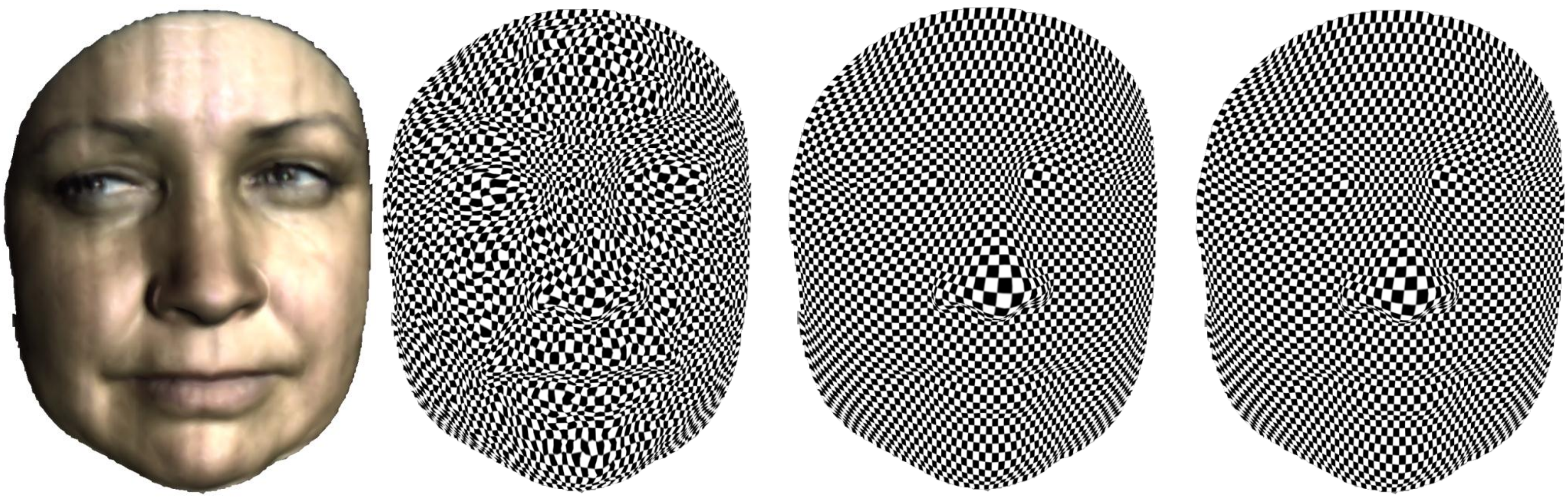
$$D_{ij} = \frac{\tan \alpha_{ji} + \tan \beta_{ij}}{r_{ij}}$$



Any symmetric weights ($D_{ij}=D_{ji}$) minimizes a spring energy with physical explanation.

Three different popular formula

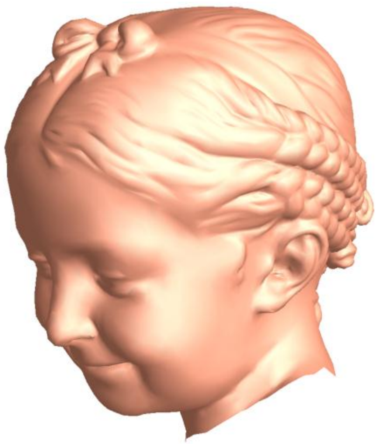
- Graph Embedding: [Tutte 1963]
- Discrete Harmonic Mapping: [Eck 1995]
- Meanvalue Coordinates: [Floater 1997]



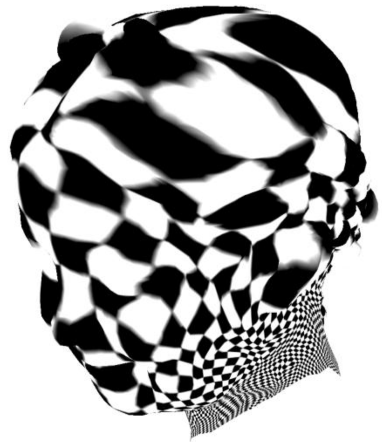
Susan Surface   Graph Embedding   Harmonic   Mean Value

Three different popular formula

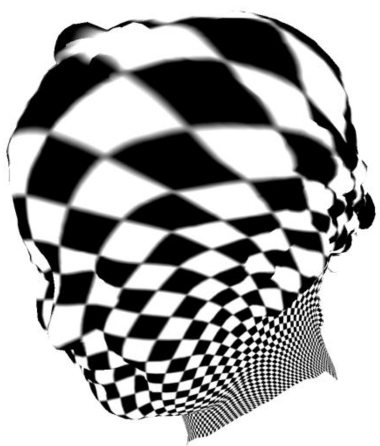
- On another surface:



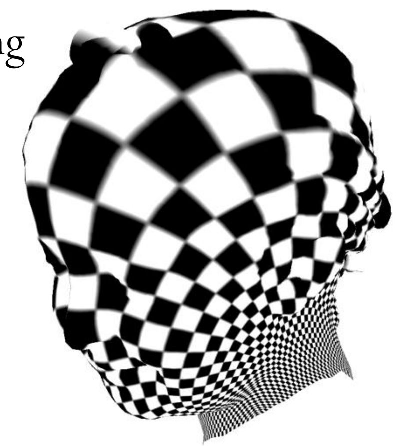
Bimba Surface



Graph Embedding



Mean Value



Harmonic

❑Carefully Read & Understand Previous Slides

❑The following materials/slides are optional

- Visually, we can tell the difference.
- But how do we measure the distortion numerically? And where do these weight formula come from?
  - E.g. why the harmonic mapping looks conformal?
- How do we design (or choose to use) a mapping technique?
  - E.g. shall we always use harmonic?
- Purely Conformal?
  - Applications needs angle-preserving
  - Applications that also needs area-preserving
- What about more general surfaces?
  - Closed Genus-0 surfaces → spherical mapping
  - Higher genus surfaces → global parameterization
  - Surface to surface → inter-surface mapping

Differential Geom. Background Review

- A surface $S \subset \mathbb{R}^3$ (2-manifold), has the parametric representation:

$$\mathbf{x}(u^1, u^2) = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2))$$

for points (u^1, u^2) in some domains in $\mathbb{R}^2$

- A representation is regular if
  i. The functions x_1, x_2, x_3 are smooth (differentiable when we need)
  ii. The vectors $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}$ are linearly independent

- 1st fundamental form (quadratic inner product on the tangent space) :

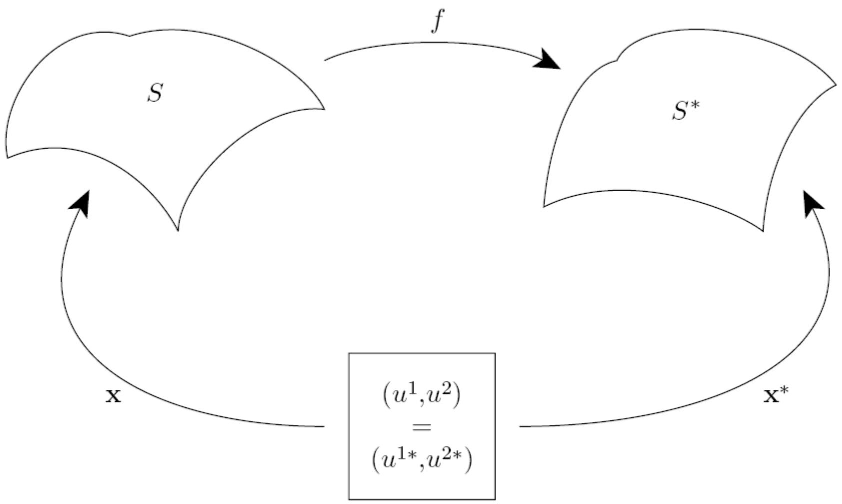
  → permits the calculation of surface metric

$$ds^2 = \mathbf{x}_1 \cdot \mathbf{x}_1 \, (du^1)^2 + 2\,\mathbf{x}_1 \cdot \mathbf{x}_2 \, du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 \, (du^2)^2$$

denoting $g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \qquad \alpha = 1, 2, \quad \beta = 1, 2,$

We have $ds^2 = (du^1 \; du^2)\, \mathbf{I} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix}$, where $\mathbf{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$

Differential Geom. Background (cont.)



f is allowable if the parameterizations x and x* are both regular.

Isometric mappings

Isometric ⇔ length-preserving

(e.g. cylinder → plane (cylindrical coordinates → Cartesian coordinates))

Theorem 1. *An allowable mapping from S to S^* is isometric if and only if the coefficients of the first fundamental forms are the same, i.e.,*

$$\mathbf{I} = \mathbf{I}^*.$$

Under an isometry:

- Curve-lengths don't change
- Angles don't change
- Areas don't change
- Gaussian curvatures don't change

Conformal mappings

Conformal ⇔ angle-preserving

(e.g. stereographic and Mercator projections)

Theorem 2. *An allowable mapping from S to S^* is conformal or angle-preserving if and only if the coefficients of the first fundamental forms are proportional, i.e.,*

$$\mathbf{I} = \eta(u^1, u^2)\,\mathbf{I}^*, \tag{1}$$

for some scalar function $\eta \neq 0$.

Under an conformal map:

- Angles don't change
- Circle → another circle (only scaling allowed)

Equiareal mappings

Equiareal ⇔ area-preserving

(e.g. Lambert projections)

Theorem 3. *An allowable mapping from S to S^* is equiareal if and only if the discriminants of the first fundamental forms are equal, i.e.,*

$$g = g^*. \tag{2}$$

(Note that: $g = \det \mathbf{I} = g_{11}g_{22} - g_{12}^2$)

Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\textbf{isometric} \Leftrightarrow \textbf{conformal} + \textbf{equiareal}.$$

An example: planar mappings

A planar mapping is a special type of the surface mapping:

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (u(x,y), v(x,y))$$

its 1st fundamental form: $\mathbf{I} = J^T J$

where $J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is the Jacobian of f.

Proposition 1. *For a planar mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ the following equivalencies hold:*

1. f is isometric $\Leftrightarrow$ $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Leftrightarrow$ $\lambda_1 = \lambda_2 = 1$
2. f is conformal $\Leftrightarrow$ $\mathbf{I} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$ $\Leftrightarrow$ $\lambda_1 / \lambda_2 = 1$
3. f is equiareal $\Leftrightarrow$ $\det \mathbf{I} = 1$ $\Leftrightarrow$ $\lambda_1 \lambda_2 = 1$

eigenvalues of $\mathbf{I}$

Planar Mappings (cont.): Conformal→Harmonic

A conformal mapping

– a complex function satisfies the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

⇩

$\Delta u = 0, \qquad \Delta v = 0,$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

A harmonic mapping

– a complex function satisfies these two Laplace equations

Isometric → Conformal → Harmonic

Harmonic Mapping with Boundary Mapping Fixed

- Easy to compute, easy to approximate
- Guaranteed existence (when suitable boundary mapping is provided)
- Minimizing deformation (minimizing the Dirichlet energy)

Theorem 5 (RKC). *If $f : S \to \mathbb{R}^2$ is harmonic and maps the boundary ∂S homeomorphically into the boundary ∂S^* of some convex region $S^* \subset \mathbb{R}^2$, then f is one-to-one;*

- Conformality depends on the boundary condition
- One-sidedness

Harmonic Map on Mesh

- Following the smooth case definition → discrete setting:

$$E(f)=\int_S \|\nabla f\|^2\, ds=\sum_{\Delta\in F} \underline{<\nabla f_\Delta, \nabla f_\Delta>} A_\Delta$$

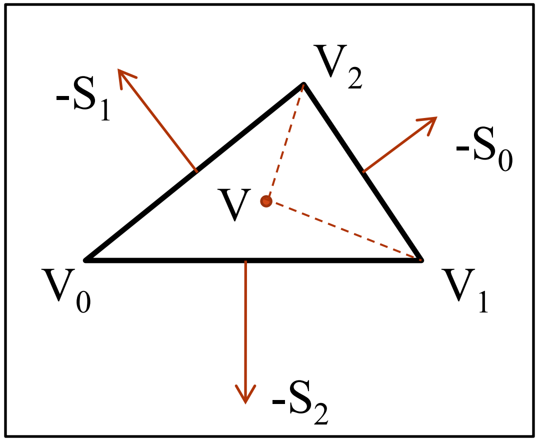
Inner product

- Look at one triangle (V_1, V_2, V_3):
  - Define: $S_i = \underline{n}\times(\underline{V_{i+2}} - \underline{V_{i+1}})$

    Normalized normal   index mod 3

  - We have: $S_0+S_1+S_2 = n\times(V_2-V_1+V_0-V_2+V_1-V_0)=0$

    → $<S_i,S_i> = <S_i, -\sum_{j\neq i} S_j> = -\sum_{j\neq i} <S_i,S_j>$



- An interior point V can be represented by barycentric coordinates:

$$V=\sum_i \lambda_i V_i,\quad \lambda_i = A_i/A \quad \text{and} \quad A_i=\frac{1}{2}|VV_{i+1}\|V_{i+1}V_{i+2}|\sin(\angle VV_{i+1}V_{i+2}) = <-S_i, V_{i+1}-V>$$

Linear function: $f(V)=\sum_i f(\lambda_i V_i) = \sum_i \lambda_i f(V_i) = \sum_i \frac{f(V_i)}{2A}<S_i,V> - \underline{\sum_i \frac{f(V_i)}{2A}<S_i,V_{i+1}>}$

$$\nabla f(V)=\sum_i \frac{1}{2A} f_i S_i \;,\; f_i \leftarrow f(V_i)$$

Harmonic Map on Mesh (cont.)

- The local energy: $<\nabla f_\Delta, \nabla f_\Delta> A = \frac{1}{4A} < \sum_i f_i S_i, \sum_j f_j S_j >$

$$= \frac{1}{4A}(\sum_i f_i^2 < S_i, S_i > + 2\sum_{i<j} f_i f_j < S_i, S_j >)$$

(because $< S_i, S_i > = -\sum_{j \neq i} < S_i, S_j >$)

$$= \frac{1}{4A}(-f_0^{\,2}(< S_0, S_1 > + < S_0, S_2 >) ... + 2\sum_{i<j} f_i f_j < S_i, S_j >)$$

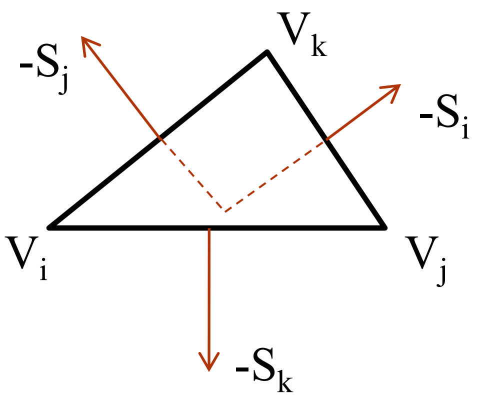
$$= \frac{-1}{4A}((f_0 - f_1)^2 < S_0, S_1 > + ...)$$

$$= \frac{-1}{4A}\sum_{i<j}(f_i - f_j)^2 < S_i, S_j >$$

Therefore : $E_\Delta(f) = \frac{1}{2}\sum_{i<j} w_{ij}(f_j - f_i)^2$

where $w_{ij} = -\frac{< S_i, S_j >}{2A}$

$$= -\frac{e_i e_j \cos(\pi - \theta_k)}{e_i e_j \sin\theta_k} = ctg(\theta_k)$$



Harmonic Map on Mesh (cont.)

- Total discrete harmonic energy:

$$E(f)=\frac{1}{2}\sum_{halfedge(i,j)} w_{ij}(f_j - f_i)^2$$

- It is minimized when

$$\frac{\partial E(f)}{\partial f_i}=\sum_{halfedge(i,j)} w_{ij}(f_j - f_i)=0$$

$$f_i=\frac{\sum (ctg\,\theta_{ij}+ctg\,\theta_{ji})\,f_j}{\sum (ctg\,\theta_{ij}+ctg\,\theta_{ji})}$$

Cotangent Weights of Discrete Harmonic Map

Mean Value Coordinates

- A problem of the cotangent weight

$$D_{ij} = \cot\gamma_{ji} + \cot\gamma_{ij}$$

Need remeshing? or

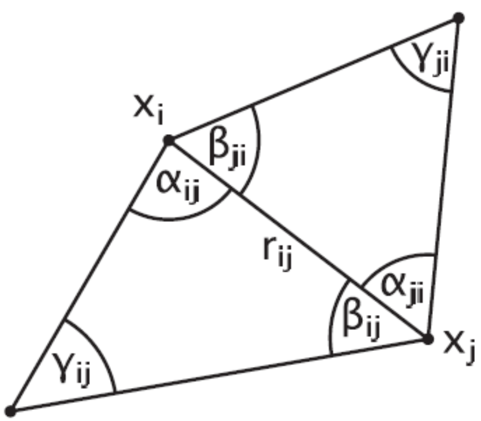
- Weights with "barycentric" property:

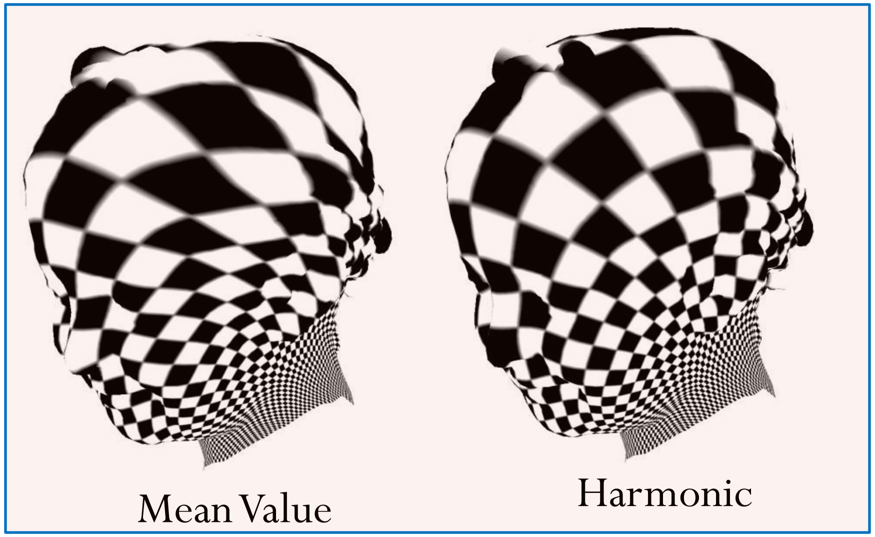
$$\begin{cases} V = \sum \lambda_i V_i \\ \sum \lambda_i = 1, \forall \lambda_i > 0 \end{cases}$$

Using Mean Value Property of the Harmonic Function

Mean Value Weights

$$D_{ij} = \frac{\tan\alpha_{ji} + \tan\beta_{ij}}{r_{ij}}$$





Metric Distortion

- Look at surface point f(u,v), move a little away from (u,v):

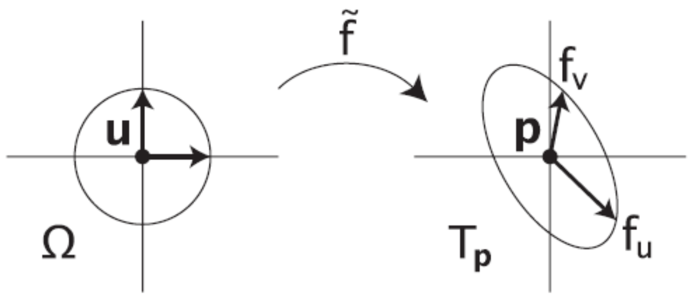
  Displacement: $(\Delta u, \Delta v)$ → new point: $f(u+\Delta u, v+\Delta v)$

  approximated by 1st order Taylor expansion:

  $$\tilde{f}(u+\Delta u, v+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v$$

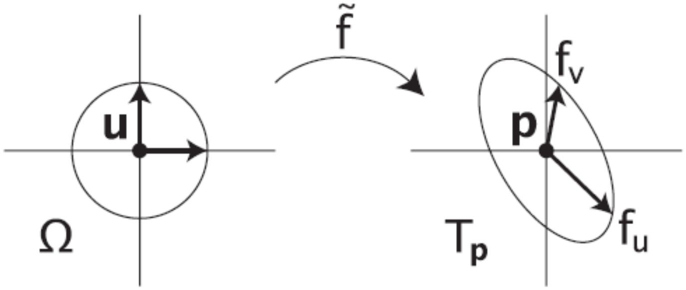
Planar local region: the vicinity of $\boldsymbol{u} = (u,v)$ ⇓ Region on tangent plane T_p at $\boldsymbol{p} = f(u,v) \in S$	Circles around **u** ⇓ ellipses around **p**

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v} \quad \text{where } J_f = (f_u \; f_v) \text{ is the } \textit{Jacobian} \text{ of } f$$



Metric Distortion (cont.)

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v}$$

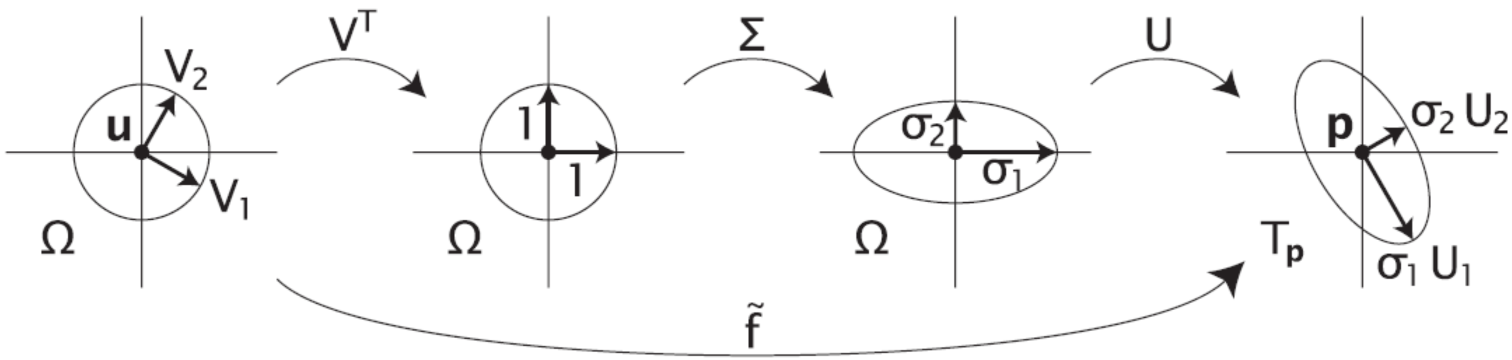


Decompose the Jacobian (3*2) matrix by SVD:

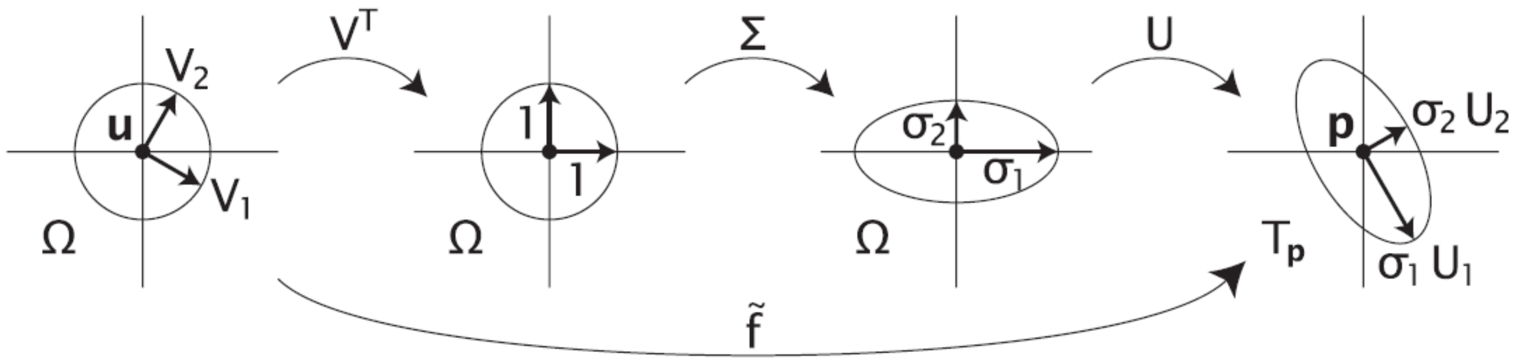
$$J_f = U\Sigma V^T = U\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{pmatrix}V^T$$

unitary, orthonormal $U \in \mathbb{R}^{3\times 3}$ $V \in \mathbb{R}^{2\times 2}$

$singular\ values\ \sigma_1 \geq \sigma_2 > 0$



Metric Distortion (cont.)



$$J_f = U\Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

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(1) 2D Rotation V → planar rotation around **u**;

(2) Stretching matrix Σ → stretches by factor σ_1 and σ_2 in the u and v directions;

(3) 3D rotation U → map the planar region onto the tangent plane

Tiny sphere with radius-r → ellipse with semi-axes of length $r\sigma_1$ and $r\sigma_2$

$\sigma_1 = \sigma_2$	⟶	Local scaling, circles to circles	: **Confomal**
$\sigma_1\sigma_2 = 1$	⟶	Area preserved	: **Equiareal**

Metric Distortion (cont.)

Singular values of any matrix A are the square roots of the eigenvalues of the matrix A^TA

Look at ${J_f}^T J_f$ $${J_f}^T J_f = \begin{pmatrix} {f_u}^T \\ {f_v}^T \end{pmatrix} (f_u \; f_v) = \mathbf{I}_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

The symmetric 2*2 matrix's eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left((E+G) \pm \sqrt{4F^2 + (E-G)^2}\right)$$

$$\begin{aligned} f \text{ is } \textit{isometric} \text{ or } \textit{length-preserving} &\iff \sigma_1 = \sigma_2 = 1 &&\iff \lambda_1 = \lambda_2 = 1, \\ f \text{ is } \textit{conformal} \text{ or } \textit{angle-preserving} &\iff \sigma_1 = \sigma_2 &&\iff \lambda_1 = \lambda_2, \\ f \text{ is } \textit{equiareal} \text{ or } \textit{area-preserving} &\iff \sigma_1\sigma_2 = 1 &&\iff \lambda_1\lambda_2 = 1. \end{aligned}$$

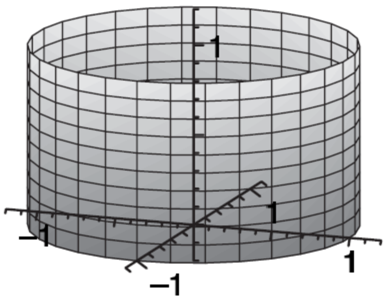
$$\text{isometric} \iff \text{conformal} + \text{equiareal}$$

Metric Distortion Example

(1) Cylinder

- *parameterization:* $f(u, v) = (\cos u, \sin u, v)$
- *Jacobian:* $J_f = \begin{pmatrix} \cos u & 0 \\ -\sin u & 0 \\ 0 & 1 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = 1$

Isometry

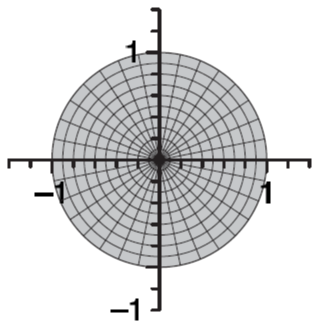


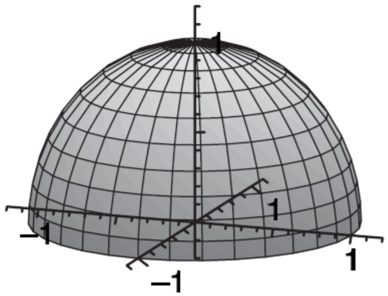
Metric Distortion Example

(2) Hemisphere (stereographic)

- *parameterization:* $f(u,v) = (2ud, 2vd, (1-u^2-v^2)d)$ where $d = \frac{1}{1+u^2+v^2}$
- *Jacobian:* $J_f = \begin{pmatrix} 2d-4u^2d^2 & -4uvd^2 \\ -4uvd^2 & 2d-4v^2d^2 \\ -4ud^2 & -4vd^2 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 4d^2 & 0 \\ 0 & 4d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 4d^2, \quad \lambda_2 = 4d^2$

Conformal



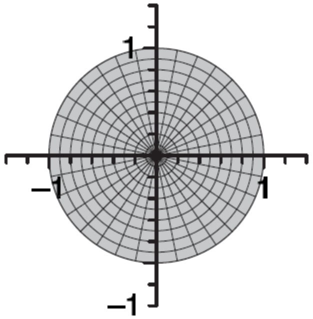


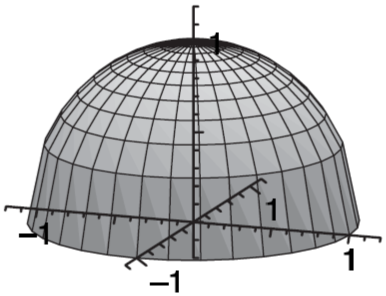
Metric Distortion Example

(3) Hemisphere (orthographic)

- *parameterization:* $f(u,v) = (u, v, \frac{1}{d})$ where $d = \frac{1}{\sqrt{1-u^2-v^2}}$
- *Jacobian:* $J_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -ud & -vd \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 1+u^2d^2 & uvd^2 \\ uvd^2 & 1+v^2d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = d^2$

Not conformal, not equiareal





Minimizing Metric Distortion

Overall distortion of a parameterization f can be generally defined by:

$$\bar{E}(f) = \int_\Omega E(\sigma_1(u,v), \sigma_2(u,v))\, du\, dv \Big/ A(\Omega)$$

Minimizing $\bar{E}(f)$ over the space of all admissible parameterizations → best parameterization

Discretely, we can look at linear function f:

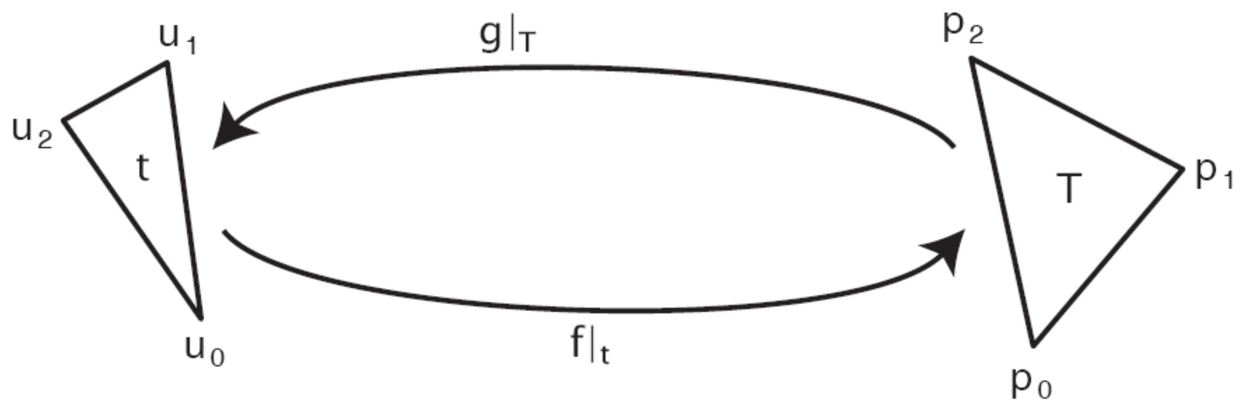
from parameter triangles $t \in \Omega$ to surface triangles $T \in \mathcal{T}$

$$\bar{E}(f) = \sum_{t\in\Omega} E(\sigma_1^t, \sigma_2^t) A(t) \Big/ \sum_{t\in\Omega} A(t)$$

Or we can look at inverse function $g=f^{-1}$: $\sigma_1^T = 1/\sigma_2^t$ and $\sigma_2^T = 1/\sigma_1^t$

$$\bar{E}(g) = \sum_{T\in\mathcal{T}} E(\sigma_1^T, \sigma_2^T) A(T) \Big/ \sum_{T\in\mathcal{T}} A(T)$$

Minimizing Metric Distortion (cont.)



$$A(t) = \frac{1}{2}\det(\boldsymbol{u}_1 - \boldsymbol{u}_0, \boldsymbol{u}_2 - \boldsymbol{u}_0) \qquad A(T) = \frac{1}{2}\|(\boldsymbol{p}_1 - \boldsymbol{p}_0) \times (\boldsymbol{p}_2 - \boldsymbol{p}_0)\|$$

$$(\sigma_1^t)^2 + (\sigma_2^t)^2 = \frac{1}{A(t)^2}\sum_{i=0}^{2}\|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2\left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)\cdot(\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^t\sigma_2^t = \frac{A(T)}{A(t)}$$

$$(\sigma_1^T)^2 + (\sigma_2^T)^2 = \frac{1}{A(T)^2}\sum_{i=0}^{2}\|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2\left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)\cdot(\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^T\sigma_2^T = \frac{A(t)}{A(T)},$$

Minimizing Metric Distortion (cont.)

Discrete Harmonic Map [Pinkall EM'93] [Eck SIG'95]:

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) = \frac{1}{2}({\sigma_1}^2 + {\sigma_2}^2)$$

Least Square Conformal Map [Desbrun SIG'02] [Levy SIG'02]:

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \frac{1}{2}(\sigma_1 - \sigma_2)^2$$

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) - E_{\mathrm{C}}(\sigma_1, \sigma_2) = \sigma_1\sigma_2$$

$$\Rightarrow \quad \bar{E}_{\mathrm{D}}(g) - \bar{E}_{\mathrm{C}}(g) = \sum_{t \in \Omega} A(t) \Big/ \sum_{T \in \mathcal{T}} A(T) = \frac{A(\Omega)}{A(S_{\mathcal{T}})}$$

Therefore, if we take a conformal map, fix its boundary and thus the area of the parameter domain Ω, and then compute the harmonic map with this boundary, then we get the same mapping, which illustrates the well-known fact that any conformal mapping is harmonic, too.

Minimizing Metric Distortion (cont.)

Conformal Mapping: → try to make $\sigma_1 = \sigma_2$

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \frac{1}{2}(\sigma_1 - \sigma_2)^2 \qquad E_{\mathrm{D}}(\sigma_1, \sigma_2) = \frac{1}{2}\left({\sigma_1}^2 + {\sigma_2}^2\right)$$

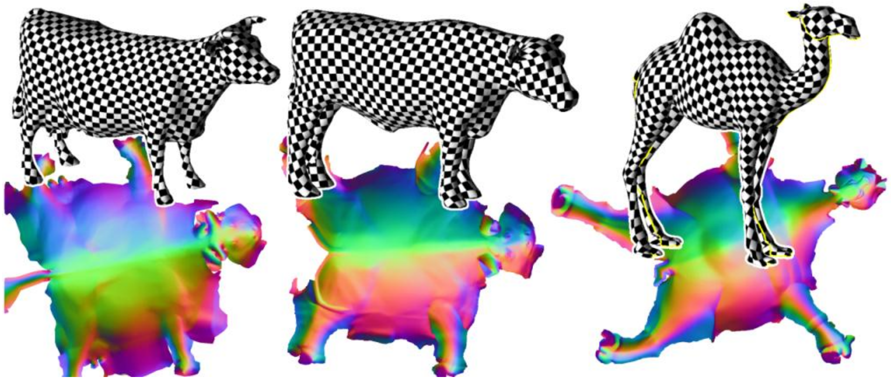
Another one: MIPS energy [Hormann 02]

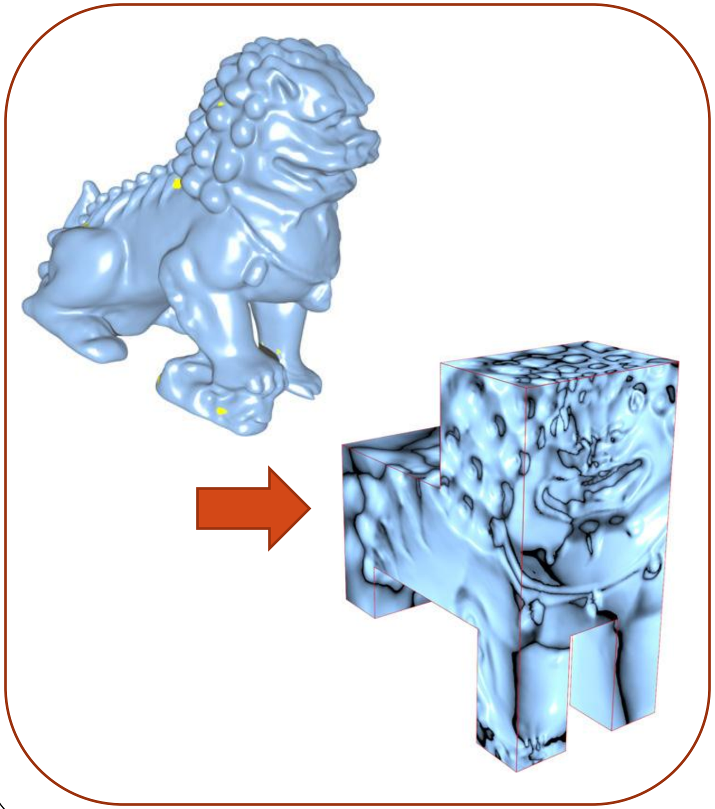
$$E_{\mathrm{M}}(\sigma_1, \sigma_2) = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \frac{{\sigma_1}^2 + {\sigma_2}^2}{\sigma_1 \sigma_2}$$

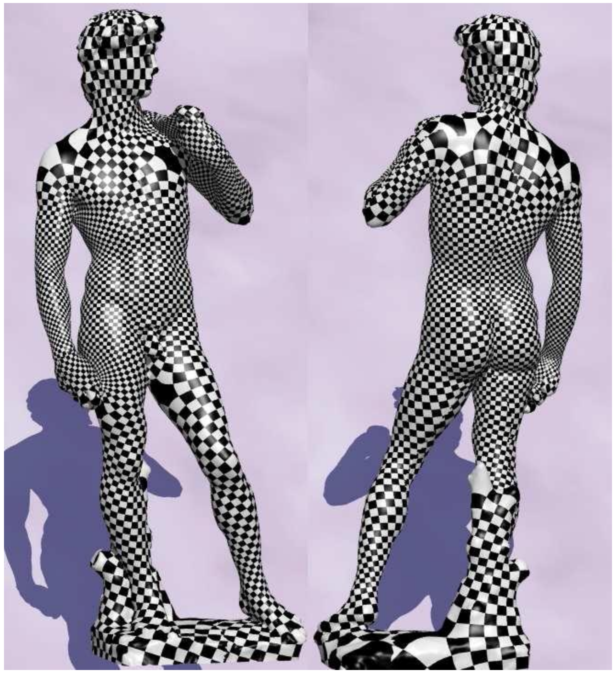
- Advantage: (1) symmetry: $E_{\mathrm{M}}(\sigma_1^T, \sigma_2^T) = E_{\mathrm{M}}(\sigma_1^t, \sigma_2^t)$
  (2) bijectivity
- Disadvantage: non-linear

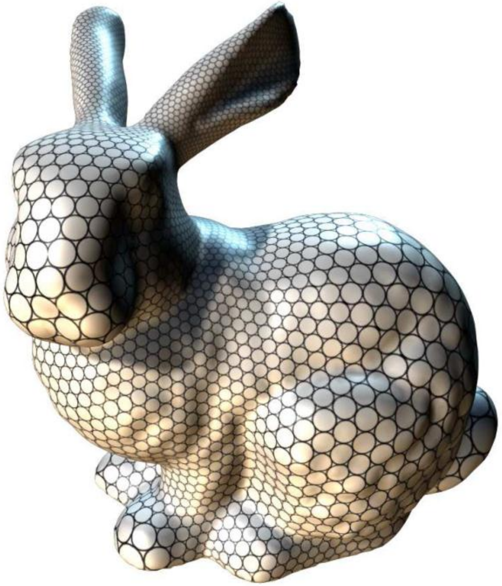
Many more about mapping…

- Free-boundary mapping
- Deforming the metric
- Global parameterization
- Inter-shape mapping









Application on meshing

With the parameterization, we can do

Remeshing – to generate high quality mesh