Lecture 4,5
Geometry of Curves and Surfaces

Basic Geometry of Curves and Surfaces

- Start with geometric properties of smooth curves and surfaces
- Then discuss their computation on polygonal meshes

For more properties or proofs of these geometric concepts, refer to standard differential geometry textbooks :

e.g. [do Carmo 76: Differential Geometry of Curves and Surfaces, Prentice Hall]

Curves

- Consider smooth planar curves: differentiable 1-manifolds embedded in R^2
  - Parametric form: $\mathbf{x}: [a,b] \to \mathbb{R}^2$ with $\mathbf{x}(u) = (x(u), y(u))^T$

    $u \in [a,b] \subset \mathbb{R}$
  - Coordinates x and y are differentiable functions of u
  - Tangent vector x'(u) to the curve at a point x(u) is defined as the first derivative of the coordinate function: $\mathbf{x}'(u) = (x'(u), y'(u))^T$
  - The trajectory of a point is a curve parameterized by time (u=t) the tangent vector x'(t) → the velocity vector at time t
  - Assume parameterization to be <u>regular</u>, s.t. $\mathbf{x}'(u) \neq \mathbf{0}$ for all $u \in [a,b]$
  - A <u>normal vector</u> n(u) at x(u) can be computed as

    $$\mathbf{n}(u) = \mathbf{x}'(u)^{\perp} / \|\mathbf{x}'(u)^{\perp}\|$$

    where $^{\perp}$ denotes rotation by 90 degree ccw.

Parameterization of a Curve

- A curve is the image of a function $\mathbf{x}$
- Same curve can be obtained with different parameterizations:

  → same trajectory using different speeds
- With different parameterizations x_1 and x_2, we usually have

  $\mathbf{x}_1(u) \neq \mathbf{x}_2(u)$ on a given u
- Different representations for a same shape
  - Can reparameterize a curve using a different mapping function

    with $g: u \rightarrow t$, $\mathbf{x}_1(u) \rightarrow \mathbf{x}_2(t)$
- We want to extract properties of a curve that are independent of its specific parameterization, e.g. length, curvature…

$\perp$

Arc Length Parameterization

- Curve length: $l(c,d) = \int_c^d \|\mathbf{x}'(u)\|\mathrm{d}u$
- A unique parameterization that can be defined as a length-preserving mapping, i.e., <u>isometry</u>, between the parameter interval and the curve using the parameterization

$$s = s(u) = \int_a^u \|\mathbf{x}'(t)\|\mathrm{d}t.$$

- Arc length parameterization x(s) :
  - the length of the curve from x(0) to x(s) is equal to s
  - independent of specific representation of the curve, maps the parameter interval [a,b] to [0,L]
  - Any regular curve can be parameterized using arc length (isometry)

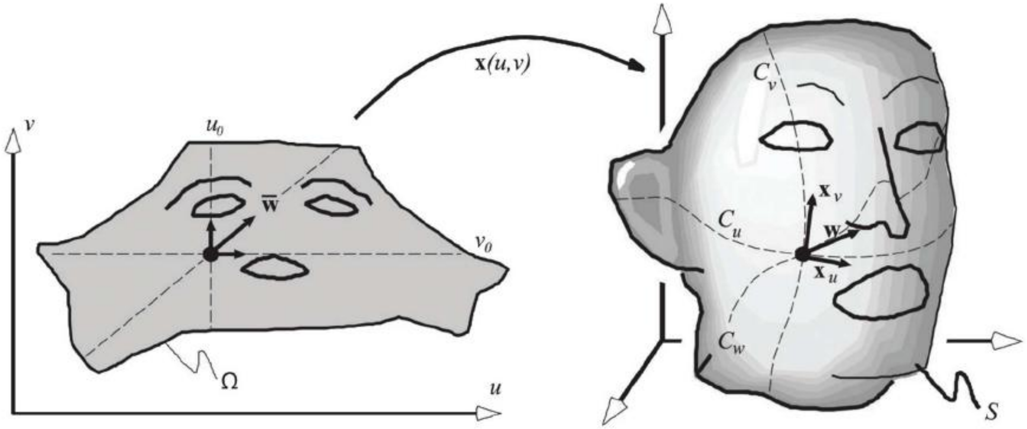
    → ideal parameterization, many computations simplified

    → doesn't work for surfaces (later)

Surfaces

- Consider a smooth surface patch: differentiable 2-manifold embedded in R^3
  - Parametric form: $$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \Omega \subset \mathbb{R}^2,$$
    where x,y,z are differentiable functions in u and v,
  - Scalars (u,v) are called coordinates in parameter space
  - In the following, we use a function $\mathbf{x}$ or $\mathbf{f}$ to represent a surface

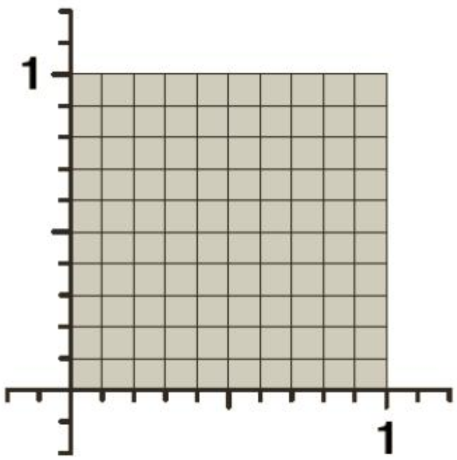
- Like tangent vectors of curves determine the metric of the curve,
- The first derivatives of $\mathbf{X}$ determines the metric of the surface

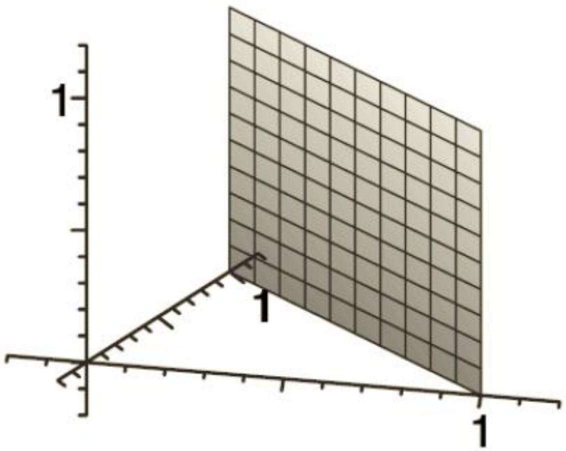


Surface Example (1)

- Simple linear function:

$$
\begin{aligned}
\textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u, v \in [0,1]\} \\
\textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x, y, z \in [0,1], x + y = 1\} \\
\textit{parameterization:} \quad & f(u,v) = (u, 1-u, v) \\
\textit{inverse:} \quad & f^{-1}(x,y,z) = (x,z)
\end{aligned}
$$

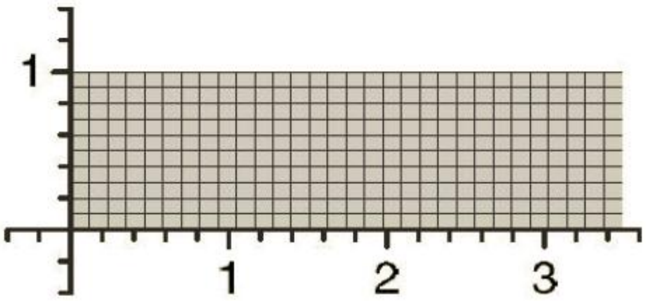


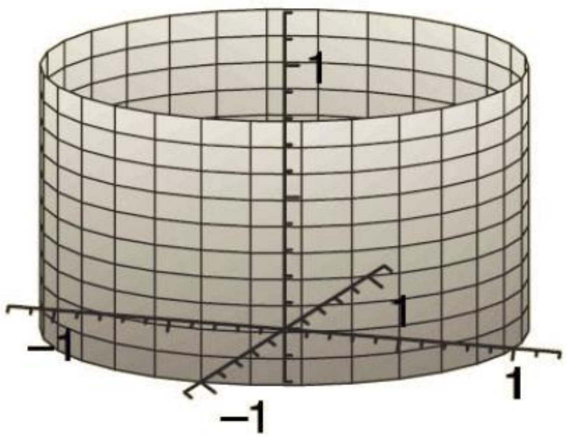


Surface Example (2)

- Cylinder:

$$\begin{aligned} \textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u \in [0, 2\pi), v \in [0,1]\} \\ \textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0,1]\} \\ \textit{parameterization:} \quad & f(u,v) = (\cos u, \sin u, v) \\ \textit{inverse:} \quad & f^{-1}(x,y,z) = (\arccos x, z) \end{aligned}$$

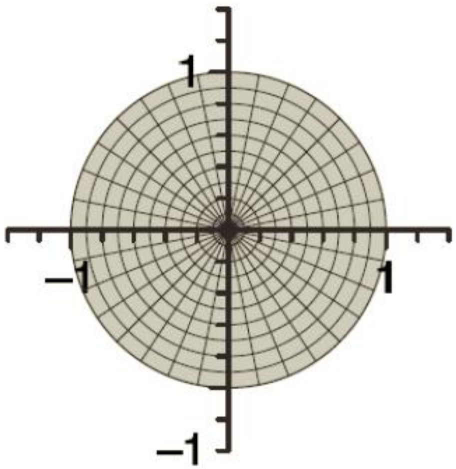


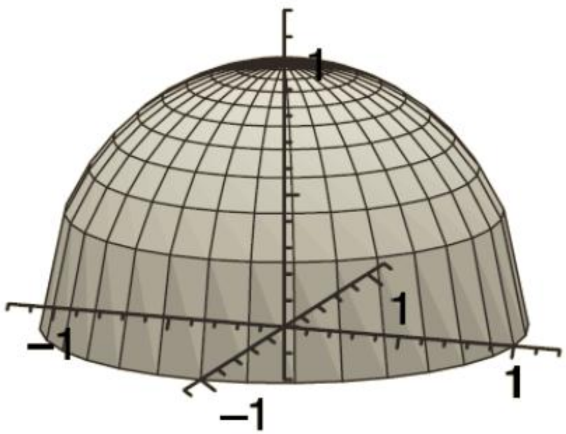


Surface Example (3)

- Hemisphere (orthographic definition) :

$$\begin{aligned} \textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\ \textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\ \textit{parameterization:} \quad & f(u,v) = (u, v, \sqrt{1 - u^2 - v^2}) \\ \textit{inverse:} \quad & f^{-1}(x,y,z) = (x,y) \end{aligned}$$

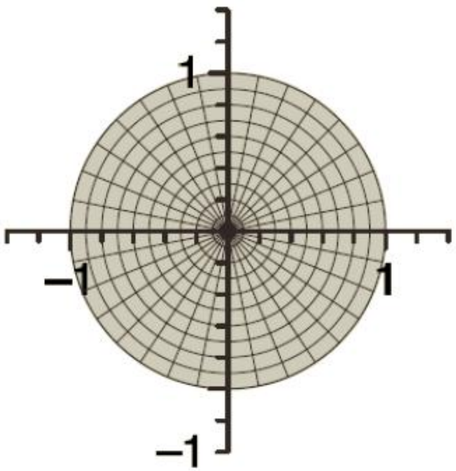


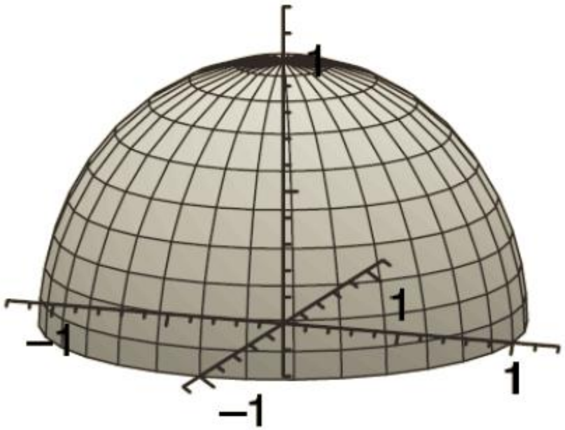


Surface Example (4)

- Hemisphere (stereographic definition) :

$$\begin{aligned}
\textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\
\textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\
\textit{parameterization:} \quad & f(u,v) = \left(\tfrac{2u}{1+u^2+v^2}, \tfrac{2v}{1+u^2+v^2}, \tfrac{1-u^2-v^2}{1+u^2+v^2}\right) \\
\textit{inverse:} \quad & f^{-1}(x,y,z) = \left(\tfrac{x}{1+z}, \tfrac{y}{1+z}\right)
\end{aligned}$$





Reparameterization

- Example (3) and (4):

  → There can be more than one parameterizations of S over Ω

- Any bijection $\varphi : \Omega \to \Omega$

  induces a reparameterization: $g = f \circ \varphi$

- Exercise: write the reparameterization $\varphi(u, v)$ between (3) and (4)

(3) $f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$

⬇ $\varphi(u, v) = ?$

(4) $f(u, v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$

$$\varphi = f_2^{-1} \circ f_1 = \left(\frac{u}{1+\sqrt{1-u^2-v^2}}, \frac{v}{1+\sqrt{1-u^2-v^2}}\right)$$

Like curves, finding a good parameterization for surfaces

→ find a good reparameterization

Tangent Plane

- Two partial derivatives:

$$\mathbf{x}_u(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) \quad \text{and} \quad \mathbf{x}_v(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$$

are the 2 tangent vectors of the two iso-parameter curves:

$$\mathbf{C_u}(t) = \mathbf{x}(u_0 + t, v_0) \quad \text{and} \quad \mathbf{C_v}(t) = \mathbf{x}(u_0, v_0 + t)$$

- Assuming a regular parameterization, i.e., $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$
- The tangent plane at this point is spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$
- The surface normal vector is orthogonal to both tangent vectors and can be computed as

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Tangent Plane (Examples)

- Surface example (3)
  - Given a point (u,v) on the orthographic hemisphere, to compute the tangent plane and normal vector:

$$f(u,v) = (u, v, \sqrt{1-u^2-v^2}) \qquad f_u(u,v) = (1,0,\frac{-u}{\sqrt{1-u^2-v^2}})$$

$$f_v(u,v) = (0,1,\frac{-v}{\sqrt{1-u^2-v^2}}) \qquad n_f(u,v) = (u, v, \sqrt{1-u^2-v^2}) = (x, y, z)$$

- Surface example (4) (exercise)
  - Given a point (u,v) on the stereographics hemisphere, to compute the tangent plane and normal vector.

- The computed Normal and tangent plane are independent of the parameterization (following our intuition)

Directional Derivatives

- Consider the straight line passing (u_0, v_0)

$$(u, v) = (u_0, v_0) + t\bar{\mathbf{w}}$$

  and a direction vector $\bar{\mathbf{w}} = (u_w, v_w)^T$ defined in parameter space
- Its corresponding curve on the surface is

$$\mathbf{C}_{\mathbf{w}}(t) \;=\; \mathbf{x}(u_0 + tu_w, v_0 + tv_w).$$

- The directional derivative $\mathbf{w}$ of $\mathbf{x}$ at (u_0, v_0) relative to the direction $\bar{\mathbf{w}}$ is defined to be the tangent to

  $\mathbf{C}_{\mathbf{w}}$ at $t = 0$, given by $\mathbf{w} = \partial\mathbf{C}_{\mathbf{w}}(t)/\partial t$

- The parameterization maps a parametric velocity vector $\bar{\mathbf{w}}$ to a vector $\mathbf{w}$ on tangent plane: $\mathbf{w} = \mathbf{J}\bar{\mathbf{w}}$

Where J: Jacobian Matrix of x :

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \left[\mathbf{x}_u \, , \, \mathbf{x}_v\right]$$

First Fundamental Form

- J encodes the metric of the surface, namely, it allows measuring how angles, distances, and areas are transformed by the mapping.
- Let $\bar{\mathbf{w}}_1$, $\bar{\mathbf{w}}_2$ be two unit direction vectors in the parameter space
- The cosine of the angle on the surface between them is:

$$\mathbf{w}_1^T \mathbf{w}_2 = (\mathbf{J}\bar{\mathbf{w}}_1)^T (\mathbf{J}\bar{\mathbf{w}}_2) = \bar{\mathbf{w}}_1^T \left(\mathbf{J}^T \mathbf{J}\right) \bar{\mathbf{w}}_2$$

- The matrix product is known as the first fundamental form:

$$\mathbf{I} = \mathbf{J}^T \mathbf{J} = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

First Fundamental Form (cont.)

- The first fundamental form $\mathbf{I}$
  - Determines the squared length of a tangent vector

$$||\mathbf{w}||^2 = \bar{\mathbf{w}}^T \mathbf{I} \bar{\mathbf{w}}$$

  - Used to measure the length of a curve $\mathbf{x}(t) = \mathbf{x}(\mathbf{u}(t))$ (image of a planar regular curve: $\mathbf{u}(t) = (u(t), v(t))$)

1) The tangent vector of the curve:

$$\frac{\mathrm{d}\mathbf{x}(\mathbf{u}(t))}{\mathrm{d}t} = \frac{\partial \mathbf{x}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial \mathbf{x}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} = \mathbf{x}_u u_t + \mathbf{x}_v v_t$$

2) So the length: $l(a,b)$ of $\mathbf{x}(\mathbf{u}(t))$ is

$$l(a,b) = \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T}\mathrm{d}t$$

$$= \int_a^b \sqrt{Eu_t^2 + 2Fu_t v_t + Gv_t^2}\mathrm{d}t.$$

First Fundamental Form (cont.)

- Used to measure the surface area: $$A = \iint_U \sqrt{\det(\mathbf{I})}\mathrm{d}u\mathrm{d}v = \iint_U \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v.$$
- Area element: $$dA = |f_u \times f_v|\, dudv = \sqrt{(f_u \cdot f_u)(f_v \cdot f_v) - (f_u \cdot f_v)^2}\, dudv = \sqrt{EG - F^2}\, dudv$$
- Example: area of a unit hemisphere (ohrographic parameterization)

$$f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

$$EG - F^2 = \frac{1}{1 - u^2 - v^2}$$

→

$$\begin{aligned} A(S) &= \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{1}{\sqrt{1 - u^2 - v^2}}\, du\, dv \\ &= \int_{-1}^{1} \left[\arcsin \frac{u}{\sqrt{1 - v^2}}\right]_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} dv \\ &= \int_{-1}^{1} \pi\, dv \\ &= 2\pi, \end{aligned}$$

- **I** allows measuring *angles, distances, and areas* → a useful geometric tool.
- Sometimes denoted by the letter **G** and called the metric tensor.

Metric Distortion

- On a surface point f(u,v)
  - A displacement on the parametric domain $(\Delta u, \Delta v)$
  - → a new point $f(u+\Delta u, v+\Delta v)$
  - Approximated by 1st order Taylor expansion:

$$\tilde{f}(u+\Delta u, v+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v$$

Planar local region: the vicinity of $\boldsymbol{u} = (u,v)$

↓

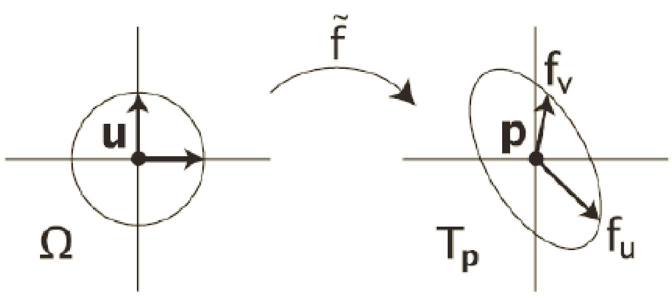
Region on tangent plane T_p at $\boldsymbol{p} = f(u,v) \in S$

Circles around **u**

↓

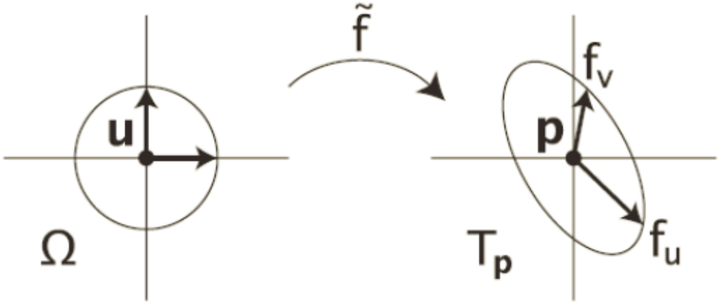
ellipses around **p**

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v} \quad \text{where } J_f = (f_u\ f_v) \text{ is the } \textit{Jacobian} \text{ of } f$$



Metric Distortion (cont.)

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v}$$

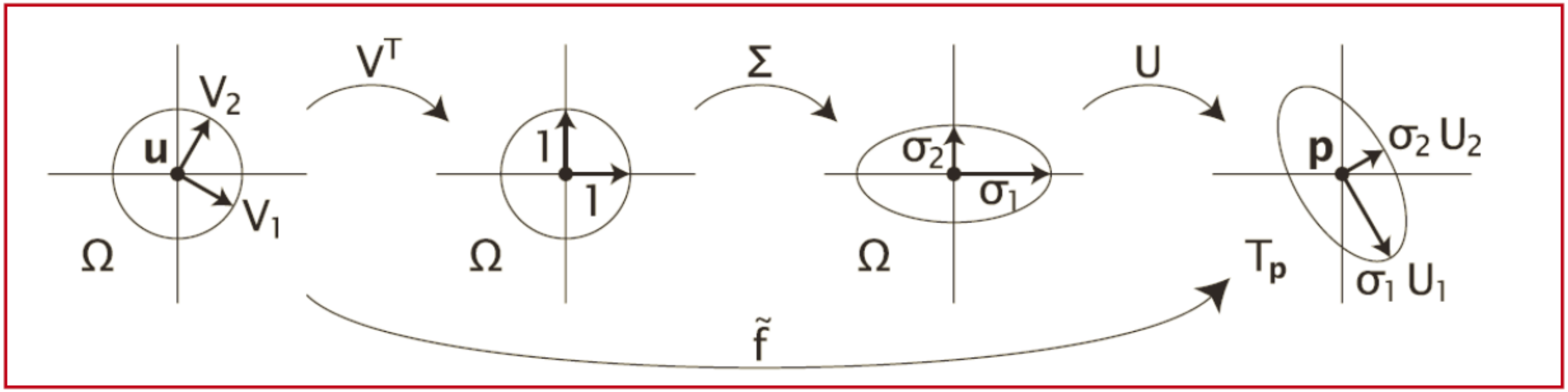


Decompose the Jacobian (3*2) matrix by SVD:

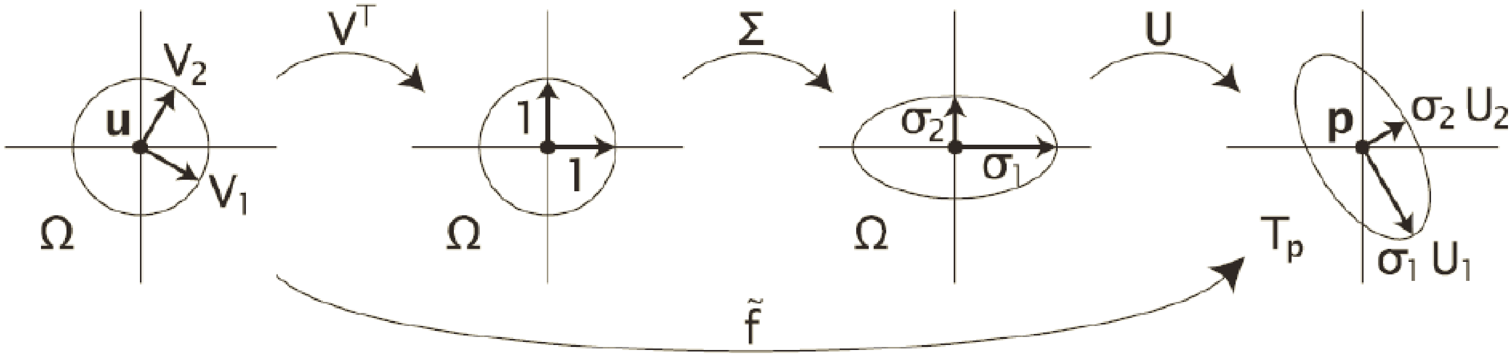
$$J_f = U\Sigma V^T = U\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{pmatrix}V^T$$

unitary, orthonormal $U \in \mathbb{R}^{3\times3}$ $V \in \mathbb{R}^{2\times2}$

$singular\ values\ \sigma_1 \geq \sigma_2 > 0$



Metric Distortion (cont.)



$$J_f = U\Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

(1) 2D Rotation V → planar rotation around **u**;

(2) Stretching matrix Σ → stretches by factor σ_1 and σ_2 in the u and v directions;

(3) 3D rotation U → map the planar region onto the tangent plane

Tiny sphere with radius-r → ellipse with semi-axes of length $r\sigma_1$ and $r\sigma_2$

$\sigma_1 = \sigma_2$	⟶	Local scaling, circles to circles	: **Confomal**
$\sigma_1\sigma_2 = 1$	⟶	Area preserved	: **Equiareal**

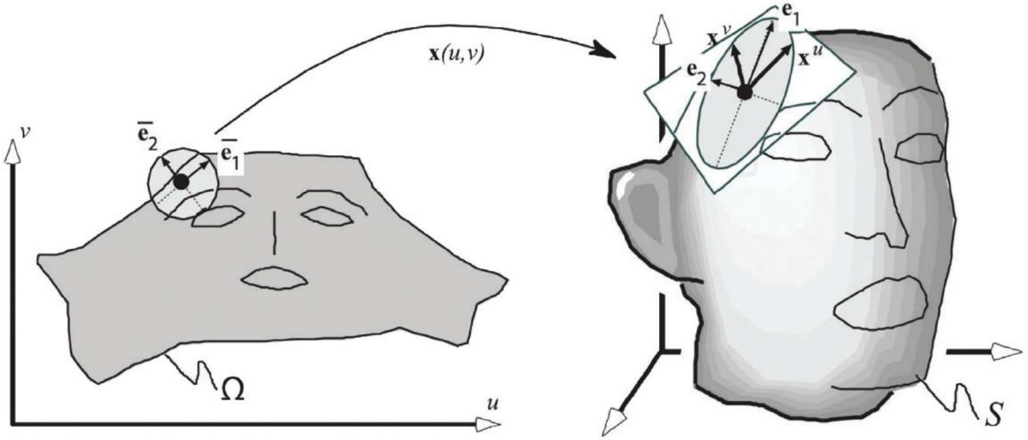
Anisotropy

- Under the Jacobian matrix, a vector $\bar{\mathbf{w}}$ is transformed into a tangent vector $\mathbf{w}$
- A unit circle → an ellipse (called anisotropy ellipse)
  - The axes of the ellipse: $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
  - The lengths of the axes: $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

    singular values of the Jacobian matrix J

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$

$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

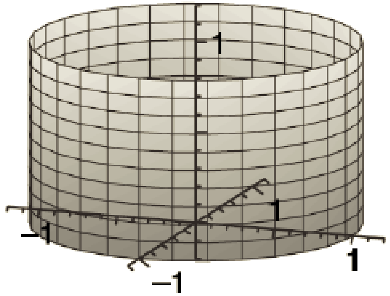


Metric Distortion Example

(1) Cylinder

- $$\textit{parameterization:} \quad f(u,v) = (\cos u, \sin u, v)$$
- $$\textit{Jacobian:} \quad J_f = \begin{pmatrix} \cos u & 0 \\ -\sin u & 0 \\ 0 & 1 \end{pmatrix}$$
- $$\textit{first fundamental form:} \quad \mathbf{I}_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
- $$\textit{eigenvalues:} \quad \lambda_1 = 1, \quad \lambda_2 = 1$$

Isometry

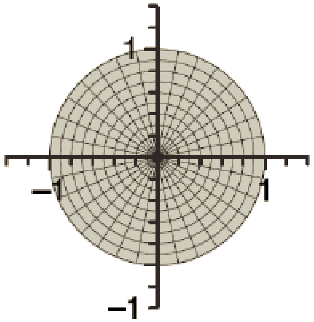


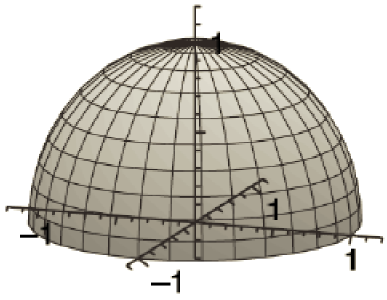
Metric Distortion Example

(2) Hemisphere (stereographic)

- *parameterization:* $f(u,v) = (2ud, 2vd, (1-u^2-v^2)d)$ where $d = \frac{1}{1+u^2+v^2}$
- *Jacobian:* $J_f = \begin{pmatrix} 2d-4u^2d^2 & -4uvd^2 \\ -4uvd^2 & 2d-4v^2d^2 \\ -4ud^2 & -4vd^2 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 4d^2 & 0 \\ 0 & 4d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 4d^2, \quad \lambda_2 = 4d^2$

Conformal



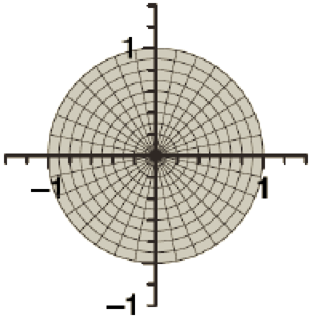


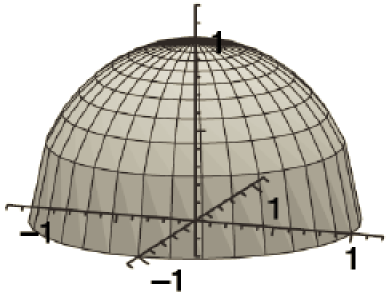
Metric Distortion Example

(3) Hemisphere (orthographic)

- *parameterization:* $f(u,v) = (u, v, \frac{1}{d})$ where $d = \frac{1}{\sqrt{1-u^2-v^2}}$
- *Jacobian:* $J_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -ud & -vd \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 1+u^2d^2 & uvd^2 \\ uvd^2 & 1+v^2d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = d^2$

Not conformal, not equiareal



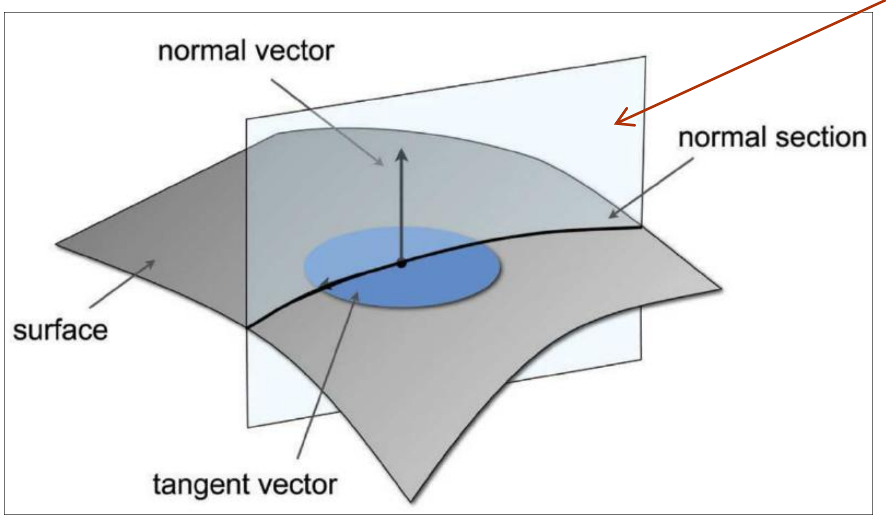


2^{nd} Order Derivatives — Surface Curvature: Normal Curvature

- How curved a surface is on a point → look at the curvature of curves embedded in the surface
- At a surface point $\mathbf{p} \in \mathcal{S}$ (parameter: $\bar{\mathbf{t}} = (u_t, v_t)^T$)
  - Pick a tangent vector $\mathbf{t} = u_t \mathbf{x}_u + v_t \mathbf{x}_v$
  - Get the surface normal vector n

  } Determines a plane

Normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$ = curvature of planar curve created by intersection of the surface and the plane



$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$$

where II denotes the 2^{nd} fundamental form:

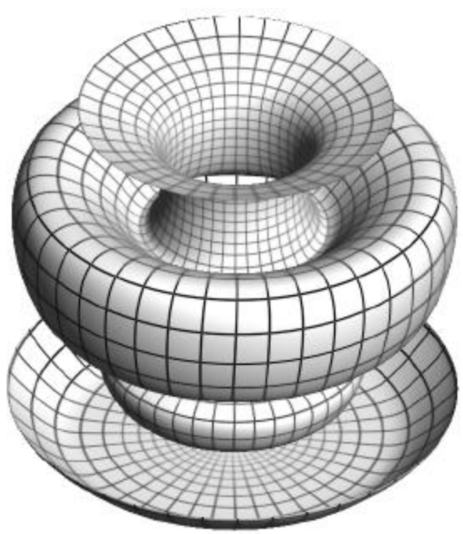
$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

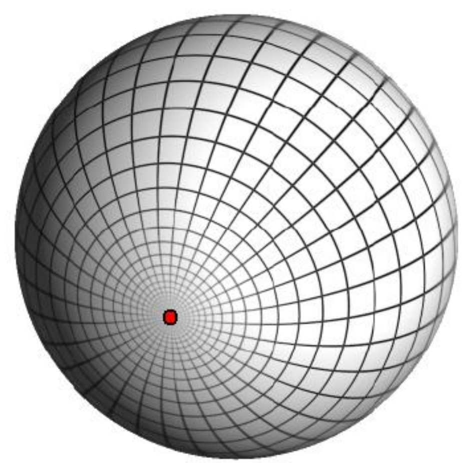
Surface Curvature: Principal Curvatures

- The curvature properties of the surface
  - Looking at all normal curvatures from rotating the tangent vector around the normal at p
- The rational quadratic function of $\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{eu_t^2 + 2fu_tv_t + gv_t^2}{Eu_t^2 + 2Fu_tv_t + Gv_t^2},$ has 2 distinct extremal values → **principal curvatures** (maximum curvature κ_1 and minimum curvature κ_2)

$\kappa_1 \neq \kappa_2$

max/min curvature
→
2 corresponding principal directions





$\kappa_1 = \kappa_2$

Isotropic curvature

Umbilical points

Euler Theorem and Curvature Tensor

- Relates principal curvatures to the normal curvature

$$\kappa_n(\bar{\mathbf{t}}) \;=\; \kappa_1 \cos^2\psi + \kappa_2 \sin^2\psi,$$

- Surface curvature encoded by two principal curvatures
- Any normal curvature is a convex combination of them

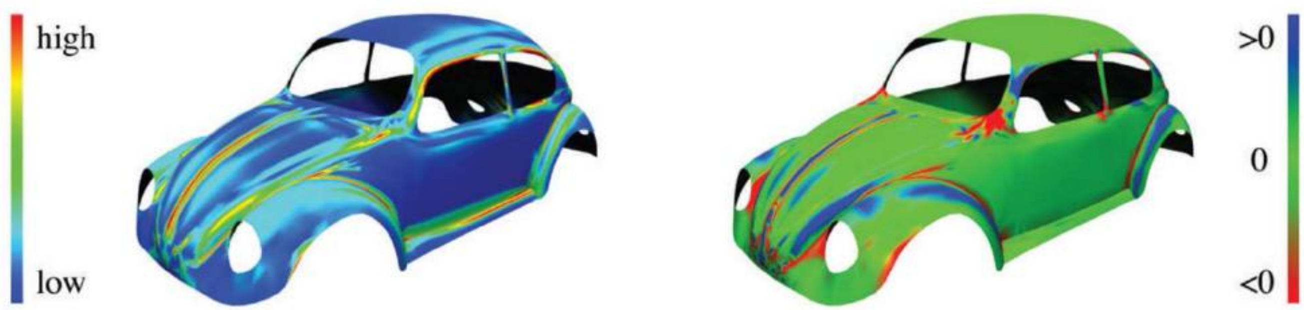
- Curvature Tensor $\mathbf{C}$
  - A symmetric 3*3 matrix with eigenvalues $\kappa_1,\ \kappa_2,\ 0$ and corresponding eigenvectors $\mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{n}$
  - Computed by
    - $\mathbf{C}=\mathbf{PDP}^{-1}$, where $\mathbf{P} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}]$ and $\mathbf{D} = \mathrm{diag}(\kappa_1, \kappa_2, 0)$

Mean and Gaussian Curvature

- Two other extensively used curvatures:
  - Mean curvature H: the average of the principal curvatures
  - Gaussian curvature K: the product of the principal curvatures

$$H = \frac{\kappa_1 + \kappa_2}{2} \qquad K = \kappa_1 \kappa_2$$

Widely use as local descriptor to analyze properties of surfaces



Another example: used for visual inspection in computer-aided geometric design.
Left: mean curvature; right: Gaussian curvature.

Intrinsic Geometry

- Intrinsic Geometry:
  - About the shape itself, not about its representation and location
  - Properties that can be perceived by 2D creatures that live on it (without knowing the 3rd dimension)
  - in differential geometry: properties that only depend on the first fundamental form (e.g. length and angles of curves on the surface, Gaussian curvature)
  - Invariant under isometries

- Extrinsic Geometry:
  - depends not only on the metrics but also the embedding of the surface
  - Could change under isometries
  - e.g. Mean curvature

Discrete Geometric Computations

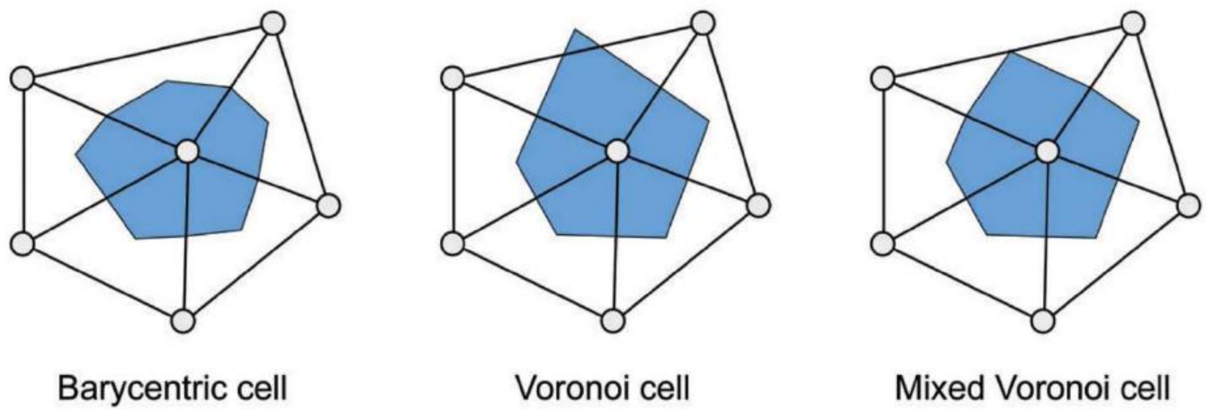
- Some integral computations on triangle meshes are straightforward:
  - Length of a discrete curve
    - Lengths of edge segments
  - Area of a discrete surface patch
    - Areas of triangle meshes
  - Volume of a solid object
    - Volumes of tetrahedral meshes

Discrete Differential Operators

- Slightly more difficult:
  - we have discussed the differential properties on a differentiable surface (e.g. at least existence of 2^{nd} derivatives)
  - How to compute them on **Polygonal meshes** which represent piecewise linear surfaces
  - to compute the approximations of the differential properties of the underlying surface
  - General idea : to compute discrete differential properties as spatial averages over a local neighborhood $N(x)$ of a point x on the mesh

Local Averaging Region

- A straightforward approximation:
  - x → mesh vertex v_i
  - $N(x)$ → one-ring (n-ring) neighborhoods $N_n(v_i)$
- Size of local neighborhoods → stability and accuracy of evaluation
  - Bigger: more smoothing, more stable against of noise
  - Smaller: more accurately capture fine-scale variations; preferable for clean data
- More accurate approximation
  - Barycentric cell: connect triangle barycenters + edge midpoints
  - Voronoi cell: triangle circumcenters + …
  - Mixed-voronoi cell: midpoint of edge opposing obtuse angle on center vertex + …



Normal Vectors

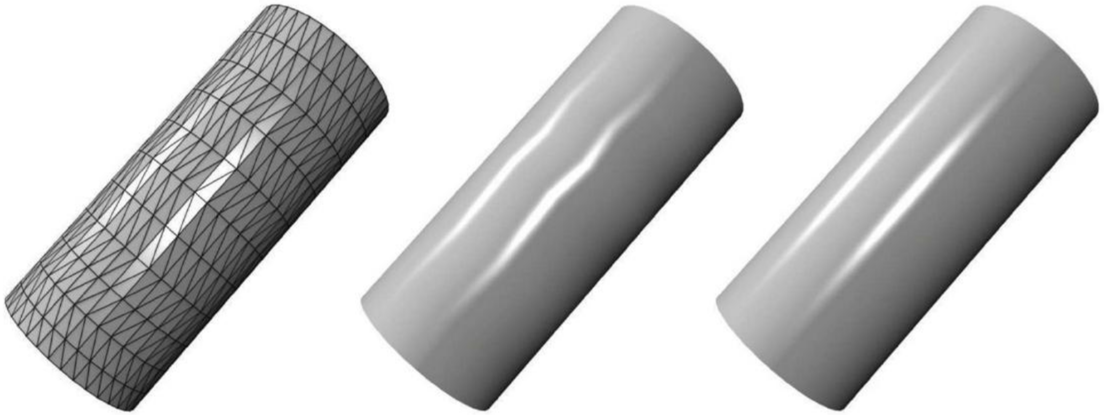
- Many operations in computer graphics require normal vectors (per face or per vertex), e.g. phone shading
- Face Normal vector: the normalized cross-product of two triangle edges:

$$\mathbf{n}(T) = \frac{(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)}{\|(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)\|}$$

- Vertex Normal: (spatial averages of normal vectors in a local one-ring neighborhood)

$$\mathbf{n}(v) = \frac{\sum_{T \in \mathcal{N}_1(v)} \alpha_T \, \mathbf{n}(T)}{\left\| \sum_{T \in \mathcal{N}_1(v)} \alpha_T \, \mathbf{n}(T) \right\|}$$

  - Different weights used:
    - Constant weights: $\alpha_T = 1$ (efficient, not good on irregular meshes)
    - Triangle area: $\alpha_T = |T|$ (efficient, may be problematic on obtuse triangles)
    - Incident triangle angles: $\alpha_T = \theta_T$ (usually natural, slightly expensive)



Gradients (1st order derivatives)

- A piecewise linear function f defined on vertex

$$f(v_i) = f(\mathbf{x}_i) = f(\mathbf{u}_i) = f_i$$

- The function is interpolated linearly within the triangle $(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)$

$$f(\mathbf{u}) = f_i B_i(\mathbf{u}) + f_j B_j(\mathbf{u}) + f_k B_k(\mathbf{u})$$

where $B_i(u)$ is the barycentric coordinate

- In this triangle: $\nabla f(\mathbf{u}) = f_i \nabla B_i(\mathbf{u}) + f_j \nabla B_j(\mathbf{u}) + f_k \nabla B_k(\mathbf{u})$ (1)
- Partition of unity → $\nabla B_i(\mathbf{u}) + \nabla B_j(\mathbf{u}) + \nabla B_k(\mathbf{u}) = 0$ (2)

(1),(2) → $\nabla f(\mathbf{u}) = (f_j - f_i)\nabla B_j(\mathbf{u}) + (f_k - f_i)\nabla B_k(\mathbf{u})$

$$B_i(\vec{u}) = \frac{(\vec{u}_j - \vec{u}) \times (\vec{u}_k - \vec{u}_j)}{2A_T} \quad \Rightarrow \quad \nabla B_i(u) = \frac{(\vec{u}_k - \vec{u}_j)^{\perp}}{2A_T}$$

- The gradient is constant in each triangle

Laplace-Beltrami Operator and Curvature (2nd order derivatives)

- Review **Laplace operator** in continuous case:
  - Defined as the divergence of the gradient: $\Delta = \nabla^2 = \nabla \cdot \nabla$
  - For a 2-parameter function f(u,v)
  - In Euclidean space:

$$\Delta f = \mathrm{div} \nabla f = \mathrm{div} \begin{pmatrix} f_u \\ f_v \end{pmatrix} = f_{uu} + f_{vv}$$

  - On surfaces: Laplace-Beltrami operator $\Delta_S f = \mathrm{div}_S \nabla_S f,$

    (imagine a gradient vector field on a surface, then think about its divergence)

- Applied to the coordinate function x of the surface
  - The Laplace-Beltrami operator = mean curvature normal ([do Carmo 76])

$$\Delta_S \mathbf{x} = -2H\mathbf{n}.$$

Often, we directly write it as Δ for simplicity

Discrete Curvature

1) Discrete Mean Curvature: $H(v_i) = \frac{1}{2}\|\Delta \mathbf{x}_i\|$

2) Discrete Guassian Curvature [Mayer et al. 03]:

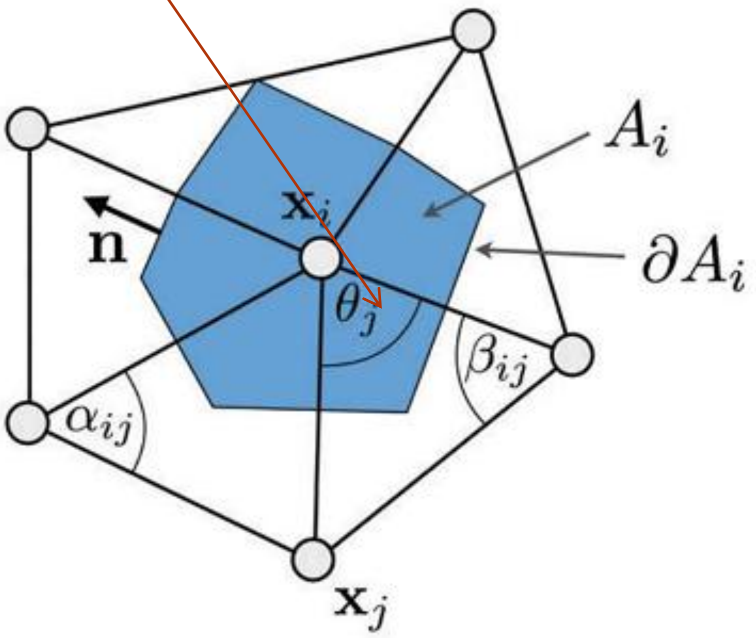
$$K(v_i) = \frac{1}{A_i}\left(2\pi - \sum_{v_j \in \mathcal{N}_1(v_i)} \theta_j\right)$$

3) Principal Curvature:

$$\kappa_{1,2}(v_i) = H(v_i) \pm \sqrt{H(v_i)^2 - K(v_i)}$$

Recall that:

$$H = \frac{\kappa_1 + \kappa_2}{2} \qquad K = \kappa_1 \kappa_2$$



Discrete Laplace-Betrami Operator (1)

- Uniform Laplacian ([Taubin 95], suitable for uniformly sampled surfaces)

$$\Delta f(v_i) = \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (f_j - f_i),$$

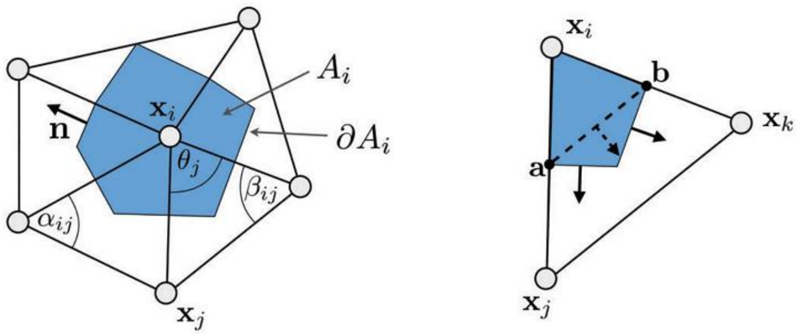
  - Applied to the coordinate function x:
    a vector pointing from the center vector to the average of the one-ring vertices
  - Not a good approximation for irregular triangle meshes
    - E.g. On a planar triangle mesh, this vector is often not zero. But according to its mean curvature, it should be.

Discrete Laplace-Beltrami Operator (2)

- Cotangent Formula (more accurate, most widely used)
  - To integrate the divergence of the gradient over a local averaging domain Ai,
  - by Divergence Theorem

$$\int_{A_i} \operatorname{div} \mathbf{F}(\mathbf{u})\, \mathrm{d}A = \int_{\partial A_i} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u})\, \mathrm{d}s$$

where n is the outward pointing unit normal

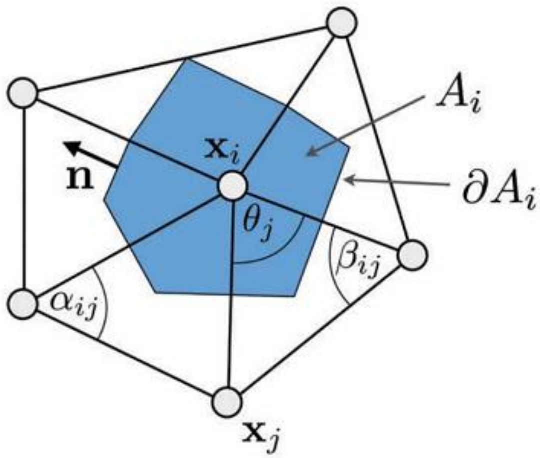


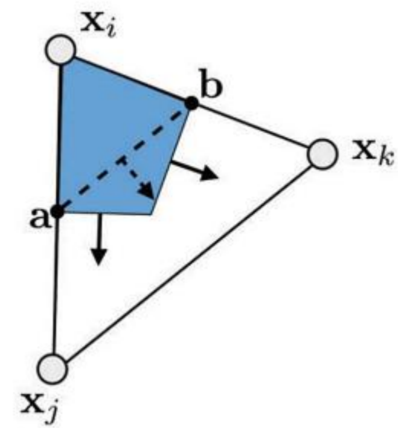
for Lapalacian we have:

$$\int_{A_i} \Delta f(\mathbf{u})\, \mathrm{d}A = \int_{A_i} \operatorname{div} \nabla f(\mathbf{u})\, \mathrm{d}A = \int_{\partial A_i} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u})\, \mathrm{d}s$$

Discrete Laplace-Beltrami Operator (2)

Now consider this integration on triangle mesh:





The integral on one triangle: (the boundary of the local Voronoi region passes through the midpoints a and b of the two triangle edges, gradient in a triangle is constant → equals integral through ab)

$$\int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \mathrm{d}s = \nabla f(\mathbf{u}) \cdot (\mathbf{a} - \mathbf{b})^{\perp}$$

$$= \frac{1}{2} \nabla f(\mathbf{u}) \cdot (\mathbf{x}_j - \mathbf{x}_k)^{\perp}.$$

Discrete Laplace-Beltrami Operator (2)

Plugging in the gradient equation, we get

$$\int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \mathrm{d}s = (f_j - f_i) \frac{(\mathbf{x}_i - \mathbf{x}_k)^\perp \cdot (\mathbf{x}_j - \mathbf{x}_k)^\perp}{4A_T} + (f_k - f_i) \frac{(\mathbf{x}_j - \mathbf{x}_i)^\perp \cdot (\mathbf{x}_j - \mathbf{x}_k)^\perp}{4A_T}.$$

Let γ_j, γ_k denote the inner triangle angles at vertices v_j, v_k, respectively. Since $A_T = \frac{1}{2} \sin \gamma_j \|\mathbf{x}_j - \mathbf{x}_i\| \|\mathbf{x}_j - \mathbf{x}_k\| = \frac{1}{2} \sin \gamma_k \|\mathbf{x}_i - \mathbf{x}_k\| \|\mathbf{x}_j - \mathbf{x}_k\|$, and $\cos \gamma_j = \frac{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{x}_j - \mathbf{x}_k)}{\|\mathbf{x}_j - \mathbf{x}_i\| \|\mathbf{x}_j - \mathbf{x}_k\|}$ and $\cos \gamma_k = \frac{(\mathbf{x}_i - \mathbf{x}_k) \cdot (\mathbf{x}_j - \mathbf{x}_k)}{\|\mathbf{x}_i - \mathbf{x}_k\| \|\mathbf{x}_j - \mathbf{x}_k\|}$, this expression simplifies to

$$\int_{\partial A_i \cap T} \nabla f(\mathbf{u}) \cdot \mathbf{n}(\mathbf{u}) \mathrm{d}s = \frac{1}{2} \left(\cot \gamma_k (f_j - f_i) + \cot \gamma_j (f_k - f_i) \right).$$

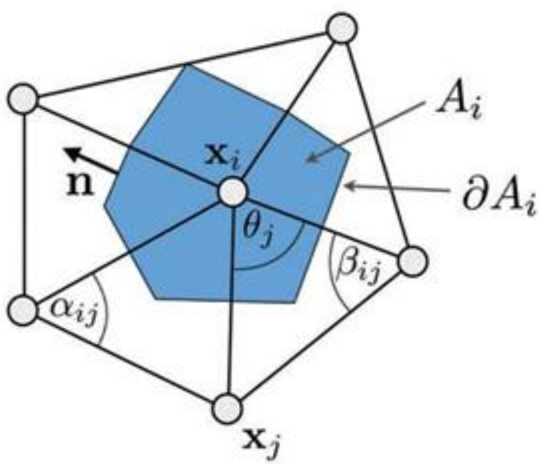
Discrete Laplace-Beltrami Operator (2)

The final integration over the entire averaging region:

$$\int_{A_i} \Delta f(\mathbf{u}) \mathrm{d}A = \frac{1}{2} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{i,j} + \cot \beta_{i,j})(f_j - f_i),$$

In other words:

$$\Delta f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{i,j} + \cot\beta_{i,j})(f_j - f_i).$$



For more details, check: ["Discrete Differential-Geometry Operators for Triangulated 2-Manifolds," by Meyer, Desbrun, Schroder, Barr, 2003]