Basic Surface Topology - II

Xin Shane Li

September 12, 2011

Orientability

Connected Sum

Product

A Mirrored Traversal

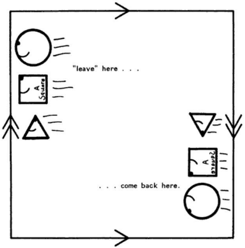
There is another planet in the 2-D universe:

- Everyone that goes north will come back normally from south;

A Mirrored Traversal

There is another planet in the 2-D universe:

- Everyone that goes north will come back normally from south;
- Everyone that goes west will come back **mirror-reflected** from east.

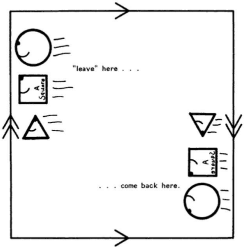


- → a Klein bottle (also imagine Mobius bands)

A Mirrored Traversal

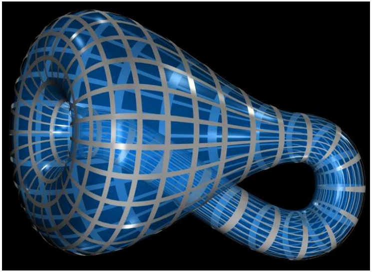
There is another planet in the 2-D universe:

- Everyone that goes north will come back normally from south;
- Everyone that goes west will come back **mirror-reflected** from east.

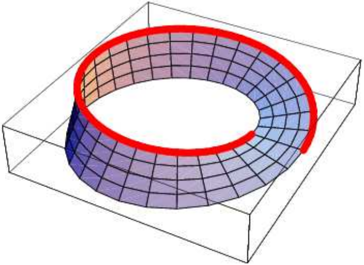


- $\rightarrow$ a Klein bottle (also imagine Mobius bands)
- Exercise: the Klein bottle tic-tac-toe game.

The Klein Bottle and Mobius Band



Klein Bottle



Mobius Band

Orientability

- Definition (Orientation-reversing Path)

  A path in a 2-manifold or 3-manifold that brings a traveler back to his starting point mirror-reversed is called an **orientation-reversing path**.

- Theorem (Orientability)

  Manifolds that don't contain orientation-reversing paths are called **orientable**, *manifolds that do are called* **nonorientable**.

Orientability

- Definition (Orientation-reversing Path)

  A path in a 2-manifold or 3-manifold that brings a traveler back to his starting point mirror-reversed is called an **orientation-reversing path**.

- Theorem (Orientability)

  Manifolds that don't contain orientation-reversing paths are called **orientable**, *manifolds that do are called* **nonorientable**.

  - Orientable manifolds: tori, infinite planes, spheres, 3-tori, ...

Orientability

- Definition (Orientation-reversing Path)

  A path in a 2-manifold or 3-manifold that brings a traveler back to his starting point mirror-reversed is called an **orientation-reversing path**.

- Theorem (Orientability)

  Manifolds that don't contain orientation-reversing paths are called **orientable**, *manifolds that do are called* **nonorientable**.

  - Orientable manifolds: tori, infinite planes, spheres, 3-tori, ...
  - Nonorientable manifolds: Klein bottle, Mobius bands, projective planes, nonorientable 3-tori, ...

Notations/Abbreviations

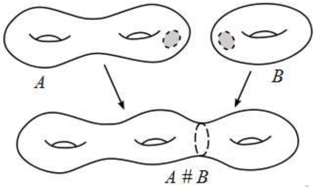
E^2 : the Euclidean plane	S^2 : the sphere
T^2 : the torus	K^2 : the Klein bottle
P^2 : the projective plane	D^2 : the disk
E^3 : the Euclidean 3D space	T^3 : the 3-torus
D^3 : a solid ball (3D disk)	P^3 : projective 3-space
E^1 : the line	S^1 : The circle
I: the interval	

The Operation of Connected Sum

Definition (Connected Sum)

The connected sum $S_1 \# S_2$ is formed by deleting the interior of disks $D_i \subset S_i$ and attaching the resulting punctured surfaces $S_i - D_i$ to each other by a homeomorphism $h : \partial D_1 \to \partial D_2$, where ∂D_i represents the boundary of D_i, so

$$S_1 \# S_2 = (S_1 - D_1) \bigcup_h (S_2 - D_2).$$



Gluing Disks, Spheres, . . .

- $S^2 \# S^2 = ?$;

Gluing Disks, Spheres, ...

- $S^2 \# S^2 = ?$;
- $T^2 \# S^2 = ?$;

Gluing Disks, Spheres, . . .

- $S^2 \# S^2 = ?$;
- $T^2 \# S^2 = ?$;
- $D^2 \# S^2 = ?$;

Gluing Disks, Spheres, . . .

- $S^2 \# S^2 = ?$;
- $T^2 \# S^2 = ?$;
- $D^2 \# S^2 = ?$;
- $D^2 \# D^2 = ?$;

Gluing Disks, Spheres, ...

- $S^2 \# S^2 = ?$;
- $T^2 \# S^2 = ?$;
- $D^2 \# S^2 = ?$;
- $D^2 \# D^2 = ?$;
- $D^2 \# T^2 = ?$.

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;
  2. a bunch of circles arranged in an interval: $C = I \times S^1$.

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;
  2. a bunch of circles arranged in an interval: $C = I \times S^1$.
- Figure.

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;
  2. a bunch of circles arranged in an interval: $C = I \times S^1$.
- Figure.
- $T^2 = ? \times ?$

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;
  2. a bunch of circles arranged in an interval: $C = I \times S^1$.
- Figure.
- $T^2 = ? \times ?$
- Figure.

Product - Cylinder and Torus

- A cylinder C is the product of a circle and a interval:
  1. a bunch of intervals arranged in a circle: $C = S^1 \times I$;
  2. a bunch of circles arranged in an interval: $C = I \times S^1$.
- Figure.
- $T^2 = ? \times ?$
- Figure.
- The torus is the only closed surface that is a product.

Product - More

Questions: What are the following:

- $I \times I = ?$

Product - More

Questions: What are the following:

- $I \times I = ?$
- $E^1 \times E^1 = ?$

Product - More

Questions: What are the following:

- $I \times I =?$
- $E^1 \times E^1 =?$
- $S^1 \times E^1 =?$

Product - More

Questions: What are the following:

- $I \times I = ?$
- $E^1 \times E^1 = ?$
- $S^1 \times E^1 = ?$
- $E^1 \times I = ?$

Product - More

Questions: What are the following:

- $I \times I = ?$
- $E^1 \times E^1 = ?$
- $S^1 \times E^1 = ?$
- $E^1 \times I = ?$
- $D^2 \times S^1 = ?$

Product - More

Questions: What are the following:

- $I \times I = ?$
- $E^1 \times E^1 = ?$
- $S^1 \times E^1 = ?$
- $E^1 \times I = ?$
- $D^2 \times S^1 = ?$
- Is the Mobius band a product?

Product - More

Questions: What are the following:

- $I \times I = ?$
- $E^1 \times E^1 = ?$
- $S^1 \times E^1 = ?$
- $E^1 \times I = ?$
- $D^2 \times S^1 = ?$
- Is the Mobius band a product?
- Is the 3-torus a product?