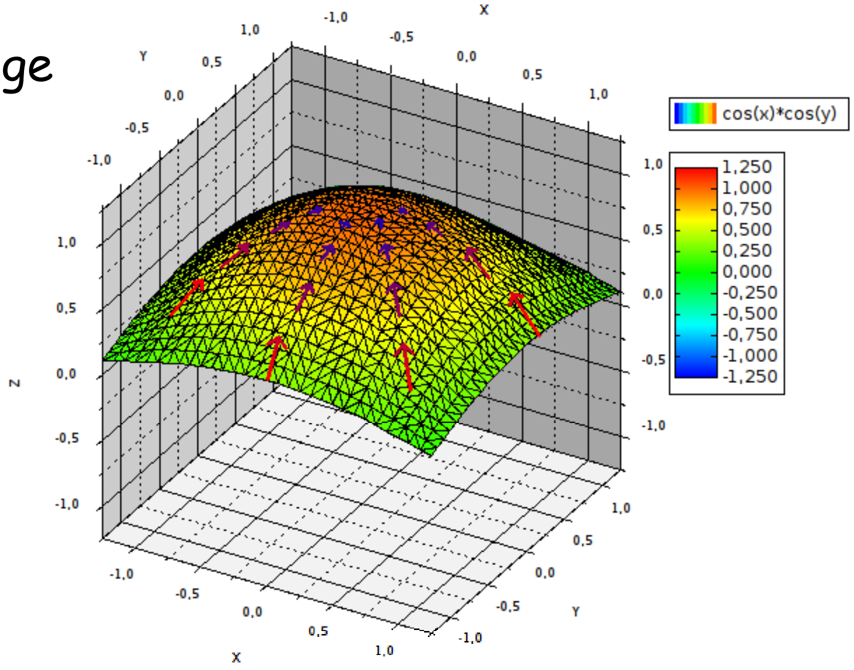
Harmonic Map & its Intuition

- Minimizing deformation

$$E_D(f) = \frac{1}{2}\int_S \|\mathrm{grad} f\|^2 = \frac{1}{2}\int_S \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)$$

-- minimize the magnitude of the change

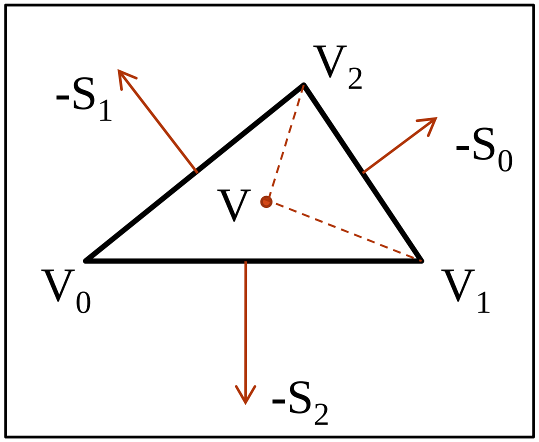
- Intuitive explanation
  - 1D
  - 2D
  - 3D



Harmonic Map on Mesh

- Following the smooth case definition → discrete setting:

$$E(f)=\int_S \|\nabla f\|^2\,ds=\sum_{\Delta\in F}<\nabla f_\Delta,\nabla f_\Delta>A_\Delta$$



- Look at one triangle (V_1, V_2, V_3):
  - Define: $S_i=\underline{n}\times(V_{\underline{i+2}}-V_{\underline{i+1}})$

    Normalized normal index mod 3
  - We have: $S_0+S_1+S_2=n\times(V_2-V_1+V_0-V_2+V_1-V_0)=0$

    ➔ $<S_i,S_i>=<S_i,-\sum_{j\neq i}S_j>=-\sum_{j\neq i}<S_i,S_j>$
- An interior point V can be represented by barycentric coordinates:

$$V=\sum_i\lambda_iV_i,\quad \lambda_i=A_i/A \quad \text{and} \quad A_i=\frac{1}{2}|VV_{i+1}||V_{i+1}V_{i+2}|\sin(\angle VV_{i+1}V_{i+2})=<-S_i,V_{i+1}-V>$$

Linear function: $f(V)=\sum_i f(\lambda_iV_i)=\sum_i\lambda_i f(V_i)=\sum_i\frac{f(V_i)}{2A}<S_i,V>-\underline{\sum_i\frac{f(V_i)}{2A}<S_i,V_{i+1}>}$

$$\nabla f(V)=\sum_i\frac{1}{2A}f_iS_i\ ,\ f_i\leftarrow f(V_i)$$

Harmonic Map on Mesh (cont.)

- The local energy: $< \nabla f_\Delta, \nabla f_\Delta > A = \frac{1}{4A} < \sum_i f_i S_i, \sum_j f_j S_j >$

$$= \frac{1}{4A}\left(\sum_i f_i^{\,2} < S_i, S_i > + 2\sum_{i<j} f_i f_j < S_i, S_j >\right)$$

(because $< S_i, S_i > = -\sum_{j \neq i} < S_i, S_j >$)

$$= \frac{1}{4A}\left(-f_0^{\,2}(< S_0, S_1 > + < S_0, S_2 >) \ldots + 2\sum_{i<j} f_i f_j < S_i, S_j >\right)$$

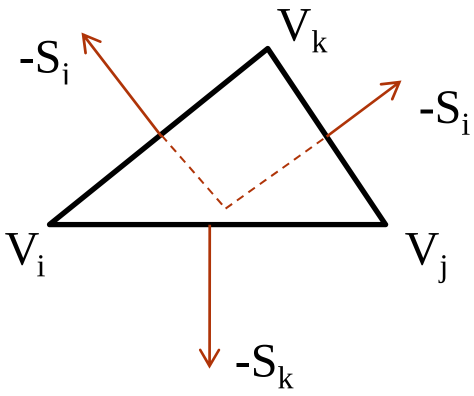
$$= \frac{-1}{4A}\left((f_0 - f_1)^2 < S_0, S_1 > + \ldots\right)$$

$$= \frac{-1}{4A}\sum_{i<j}(f_i - f_j)^2 < S_i, S_j >$$

Therefore : $E_\Delta(f) = \frac{1}{2}\sum_{i<j} w_{ij}(f_j - f_i)^2$

where $$w_{ij} = -\frac{< S_i, S_j >}{2A}$$

$$= -\frac{e_i e_j \cos(\pi - \theta_k)}{e_i e_j \sin\theta_k} = ctg(\theta_k)$$



Harmonic Map on Mesh (cont.)

- Total discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{halfedge(i,j)} w_{ij}(f_j - f_i)^2$$

- It is minimized when

$$\frac{\partial E(f)}{\partial f_i} = \sum_{halfedge(i,j)} w_{ij}(f_j - f_i) = 0$$

$$\boxed{f_i = \frac{\sum (ctg\theta_{ij} + ctg\theta_{ji}) f_j}{\sum (ctg\theta_{ij} + ctg\theta_{ji})}}$$

Cotangent Weights of Discrete Harmonic Map

Mean Value Coordinates

- ❑ A problem of the cotangent weight

$$w_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji}$$

Need remeshing? or

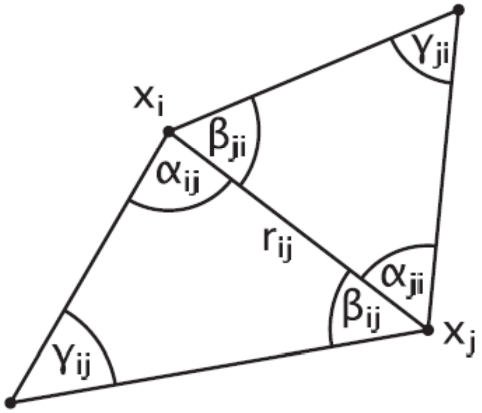
- ❑ Weights with "barycentric" property:

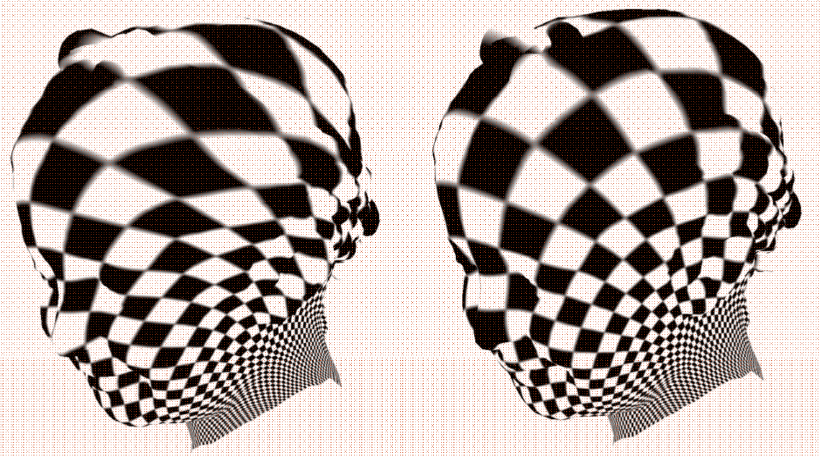
$$\begin{cases} V = \sum \lambda_i V_i \\ \sum \lambda_i = 1, \forall \lambda_i > 0 \end{cases}$$

Using Mean Value Property of the Harmonic Function

Mean Value Weights

$$w_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}}$$





Mean Value Harmonic

The definition review

- Simply connected domain $\Omega \subset \mathbb{R}^2$

$$\text{the } \textit{unit square}: \quad \Omega = \{(u,v) \in \mathbb{R}^2 : u, v \in [0,1]\}, \quad \text{or}$$

$$\text{the } \textit{unit disk}: \quad \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\},$$

- A continuous injection (no 2 distinct points → same point) $f : \Omega \to \mathbb{R}^3$
- The image S of Ω under f ➔ a surface

$$S = f(\Omega) = \{f(u,v) : (u,v) \in \Omega\},$$

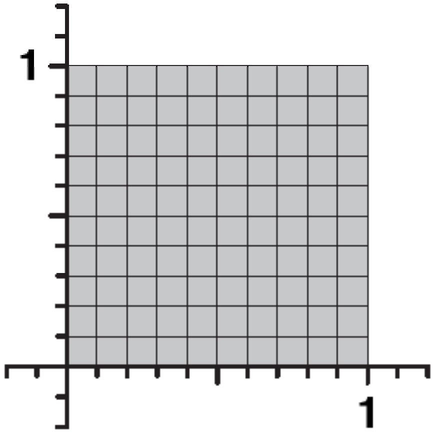
f is a parameterization of S over the parameter domain Ω

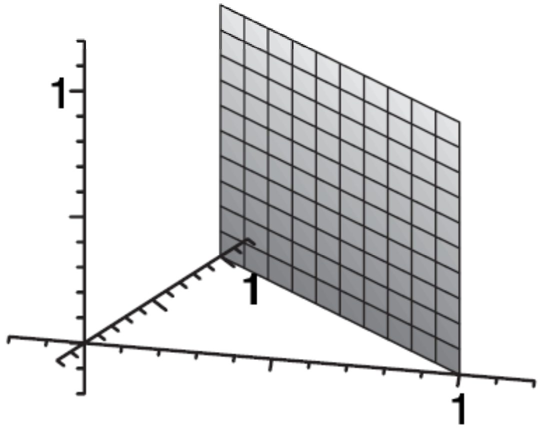
- →f is a bijection between Ω and S → $f^{-1} : S \to \Omega$

Surface Examples (1)

- Simple linear function:

$$\begin{aligned} \textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u,v \in [0,1]\} \\ \textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \in [0,1], x+y=1\} \\ \textit{parameterization:} \quad & f(u,v) = (u, 1-u, v) \\ \textit{inverse:} \quad & f^{-1}(x,y,z) = (x,z) \end{aligned}$$

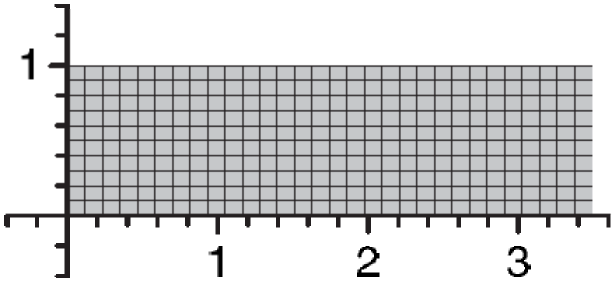


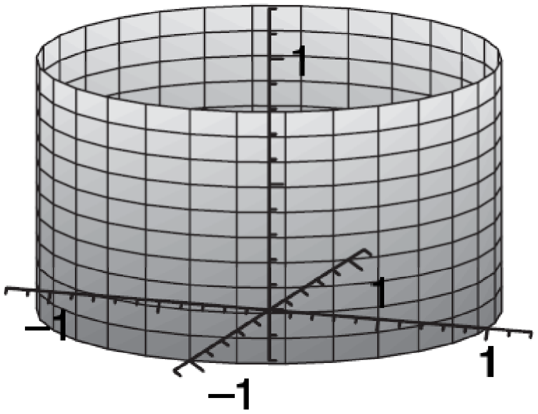


Surface Examples (2)

- Cylinder:

$$\begin{aligned}
\textit{parameter domain:}\quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u \in [0, 2\pi), v \in [0,1]\} \\
\textit{surface:}\quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0,1]\} \\
\textit{parameterization:}\quad & f(u,v) = (\cos u, \sin u, v) \\
\textit{inverse:}\quad & f^{-1}(x,y,z) = (\arccos x, z)
\end{aligned}$$

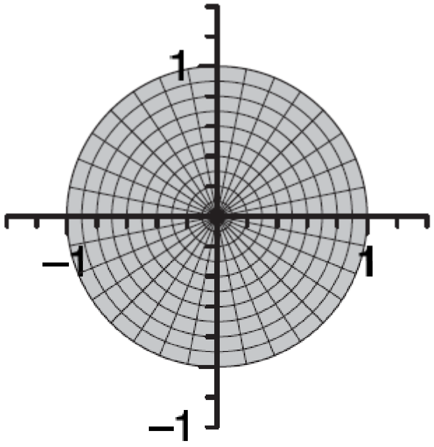


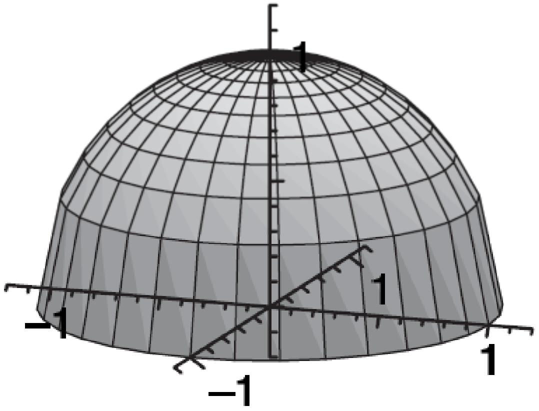


Surface Examples (3)

- Hemisphere (orthographic definition) :

$$\begin{aligned} \textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\ \textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\ \textit{parameterization:} \quad & f(u,v) = (u, v, \sqrt{1 - u^2 - v^2}) \\ \textit{inverse:} \quad & f^{-1}(x,y,z) = (x,y) \end{aligned}$$

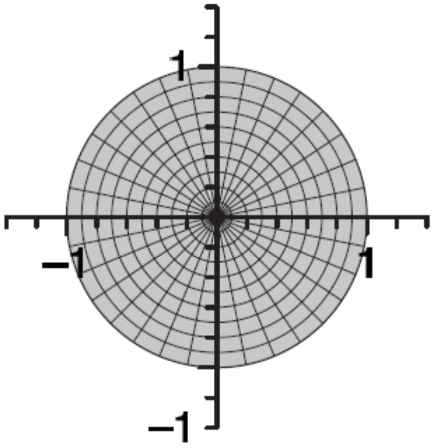


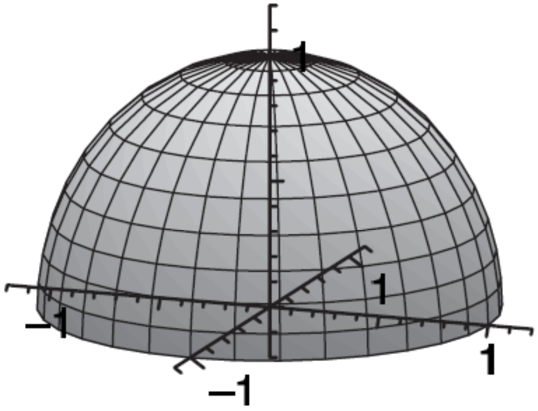


Surface Examples (4)

- Hemisphere (stereographic definition) :

$$\begin{aligned}
\textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\
\textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\
\textit{parameterization:} \quad & f(u,v) = (\tfrac{2u}{1+u^2+v^2}, \tfrac{2v}{1+u^2+v^2}, \tfrac{1-u^2-v^2}{1+u^2+v^2}) \\
\textit{inverse:} \quad & f^{-1}(x,y,z) = (\tfrac{x}{1+z}, \tfrac{y}{1+z})
\end{aligned}$$





Reparameterization

- Example (3) and (4):

  →There can be more than one parameterizations of S over Ω

- Any bijection $\varphi : \Omega \rightarrow \Omega$

  induces a reparameterization: $g = f \circ \varphi$

- Exercise: write the reparameterization $\varphi(u, v)$ between (3) and (4)

  (3) $f(u,v) = (u, v, \sqrt{1-u^2-v^2})$

  ⬇

  $\varphi(u, v) =$?

  (4) $f(u,v) = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2})$

- Surface Mapping Optimization Procedure = Reparameterization Procedure

Intrinsic Surface Properties

- Intrinsic and extrinsic
  - → intrinsic: about the shape itself, not about its representation and location
- Intrinsic property examples: curvature (Gaussian, mean), normal
- Tangent Plane: spanned by $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$
- Surface Normal: $n_f = \frac{f_u \times f_v}{\|f_u \times f_v\|}$
- Example (orthographic hemisphere):

$$f(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

$$f_u(u,v) = (1,0, \frac{-u}{\sqrt{1-u^2-v^2}})$$

$$f_v(u,v) = (0,1, \frac{-v}{\sqrt{1-u^2-v^2}})$$

$$n_f(u,v) = (u, v, \sqrt{1-u^2-v^2}) = (x, y, z)$$

(same with stereographics)

→ Following our intuition: normal is independent of the parameterization

(intrinsic property)

1st Fundamental Form and Surface Area

- Area of a surface is intrinsic too
- The first fundamental form $\mathbf{I}_f = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$
- Area element: $dA = | f_u \times f_v | dudv = \sqrt{(f_u \cdot f_u)(f_v \cdot f_v) - (f_u \cdot f_v)^2}\, dudv = \sqrt{EG - F^2}\, dudv$
- Example: Area of a unit hemisphere (orthographic parameterization)

$$f(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

$$EG - F^2 = \frac{1}{1-u^2-v^2}$$

$$\begin{aligned} A(S) &= \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{1}{\sqrt{1-u^2-v^2}}\, du\, dv \\ &= \int_{-1}^{1} \left[\arcsin \frac{u}{\sqrt{1-v^2}} \right]_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} dv \\ &= \int_{-1}^{1} \pi\, dv \\ &= 2\pi, \end{aligned}$$

- Exercise:

Area under stereographic parameterization

- Intrinsic property: Area is independent of the parameterization

2nd Fundamental Form and Curvature

- ☐ 2nd partial derivative: $f_{uu} = \frac{\partial^2 f}{\partial u^2}, \quad f_{uv} = \frac{\partial^2 f}{\partial u \partial v}, \quad \text{and} \quad f_{vv} = \frac{\partial^2 f}{\partial v^2}$
- ☐ Their dot products with the surface normal → 2nd fundamental form:

$$\mathbf{II}_f = \begin{pmatrix} f_{uu} \cdot n_f & f_{uv} \cdot n_f \\ f_{uv} \cdot n_f & f_{vv} \cdot n_f \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

- ☐ Gaussian curvature K and mean curvature H, defined as the determinant and half the trace of the matrix $\mathbf{I}_f^{-1}\mathbf{II}_f$, respectively:

$$K = \det(\mathbf{I}_f^{-1}\mathbf{II}_f) = \frac{\det \mathbf{II}_f}{\det \mathbf{I}_f} = \frac{LN - M^2}{EG - F^2}$$

$$H = \tfrac{1}{2}\,\mathrm{trace}(\mathbf{I}_f^{-1}\mathbf{II}_f) = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

Metric Distortion

- Look at surface point f(u,v), move a little away from (u,v):

  Displacement: $(\Delta u, \Delta v)$ → new point: $f(u+\Delta u, v+\Delta v)$

  approximated by 1st order Taylor expansion:

$$\tilde{f}(u+\Delta u, v+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v$$

Planar local region: the vicinity of $\boldsymbol{u} = (u,v)$

↓

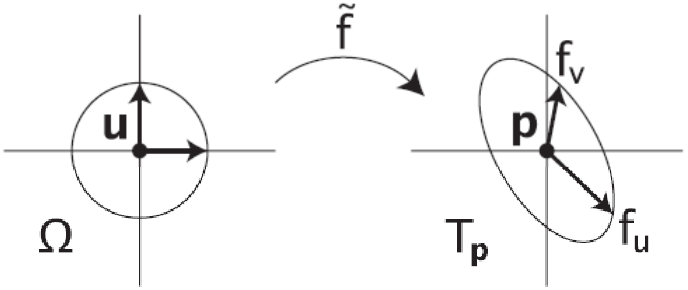
Region on tangent plane T_p at $\boldsymbol{p} = f(u,v) \in S$

Circles around **u**

↓

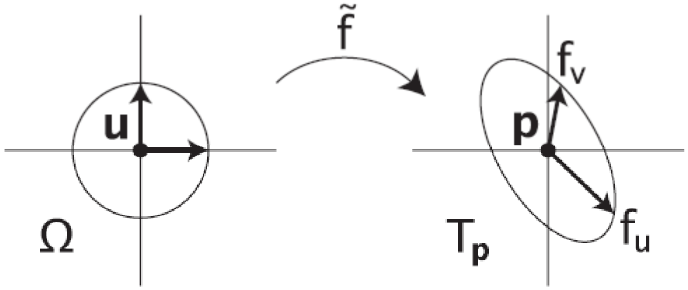
ellipses around **p**

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v} \quad \text{where } J_f = (f_u \;\, f_v) \text{ is the } \textit{Jacobian} \text{ of } f$$



Metric Distortion (cont.)

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v}$$

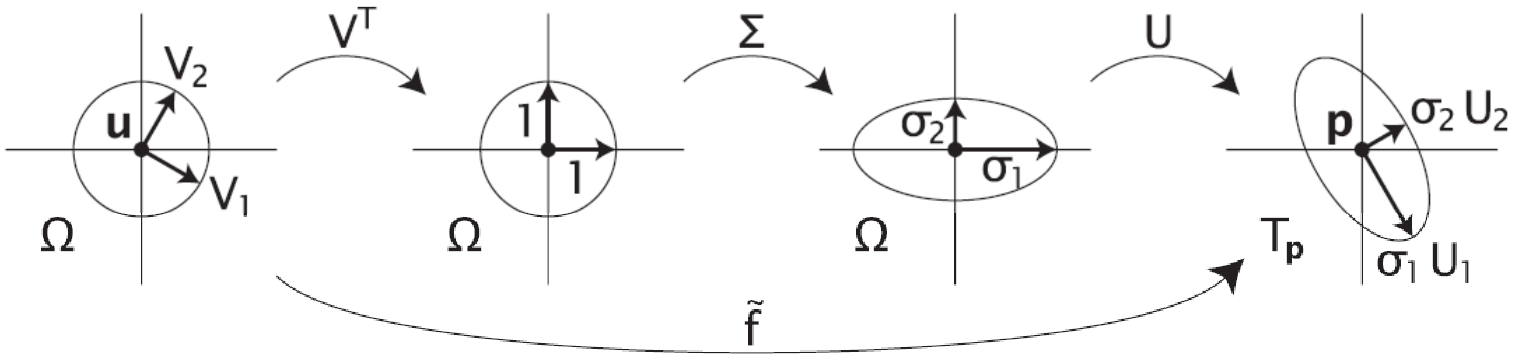


Decompose the Jacobian (3*2) matrix by SVD:

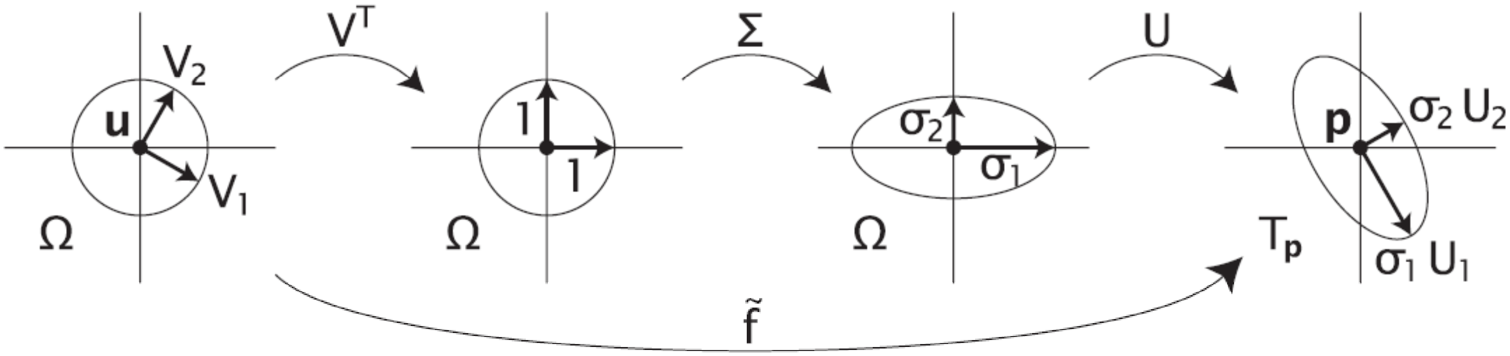
$$J_f = U\Sigma V^T = U\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2 \\ 0 & 0\end{pmatrix} V^T$$

unitary, orthonormal $U \in \mathbb{R}^{3\times 3}$ $V \in \mathbb{R}^{2\times 2}$

$singular\ values\ \sigma_1 \geq \sigma_2 > 0$



Metric Distortion (cont.)



$$J_f = U\Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

(1) 2D Rotation V → planar rotation around **u**;

(2) Stretching matrix Σ → stretches by factor σ_1 and σ_2 in the u and v directions;

(3) 3D rotation U → map the planar region onto the tangent plane

Tiny sphere with radius-r → ellipse with semi-axes of length $r\sigma_1$ and $r\sigma_2$

$\sigma_1 = \sigma_2 \longrightarrow$ Local scaling, circles to circles : **Confomal**

$\sigma_1\sigma_2 = 1 \longrightarrow$ Area preserved : **Equiareal**

Metric Distortion (cont.)

Singular values of any matrix A are the square roots of the eigenvalues of the matrix A^TA

Look at ${J_f}^T J_f$

$${J_f}^T J_f = \begin{pmatrix} {f_u}^T \\ {f_v}^T \end{pmatrix} (f_u \; f_v) = \mathbf{I}_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

The symmetric 2*2 matrix's eigenvalues:

$$\lambda_{1,2} = \tfrac{1}{2}\left((E+G) \pm \sqrt{4F^2 + (E-G)^2}\right)$$

$$\begin{array}{lcccc}
f \text{ is } \textit{isometric} \text{ or } \textit{length-preserving} & \iff & \sigma_1 = \sigma_2 = 1 & \iff & \lambda_1 = \lambda_2 = 1, \\
f \text{ is } \textit{conformal} \text{ or } \textit{angle-preserving} & \iff & \sigma_1 = \sigma_2 & \iff & \lambda_1 = \lambda_2, \\
f \text{ is } \textit{equiareal} \text{ or } \textit{area-preserving} & \iff & \sigma_1 \sigma_2 = 1 & \iff & \lambda_1 \lambda_2 = 1.
\end{array}$$

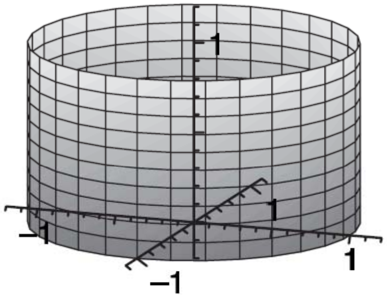
$$\text{isometric} \iff \text{conformal} + \text{equiareal}$$

Metric Distortion Example

(1) Cylinder

- ☐ *parameterization:* $f(u,v) = (\cos u, \sin u, v)$
- ☐ *Jacobian:* $J_f = \begin{pmatrix} \cos u & 0 \\ -\sin u & 0 \\ 0 & 1 \end{pmatrix}$
- ☐ *first fundamental form:* $\mathbf{I}_f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
- ☐ *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = 1$

Isometry

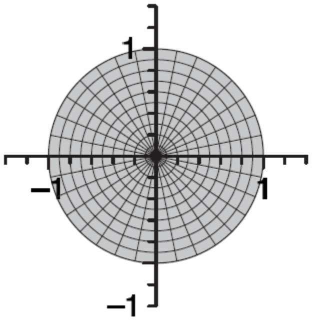


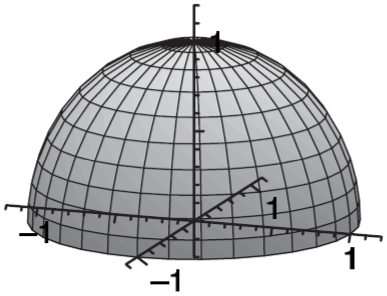
Metric Distortion Example

(2) Hemisphere (stereographic)

- *parameterization:* $f(u,v) = (2ud, 2vd, (1-u^2-v^2)d)$ where $d = \frac{1}{1+u^2+v^2}$
- *Jacobian:* $J_f = \begin{pmatrix} 2d-4u^2d^2 & -4uvd^2 \\ -4uvd^2 & 2d-4v^2d^2 \\ -4ud^2 & -4vd^2 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 4d^2 & 0 \\ 0 & 4d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 4d^2, \quad \lambda_2 = 4d^2$

Conformal



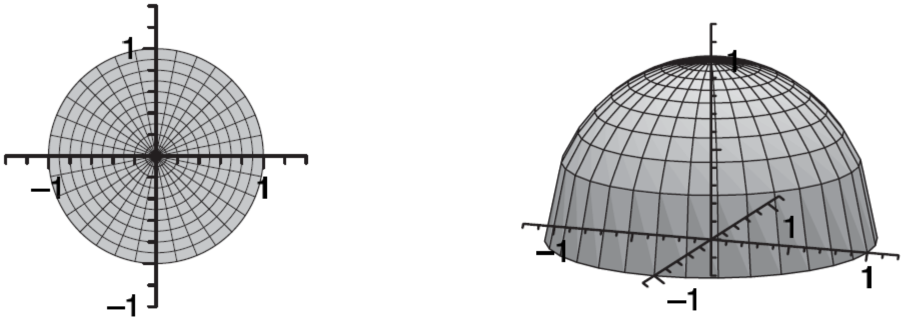


Metric Distortion Example

(3) Hemisphere (orthographic)

- *parameterization:* $f(u,v) = (u, v, \frac{1}{d})$ where $d = \frac{1}{\sqrt{1-u^2-v^2}}$
- *Jacobian:* $J_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -ud & -vd \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 1+u^2d^2 & uvd^2 \\ uvd^2 & 1+v^2d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = d^2$

Not conformal, not equiareal



Minimizing Metric Distortion

Overall distortion of a parameterization f can be generally defined by:

$$\bar{E}(f) = \int_{\Omega} E(\sigma_1(u,v), \sigma_2(u,v))\, du\, dv \Big/ A(\Omega)$$

Minimizing $\bar{E}(f)$ over the space of all admissible parameterizations → best parameterization

Discretely, we look at linear function f:

from parameter triangles $t \in \Omega$ to surface triangles $T \in \mathcal{T}$

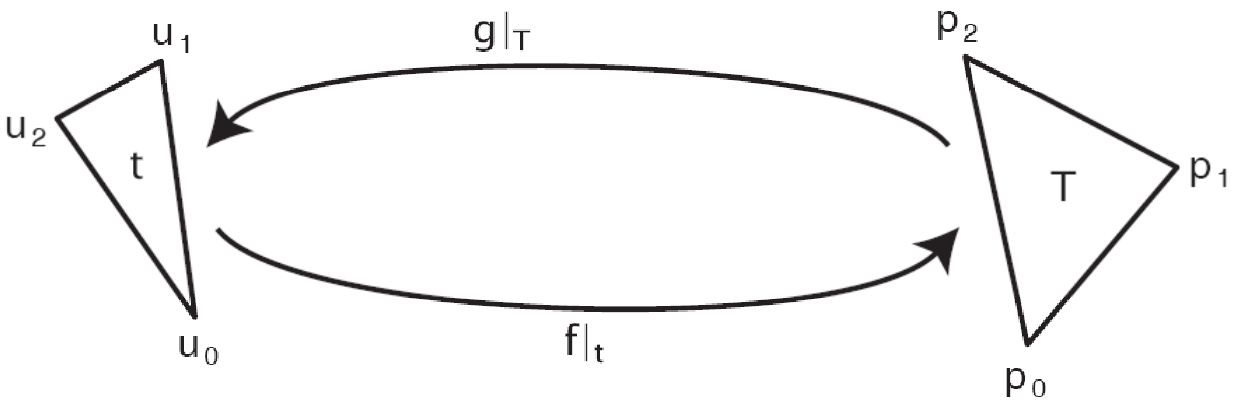
$$\bar{E}(f) = \sum_{t \in \Omega} E(\sigma_1^t, \sigma_2^t) A(t) \Big/ \sum_{t \in \Omega} A(t)$$

Or we can look at inverse function g=f^{-1}:

$$\sigma_1^T = 1/\sigma_2^t \text{ and } \sigma_2^T = 1/\sigma_1^t$$

$$\bar{E}(g) = \sum_{T \in \mathcal{T}} E(\sigma_1^T, \sigma_2^T) A(T) \Big/ \sum_{T \in \mathcal{T}} A(T)$$

Minimizing Metric Distortion (cont.)



$$A(t) = \tfrac{1}{2}\det(\boldsymbol{u}_1 - \boldsymbol{u}_0, \boldsymbol{u}_2 - \boldsymbol{u}_0) \qquad A(T) = \tfrac{1}{2}\|(\boldsymbol{p}_1 - \boldsymbol{p}_0) \times (\boldsymbol{p}_2 - \boldsymbol{p}_0)\|$$

$$\left(\sigma_1^t\right)^2 + \left(\sigma_2^t\right)^2 = \frac{1}{A(t)^2}\sum_{i=0}^{2} \|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2 \left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^t \sigma_2^t = \frac{A(T)}{A(t)}$$

$$\left(\sigma_1^T\right)^2 + \left(\sigma_2^T\right)^2 = \frac{1}{A(T)^2}\sum_{i=0}^{2} \|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2 \left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^T \sigma_2^T = \frac{A(t)}{A(T)},$$

Minimizing Metric Distortion (cont.)

Discrete Harmonic Map
[Pinkall EM'93] [Eck SIG'95]:

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) = \tfrac{1}{2}({\sigma_1}^2 + {\sigma_2}^2)$$

Least Square Conformal Map
[Desbrun SIG'02] [Levy SIG'02]:

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \tfrac{1}{2}(\sigma_1 - \sigma_2)^2$$

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) - E_{\mathrm{C}}(\sigma_1, \sigma_2) = \sigma_1 \sigma_2$$

$$\Rightarrow \quad \bar{E}_{\mathrm{D}}(g) - \bar{E}_{\mathrm{C}}(g) = \sum_{t \in \Omega} A(t) \Big/ \sum_{T \in \mathcal{T}} A(T) = \frac{A(\Omega)}{A(S_{\mathcal{T}})}$$

Therefore, if we take a conformal map, fix its boundary and thus the area of the parameter domain Ω, and then compute the harmonic map with this boundary, then we get the same mapping, which illustrates the well-known fact that any conformal mapping is harmonic, too.

Minimizing Metric Distortion (cont.)

Conformal Mapping: → try to make $\sigma_1 = \sigma_2$

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \tfrac{1}{2}(\sigma_1 - \sigma_2)^2 \qquad E_{\mathrm{D}}(\sigma_1, \sigma_2) = \tfrac{1}{2}\left({\sigma_1}^2 + {\sigma_2}^2\right)$$

Another one: MIPS energy [Hormann 02]

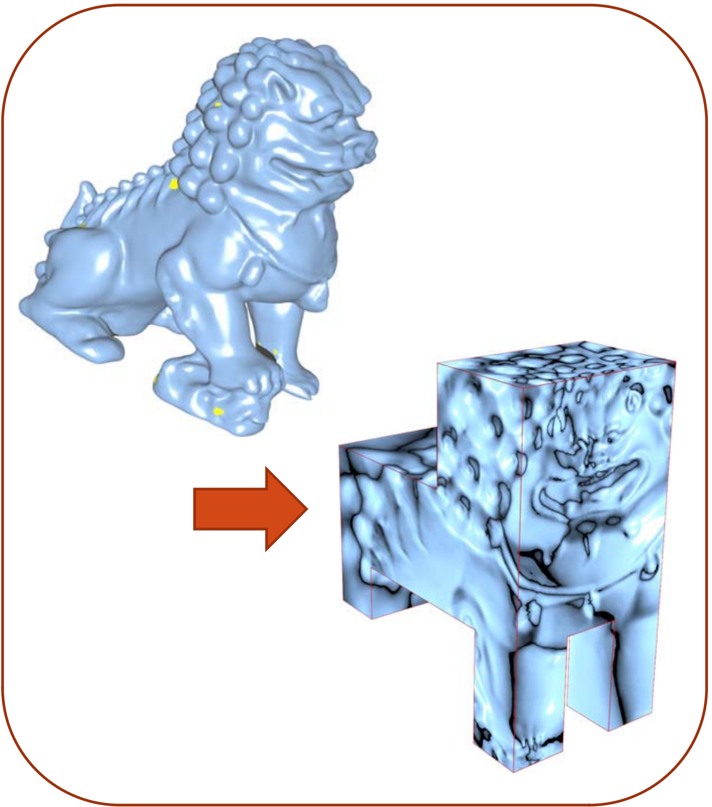
$$E_{\mathrm{M}}(\sigma_1, \sigma_2) = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \frac{{\sigma_1}^2 + {\sigma_2}^2}{\sigma_1 \sigma_2}$$

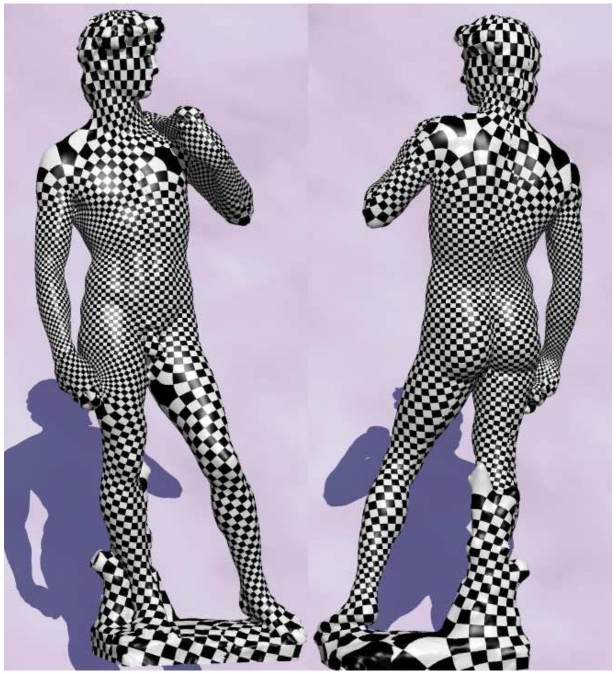
- Advantage: (1) symmetry: $E_{\mathrm{M}}(\sigma_1^T, \sigma_2^T) = E_{\mathrm{M}}(\sigma_1^t, \sigma_2^t)$

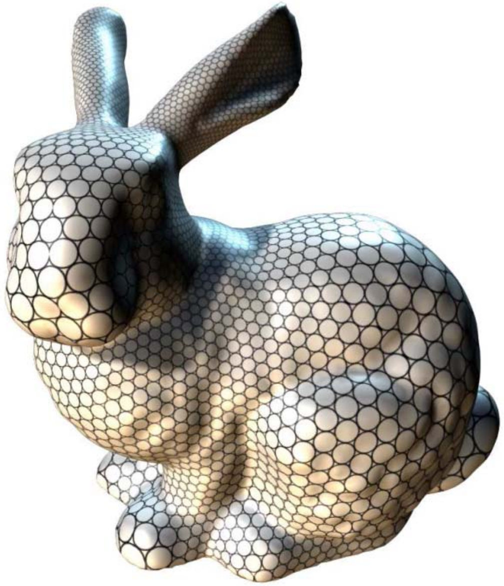
  (2) bijectivity
- Disadvantage: non-linear

Many more about mapping…

- Free-boundary mapping
- Deforming the metric
- Global parameterization
- Inter-shape mapping







Next class

Next class: an application of this class,

With the parameterization, we can do

Remeshing – to generate high quality mesh