Basic Surface Topology - III

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Definition (Simplex)

Suppose $k+1$ points $\{v_0, v_1, \ldots, v_k\}$ are in general positions in $\mathbb{R}^n$, $n \geq k+1$, the standard simplex $[v_0, v_1, \ldots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \ldots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \ldots, v_k$ as the vertices of the simplex σ. Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a facet of σ.

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- **Definition (Orientation of a Simplex)**

  Suppose $k+1$ points $\{v_0, v_1, \ldots, v_k\}$ are in the general positions in $\mathbb{R}^n, n \geq k+1$. The orientation of a simplex $[v_{i_0}, v_{i_1}, \ldots, v_{i_k}]$ is positive, if the permutation $(i_0, i_1, \ldots, i_k)$ differs from $(0, 1, \ldots, k)$ by an even number of two-element swaps; otherwise, the orientation is negative.

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  A simplicial complex Σ is a union of simplices, such that:

  1. If a simplex σ belongs to Σ, then all its facets also belongs to Σ;
  2. If $\sigma_1, \sigma_2 \subset K, \sigma_1 \bigcap \sigma_2 \neq \emptyset$, then the intersection of σ_1 and σ_2 is also a common facet.

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- Triangular meshes are simplicial complexes (vertex, oriented edges, and oriented faces are 0-simplexes, 1-simplexes, and 2-simplexes respectively).

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- Denote edges by lowercase letters, each has a direction. A curve path can be denoted as a sequence of letters. On each path if it traverses in the reversed direction along the edge a, we write an inverse a^{-1}.

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Definition (Word)

The sequence of letters for the boundary edges read ccw on a planar model is called the **word** for the planar model.

Examples

What are *words* for the following surfaces:

- Sphere :
- Projective plane :
- Torus :
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If two words W_1 and W_2 both represent the same surface, then we say that the words are equivalent, and write $W_1 \sim W_2$.
A surface could have more than one words.

Theorems

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- *Every closed surface can be triangulated. A triangulation of a surface follows the coherent rule of the simplicial complex.*
- *Every closed connected surface has a planar model. (To show it: the surface can be triangulated, the triangulation mesh can be embedded onto the plane glued together.)*
- *Each letter in a word for a planar model for a closed surface appears exactly twice. If both instances of each letter have different exponents and the planar model is triangulated, the the triangulated surface is orientable. If some letters appears twice with the same exponent and the planar model is triangulated, then the triangulates surface is nonorientable.*

Classification Theorem

Theorem

Any closed surface is homeomorphic to a sphere with g handles (i.e. g-holed torus) or sphere with k crosscaps.

- This theorem simply indicates that any closed orientable surface can be classified by its genus.
- Its proof is tedious and not required (the following slides)...

Word Concatenation and Surface Connected Sum

Definition

The concatenation of words W_1 and W_2 is the word consisting of all the letters of W_1 (in order) followed by the letters of W_2 (in order), denoted as W_1W_2.

e.g. $W_1 = aba^{-1}b^{-1}$, $W_2 = cdc^{-1}d^{-1}$, then $W_1W_2 = aba^{-1}b^{-1}cdc^{-1}d^{-1}$. (Fig.)

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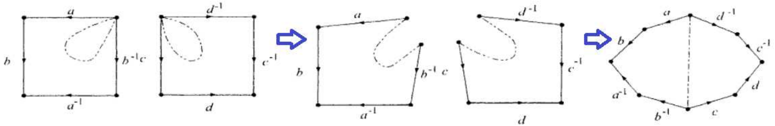
Let W_1 and W_2 be representative words for surfaces S_1 and S_2. A word for $S_1\#S_2$ is W_1W_2.

Pairs of 1st and 2nd Kinds



1. A pair of $\cdots x \cdots x \cdots$ is called a **a pair of the 1st kind**;
2. A pair of $\cdots x \cdots x^{-1} \cdots$ is called a **a pair of the 2nd kind**.

Rules-I

- **Permutation Rule**: If $x_1 x_2 \ldots x_n$ is a word for a surface S, then $x_k x_{k+1} \ldots x_n x_1 \ldots x_{k-1}$ is also a word for S.

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- **Inverse Rule**: If $W = x_1x_2\dots x_n$ is a word for a surface S, then $W^{-1} = x_n^{-1}\dots x_1^{-1}$ is also a word for S.

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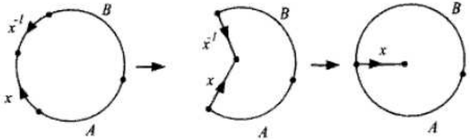
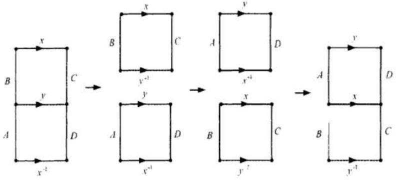
- **Cancelation Rule**: If $Axx^{-1}B$ is a word for a surface S and either A or B is nonempty, then AB is also a word for S.



Rules-II

- **Cylinder Cut-and-Paste Rule**: If $ABxCDx^{-1}$ is a word for a surface S, then $BAxDCx^{-1}$ is also a word for S.



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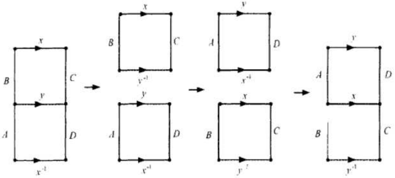

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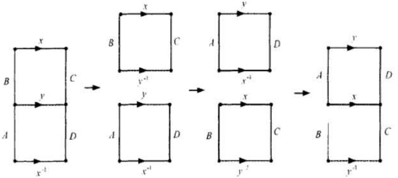

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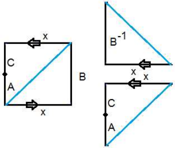

Classification Theorem and Proof

Theorem

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   Proof: suppose $b = (u \rightsquigarrow v)$,
   - if there are no other vertex u appears on the boundary ∂P, then u connects b and b^{-1}, and can be canceled .
   - otherwise, find the corresponding edge and reduce the appearance of u on ∂P by one. (Fig.)

Classification Theorem and Proof

3. **Collect crosscaps**: For each edge a that occurs twice in W in the same exponent, we can use Möbius strip cut-and-paste rule to rearrange W so that these edges appear consecutively in W. If all edges come in pairs with the same exponent, then S is a sphere with k crosscaps, where k is the number of pairs of edges.

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4. **Collect handles**: Now $W = AxBx^{-1}$. Find the closest two corresponding blocks x, then

$$W = [Ax][B]y[C][x^{-1}D]y^{-1},$$

$$W = [B][Ax]y[x^{-1}D][C]y^{-1} = x[y][]x^{-1}[DC][y^{-1}BA],$$

$$W = x[\ ][y]x^{-1}[y^{-1}BA][DC] = xyx^{-1}y^{-1}BADC.$$

Classification Theorem and Proof

$$W = a_1 a_1 a_2 a_2 \cdots a_m a_m x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_n y_n x_n^{-1} y_n^{-1}$$

$\rightarrow$ any surface is one with m Möbius strips and n handles.

5. **Combine crosscaps and handles**: If there are no crosscaps, then S is a g-holed torus. Otherwise, we can iteratively combine each handle with a crosscap to create three crosscaps, and finally get a sphere with k crosscaps.

Proposition

The direct sum of a torus with a projective plane is homeomorphic to the connected sum of three projective planes.

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(keep applying Möbius strip cut-and-paste rule:)

$$W = aba^{-1}b^{-1}cc = a^{-1}b^{-1}cb^{-1}a^{-1}c,$$

$$= abbc^{-1}ac = bbc^{-1}aca = bbc^{-1}c^{-1}aa.$$