Lecture 4
Basic Geometry of Curves and Surfaces

Basic Geometry of Curves and Surfaces

- Start with geometric properties of smooth curves and surfaces
- Then discuss their computation on polygonal meshes

For more properties or proofs of these geometric concepts, refer to standard differential geometry textbooks :

e.g. [do Carmo 76: Differential Geometry of Curves and Surfaces, Prentice Hall, 1976]

Curves

- Consider smooth planar curves: differentiable 1-manifolds embedded in R^2
  - Parametric form: $\mathbf{x} : [a,b] \to \mathbb{R}^2$ with $\mathbf{x}(u) = (x(u), y(u))^T$
    $u \in [a,b] \subset \mathbb{R}$
  - Coordinates x and y are differentiable functions of u
  - Tangent vector x'(u) to the curve at a point x(u) is defined as the first derivative of the coordinate function: $\mathbf{x}'(u) = (x'(u), y'(u))^T$
  - The trajectory of a point is a curve parameterized by time (u=t) the tangent vector x'(t) → the velocity vector at time t
  - Assume parameterization to be regular, s.t. $\mathbf{x}'(u) \neq \mathbf{0}$ for all $u \in [a,b]$
  - A normal vector n(u) at x(u) can be computed as

$$\mathbf{n}(u) = \mathbf{x}'(u)^{\perp} / \|\mathbf{x}'(u)^{\perp}\|$$

where $^{\perp}$ denotes rotation by 90 degree ccw.

Parameterization of a Curve

- A curve is the image of a function x
- Same curve can be obtained with different parameterizations:
  → <u>same trajectory using different speeds</u>
- With different parameterizations x_1 and x_2, we usually have
  $\mathbf{x}_1(u) \neq \mathbf{x}_2(u)$ on a given u
- Different representations for a same shape
  - Can reparameterize a curve using a different mapping function
    with g: u→t , $\mathbf{x}_1(u) \rightarrow \mathbf{x}_2(t)$
- We want to extract properties of a curve that are independent of its specific parameterization, e.g. length, curvature…

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Arc Length Parameterization

- Curve length: $l(c,d) = \int_c^d \|\mathbf{x}'(u)\| \mathrm{d}u$
- A unique parameterization that can be defined as a length-preserving mapping, i.e., isometry, between the parameter interval and the curve using the parameterization

$$s = s(u) = \int_a^u \|\mathbf{x}'(t)\| \mathrm{d}t.$$

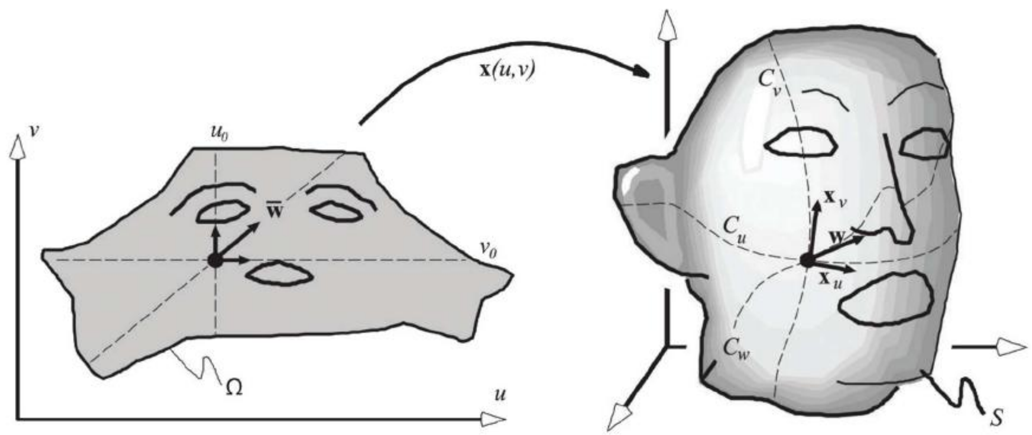
- Arc length parameterization $x(s)$:
  - the length of the curve from $x(0)$ to $x(s)$ is equal to s
  - independent of specific representation of the curve, maps the parameter interval $[a,b]$ to $[0,L]$
  - Any regular curve can be parameterized using arc length (isometry)
    - → ideal parameterization, many computations simplified
    - → doesn't work for surfaces (later)

Surfaces

- Consider a smooth surface patch: differentiable 2-manifold embedded in R^3
  - Parametric form: $$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \Omega \subset \mathbb{R}^2,$$

    where x,y,z are differentiable functions in u and v,
  - Scalars (u,v) are called coordinates in parameter space

- Like tangent vectors of curves determine the metric of the curve,
- The first derivatives of $\mathbf{X}$ determines the metric of the surface



Tangent Plane

- Two partial derivatives:

$$\mathbf{x}_u(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) \quad \text{and} \quad \mathbf{x}_v(u_0, v_0) := \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)$$

are the 2 tangent vectors of the two iso-parameter curves:

$$\mathbf{C}_\mathbf{u}(t) = \mathbf{x}(u_0 + t, v_0) \quad \text{and} \quad \mathbf{C}_\mathbf{v}(t) = \mathbf{x}(u_0, v_0 + t)$$

- Assuming a regular parameterization, i.e., $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$
- The <u>tangent plane</u> at this point is spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$
- The surface <u>normal vector</u> is orthogonal to both tangent vectors and can be computed as

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

Directional Derivatives

- Consider the straight line passing (u_0, v_0)

$$(u, v) = (u_0, v_0) + t\bar{\mathbf{w}}$$

and a direction vector $\bar{\mathbf{w}} = (u_w, v_w)^T$ defined in parameter space

- Its corresponding curve on the surface is

$$\mathbf{C}_{\mathbf{w}}(t) \;=\; \mathbf{x}(u_0 + tu_w, v_0 + tv_w).$$

- The directional derivative $\mathbf{w}$ of $\mathbf{x}$ at (u_0, v_0) relative to the direction $\bar{\mathbf{w}}$ is defined to be the tangent to

$\mathbf{C}_{\mathbf{w}}$ at $t = 0$, given by $\mathbf{w} = \partial \mathbf{C}_{\mathbf{w}}(t)/\partial t$

- Mapping the velocity vector to another: $\mathbf{w} = \mathbf{J}\bar{\mathbf{w}}$

Where J: Jacobian Matrix of x : $\mathbf{J} \;=\; \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \;=\; [\mathbf{x}_u \;,\; \mathbf{x}_v]$

First Fundamental Form

- J → linear map that transforms a vector $\bar{\mathbf{w}}$ in parametric space into a tangent vector $\mathbf{w}$ on the surface.
- J encodes the metric of the surface, namely, it allows measuring how angles, distances, and areas are transformed by the mapping.
- Let $\bar{\mathbf{w}}_1$, $\bar{\mathbf{w}}_2$ be two unit direction vectors in the parameter space
- The cosine of the angle on the surface between them is:

$$\mathbf{w}_1^T \mathbf{w}_2 = (\mathbf{J}\bar{\mathbf{w}}_1)^T (\mathbf{J}\bar{\mathbf{w}}_2) = \bar{\mathbf{w}}_1^T \left(\mathbf{J}^T \mathbf{J}\right) \bar{\mathbf{w}}_2$$

- The matrix product is known as the first fundamental form:

$$\mathbf{I} = \mathbf{J}^T \mathbf{J} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{bmatrix}$$

First Fundamental Form (cont.)

- The first fundamental form $\mathbf{I}$
  - Determines the squared length of a tangent vector

$$||\mathbf{w}||^2 = \bar{\mathbf{w}}^T \mathbf{I} \bar{\mathbf{w}}$$

  - Used to measure the length of a curve $\mathbf{x}(t) = \mathbf{x}(\mathbf{u}(t))$

(image of a planar regular curve: $\mathbf{u}(t) = (u(t), v(t))$)

1) The tangent vector of the curve:

$$\frac{\mathrm{d}\mathbf{x}(\mathbf{u}(t))}{\mathrm{d}t} = \frac{\partial \mathbf{x}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial \mathbf{x}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} = \mathbf{x}_u u_t + \mathbf{x}_v v_t$$

2) So the length: $l(a,b)$ of $\mathbf{x}(\mathbf{u}(t))$ is

$$l(a,b) = \int_a^b \sqrt{(u_t, v_t)\mathbf{I}(u_t, v_t)^T}\mathrm{d}t$$

$$= \int_a^b \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2}\mathrm{d}t.$$

First Fundamental Form (cont.)

- Used to measure the surface area:

$$A = \iint_U \sqrt{\det(\mathbf{I})}\mathrm{d}u\mathrm{d}v = \iint_U \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v.$$

- Since it allows measuring angles, distances, and areas, the first fundamental form $\mathbf{I}$ can be considered as a geometric tool.
- Sometimes denoted by the letter $\mathbf{G}$ and called the metric tensor.

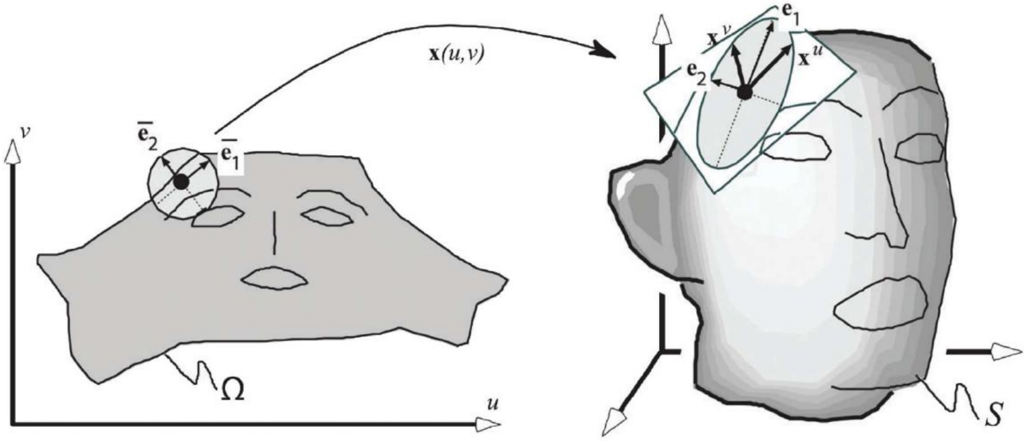
Anisotropy

- Under the Jacobian matrix, a vector $\bar{\mathbf{w}}$ is transformed into a tangent vector $\mathbf{w}$
- A unit circle → an ellipse (called anisotropy ellipse)
  - The axes of the ellipse: $\mathbf{e}_1 = \mathbf{J}\bar{\mathbf{e}}_1$ and $\mathbf{e}_2 = \mathbf{J}\bar{\mathbf{e}}_2$;
  - The lengths of the axes: $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$.

    singular values of the Jacobian matrix J

$$\sigma_1 = \sqrt{1/2(E+G) + \sqrt{(E-G)^2 + 4F^2}},$$

$$\sigma_2 = \sqrt{1/2(E+G) - \sqrt{(E-G)^2 + 4F^2}},$$

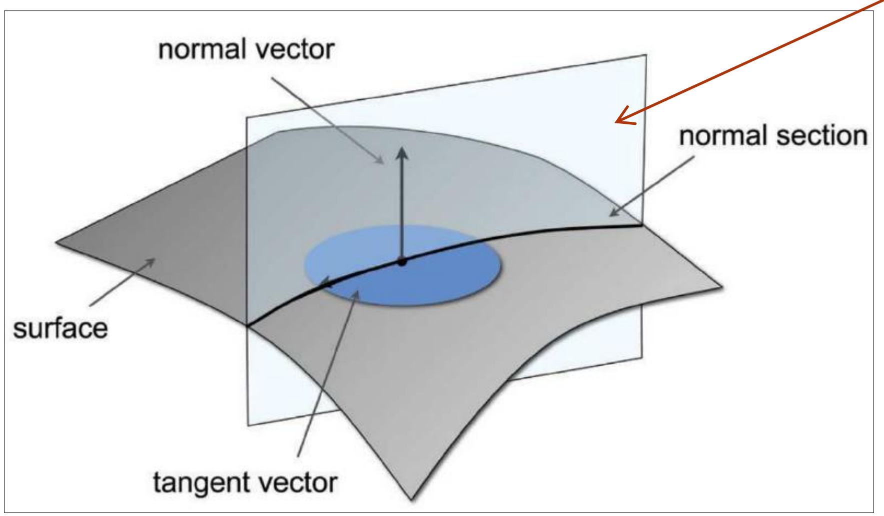


Surface Curvature: Normal Curvature

- How curved a surface is on a point → look at the curvature of curves embedded in the surface
- At a surface point $\mathbf{p} \in \mathcal{S}$ (parameter: $\bar{\mathbf{t}} = (u_t, v_t)^T$)
  - Pick a tangent vector $\mathbf{t} = u_t \mathbf{x}_u + v_t \mathbf{x}_v$
  - Get the surface normal vector n

} Determines a plane

Normal curvature $\kappa_n(\bar{\mathbf{t}})$ at $\mathbf{p}$ = curvature of planar curve created by intersection of the surface and the plane



$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$$

where II denotes the 2nd fundamental form:

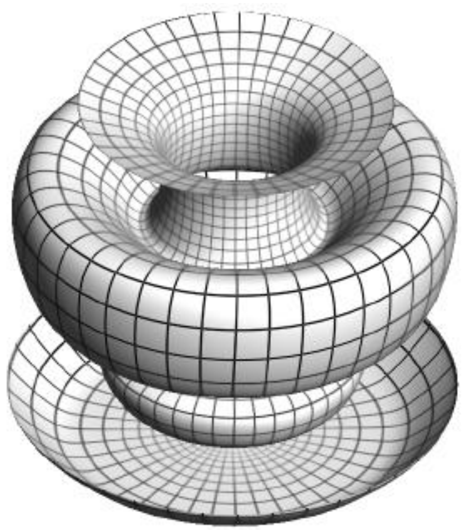
$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{bmatrix}$$

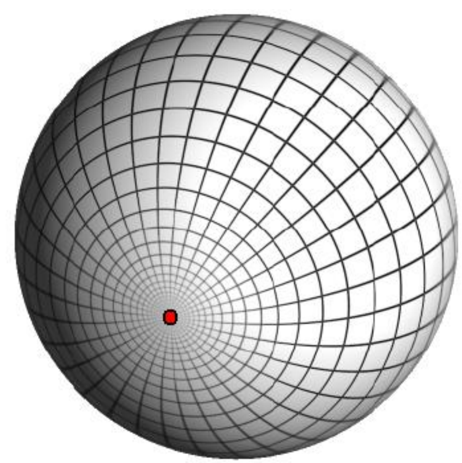
Surface Curvature: Principal Curvatures

- The curvature properties of the surface
  - Looking at all normal curvatures from rotating the tangent vector around the normal at p
- The rational quadratic function of $\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \mathbf{II}\, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{e u_t^2 + 2 f u_t v_t + g v_t^2}{E u_t^2 + 2 F u_t v_t + G v_t^2},$ has 2 distinct extremal values → **principal curvatures** (maximum curvature κ_1 and minimum curvature κ_2)

$\kappa_1 \neq \kappa_2$

max/min curvature
→
2 corresponding principal directions





$\kappa_1 = \kappa_2$

Isotropic curvature

Umbilical points

Euler Theorem and Curvature Tensor

- Relates principal curvatures to the normal curvature

$$\kappa_n(\bar{\mathbf{t}}) = \kappa_1 \cos^2 \psi + \kappa_2 \sin^2 \psi,$$

- Surface curvature encoded by two principal curvatures
- Any normal curvature is a convex combination of them

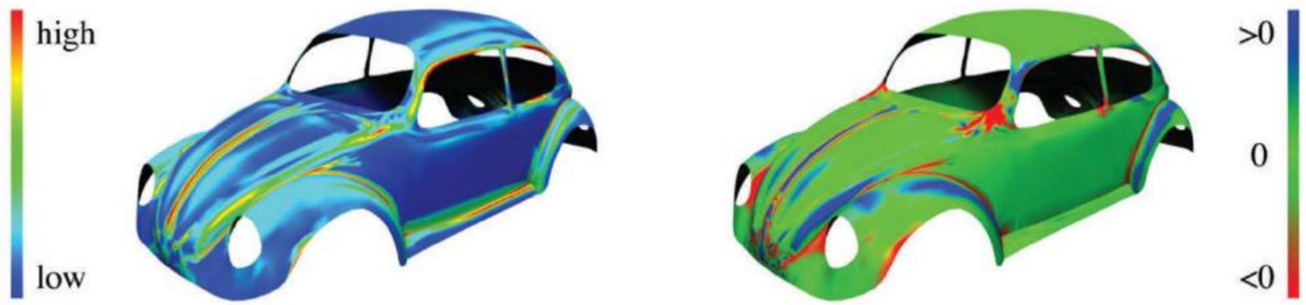
- Curvature Tensor $\mathbf{C}$
  - A symmetric 3*3 matrix with eigenvalues $\kappa_1,\ \kappa_2,\ 0$ and corresponding eigenvectors $\mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{n}$
  - Computed by
    - $\mathbf{C}=\mathbf{PDP}^{-1}$, where $\mathbf{P} = [\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}]$ and $\mathbf{D} = \mathrm{diag}(\kappa_1, \kappa_2, 0)$

Mean and Gaussian Curvature

- Two other extensively used curvatures:
  - Mean curvature H: the average of the principal curvatures
  - Gaussian curvature K: the product of the principal curvatures

$$H = \frac{\kappa_1 + \kappa_2}{2} \qquad K = \kappa_1 \kappa_2$$

Widely use as local descriptor to analyze properties of surfaces



Another example: used for visual inspection in computer-aided geometric design.
Left: mean curvature; right: Gaussian curvature.

Intrinsic Geometry

- Intrinsic Geometry: Properties of the surface that can be perceived by 2D creatures that live on it (without knowing the 3rd dimension)
  - in differential geometry: properties that only depend on the first fundamental form (e.g. length and angles of curves on the surface, Gaussian curvature)
  - Invariant under isometries

- Extrinsic Geometry:
  - depends not only on the metrics but also the embedding of the surface
  - Could change under isometries
  - e.g. Mean curvature