Basic Surface Topology - I

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Outline

About Topology

- Topology is a fundamental branch of geometry; and a key issue to better understand 3D shapes!
- It has remarkable contributions in modern mathematics, integrating analysis, geometry, and algebra, leading to many important problems and applications.
- Intuitively, how shapes behave when free-form deformations (twisting, pulling, stretching) are allowed without ripping or tearing.
- Before more details, let's start from some basic concepts.

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  - shape, area, perimeter, curvature, parallelism... ?
  - connected-component, holes, how they segment embedded spaces...?
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- Sounds very different from Euclidean geometry we imagine?
- in fact, fundamentally alike:
  - In Euclidean geometry, shapes are "invariant" under rigid transformations (congruences);
  - In Topology, shapes are "invariant" under continuous transformations (homeomorphism);
  - Rigid motions are always continuous.

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- (Figure 1)

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  - We find the exact same star in 2 different locations in the sky.
  - We are searching for signals from aliens, and receive a faint signal from a distant galaxy. It is (was) the broadcast of a 1990s TV show.
  - ...

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- The feeling of gluing $\rightarrow$ we are on it.

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  3. Maybe: the light does come back, but it is so far away, that it takes billions of years ...

A Few Important Concepts

1. Topological vs. geometric properties
2. Intrinsic vs. extrinsic properties
3. Local vs. global properties

We take surfaces (2-manifolds) as examples, you try to imagine them in higher dimensional manifolds.

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- An exercise: a torus vs. the glued screen

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  4. Whether the planet is a plane or a hemisphere.

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- Examples: Which of following discoveries in the 2D planet are local/global?
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  2. As their civilization spread, they find the area of the planet to be finite.