Surface Mapping (1,2)

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Review

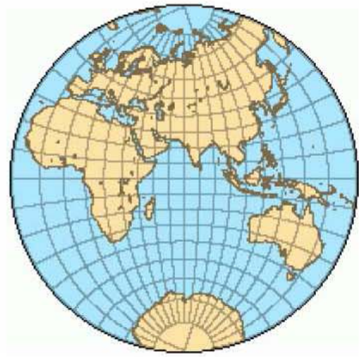
- Texture mapping: a technique that generates enhanced effects over simple geometry shapes
  - From geometry space to the texture (image) space → surface mapping
  - Quality of the mapping: dictates the effect of texture mapping (low distortion preferred)
- Spline Representation: a compact representation, good for precise computer aided design, and scientific computing
  - For spline fitting : a good parameterization is important for generating smooth spline with small # of control points

Surface Mapping Definition

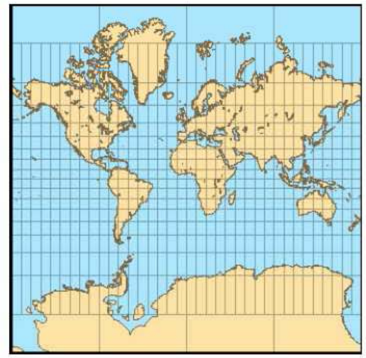
- Image Texture Mapping:
  - A one-to-one map from geometry shape S to a texture image (2D domain) D
  - D is usually a rectangular domain, e.g. $D = [0, 1] \times [0, 1]$
  - The mapping: a vector function $\vec{f} : S \rightarrow D \subset \mathbb{R}^2$,
    composed by two scalar function f_u and f_v.
  - ⇔Define a "u-v" coordinates over the surface S.
  - Infinite mapping ways, which one is good?

Historical Background

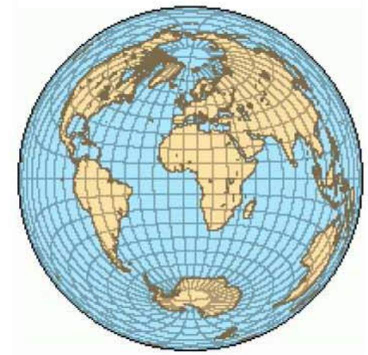
(a) Orthographic;



(b) stereographic;



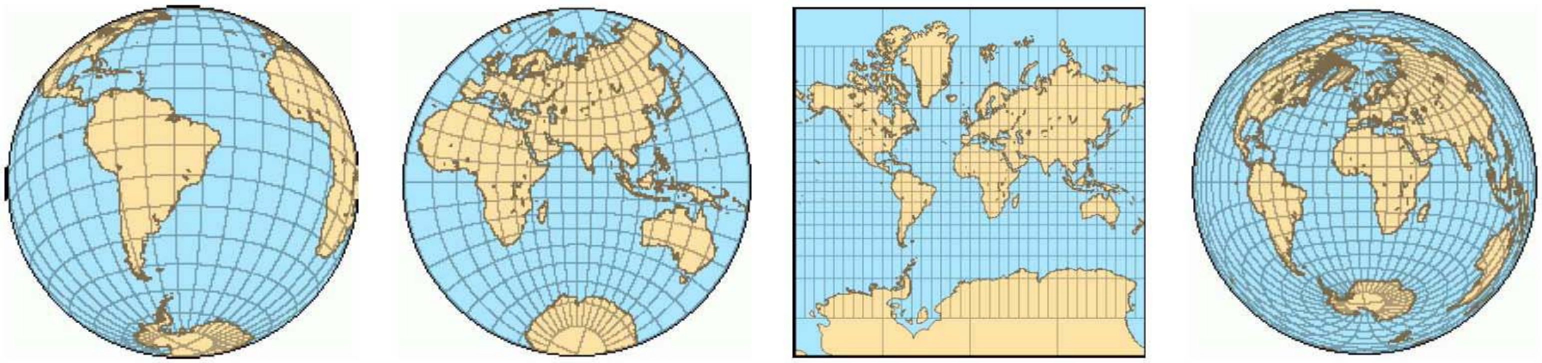
(c) Mercator; and



(d) Lambert

- Cartography
- Distortion: angles and areas distortion
  - Isometry: no distortion
  - Not all surfaces has the isometry to a planar region
  - Peeling oranges → can't be without distortion
- Ptolemy was the first known to produce the data for creating a map showing the world (100-150AD)
  - [Geography] → project a sphere by longitude and latitude

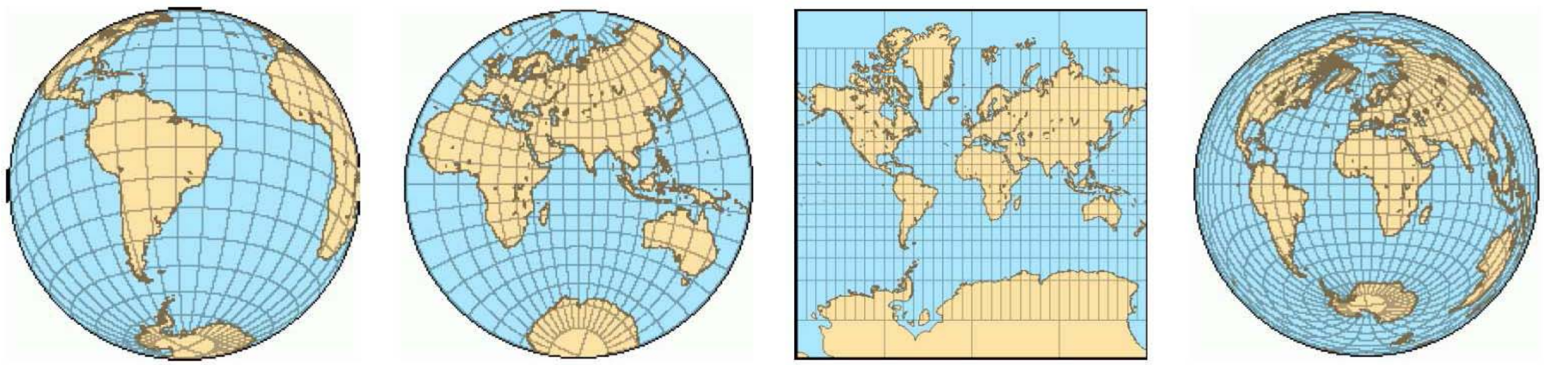
Historical Background (cont.)



(a) Orthographic; (b) stereographic; (c) Mercator; and (d) Lambert

(a) The orthographic projection (Egyptians and Greeks, > 2000 years ago) → modifies both angles and areas
(b) Stereographic projection (Hipparchus, 190-120B.C.) → preserves angles, not areas
(c) Mercator projection (Mercator 1569) → preserves angles, not areas
(d) Lambert projection (Lambert 1772) → preserves areas, not angles

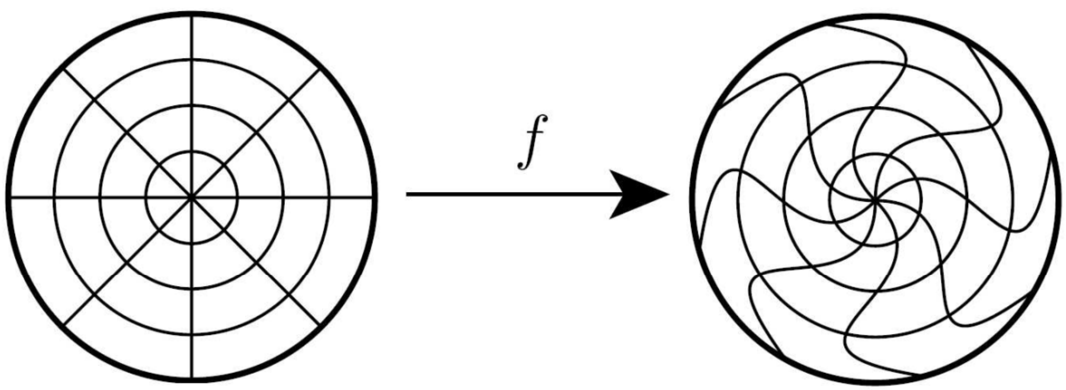
Good "UV" versus bad "UV"?



- What do we look for? What do we preserve?
- Should we map it onto a rectangle? Or a disk? Or something awkward? What do we choose?
- If the target shape is fixed (e.g. a rectangle, or a disk…), what is the best mapping then?
- At this beginning stage:
  - ❑ Source: a genus-zero open surface (topological disk)
  - ❑ Target: planar shape is fixed

Mapping Criteria

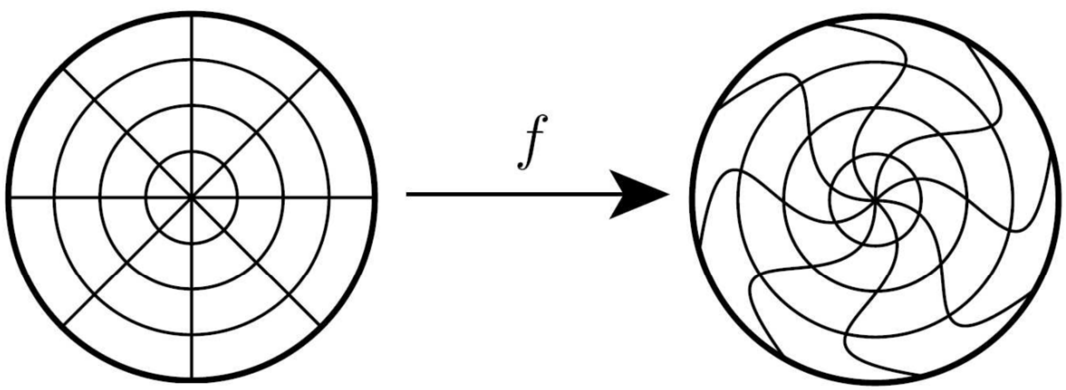
- Angle Distortion: change of the local angles
  - Conformal mapping: no angle distortion (locally, a right angle → a right angle, or a circle → a circle)
- Area Distortion: change of the local area
  - Equiareal mapping: no area change
- Isometric Mapping: neither angles nor area distortion
- Isometric ⇔ conformal + equiareal
- Isometry exists between a given surface and a planar domain, only if this surface is "developable"
- Purely Equiareal Mapping is infinitely dimensional and not necessarily useful



Mapping Criteria

- Therefore:

  Given an arbitrary topological disk surface and a planar domain
  - Isometric mapping rarely exists
  - Conformal mapping always exists (Riemann Mapping Theorem)
  - Infinitely many equiareal mapping, as a pure criterion, not easy to control and design



From discrete aspects…

- Another intuitive way to start with, unlike mapping and its distortion defined in continuous case:
  - Given a mesh of strings (topological disk), with its boundary vertices pinned along a planar rectangle (or any other simple planar convex polygon*) boundary
  - Where do we place the interior vertices on the planar region within region of the boundary

- We might simply want to find an analogy of "barycentric coordinates" locally (why? Think about the curve case…)
- →iteratively adjust each "uv" locally according to its neighboring information

$$\sum_{j \in N_i} w_{ij}(f(v_j) - f(v_i)) = 0, \qquad v_i \in V_I, \qquad \lambda_{ij} = w_{ij} \Big/ \sum_{k \in N_i} w_{ik},$$

$$f(v_i) = \sum_{j \in N_i} \lambda_{ij} f(v_j), \qquad v_i \in V_I.$$

Mesh Mapping

- A physical model:
  - Edges of the triangle mesh are springs (spring network)
  - Fix the boundary on the plane
  - Relax the interior of this network
  - Physical law being the only rule
  - Stabilized position → mapping for the interior vertices

- A simplified model – mesh with n+b (interior: 1.. n, boundary: n+1…n+b) vertices:
  - The rest string length → 0
  - Potential energy → $(Ds^2)/2$, (D-constant, s-final string length)
  - Boundary vertices p_i → u_i (2d-vector u_i)
  - Minimize spring energy:

$$E = \frac{1}{2} \sum_{i=1}^{n+b} \sum_{j \in N_i} \frac{1}{2} D_{ij} \|\boldsymbol{u}_i - \boldsymbol{u}_j\|^2,$$

where $D_{ij} = D_{ji}$ is the spring constant of the spring between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$

Mesh Mapping (cont.)

- To find the minimized solution:

$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j \in N_i} D_{ij}(\boldsymbol{u}_i - \boldsymbol{u}_j) = 0 \quad \Longrightarrow \quad \sum_{j \in N_i} D_{ij}\boldsymbol{u}_i = \sum_{j \in N_i} D_{ij}\boldsymbol{u}_j$$

(for any interior vertex i=1…n)

- Remove boundary points from the left to right hand side:

$$\boldsymbol{u}_i - \sum_{j \in N_i, j \le n} \lambda_{ij}\boldsymbol{u}_j = \sum_{j \in N_i, j > n} \lambda_{ij}\boldsymbol{u}_j,$$

- Lead to two sparse linear systems (in two axis directions):

$$AU = \bar{U} \quad \text{and} \quad AV = \bar{V},$$

$$\bar{u}_i = \sum_{j \in N_i, j > n} \lambda_{ij} u_j \quad \text{and} \quad \bar{v}_i = \sum_{j \in N_i, j > n} \lambda_{ij} v_j$$

$$A = (a_{ij})_{i,j=1,\dots,n} \; : \; a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\lambda_{ij} & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Boundary Mapping

- Don't want fold-overs → not a projection
- Flatten a curve:
  a) Choosing the shape of the planar domain boundary
  b) Choosing the distribution of the points on the boundary

a) Boundary Shape: Usually rectangle, circle …
   - Convex shape → bijectivity guarantees for many weights
   - Larger distortion when surface is highly concave

b) Distribution: Usually chord length, …
   - Working pretty well in most cases
   - Including the boundary points in the optimization → less distortion

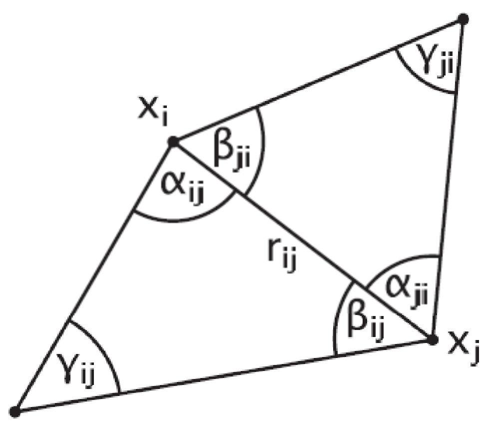
(2) Interior Mapping
– different weights

- Different D_{ij} :
  - Wachspress coordinates:
    - Earliest generalization of barycentric coordinates
    - Mainly used in finite element methods
  - Harmonic coordinates:
    - Standard piecewise linear approximation to Laplace equation
    - Minimizing deformation energy
  - Mean value coordinates:
    - Discretizing mean value theorem of harmonic function
    - Positive weights guaranteed, stable parameterization

$$w_{ij} = \frac{\cot \alpha_{ji} + \cot \beta_{ij}}{{r_{ij}}^2}$$

$$w_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji}$$

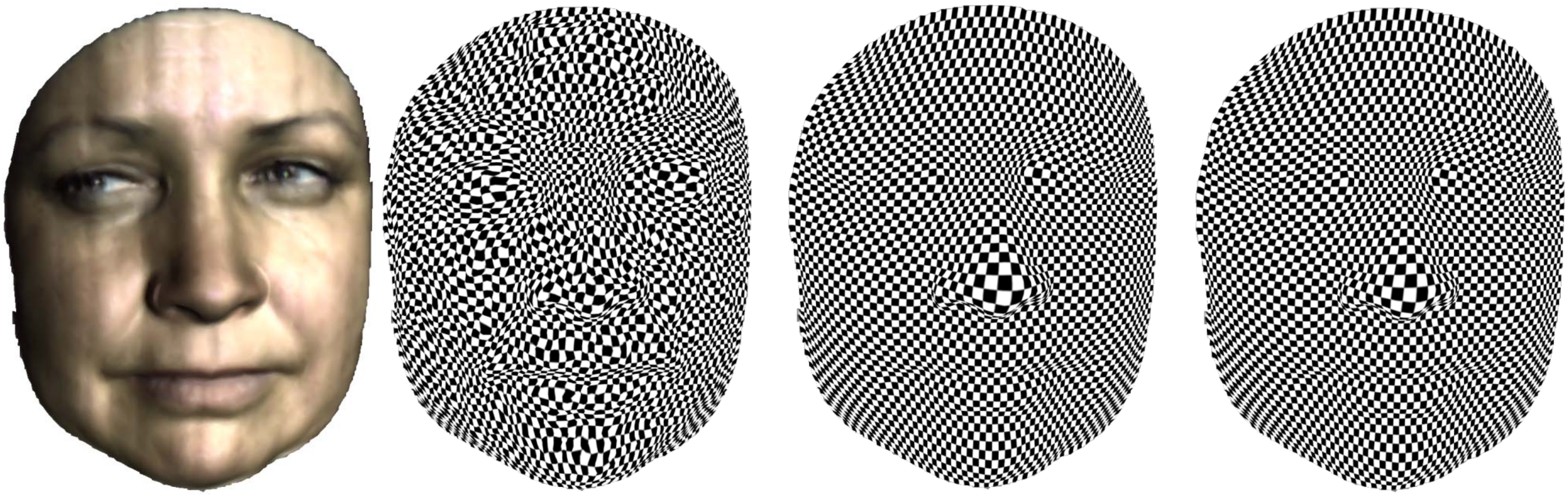
$$w_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}}$$



It has been proved that:
Any symmetric weights ($w_{ij}=w_{ji}$) minimizes a spring energy.

Three different popular formula

- Graph Embedding: [Tutte 1963]
- Discrete Harmonic Mapping: [Eck 1995]
- Meanvalue Coordinates: [Floater 1997]

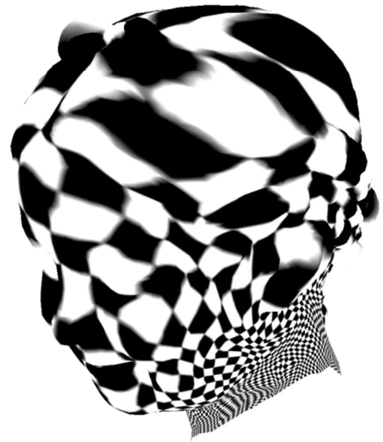


Susan Surface   Graph Embedding   Harmonic   Mean Value

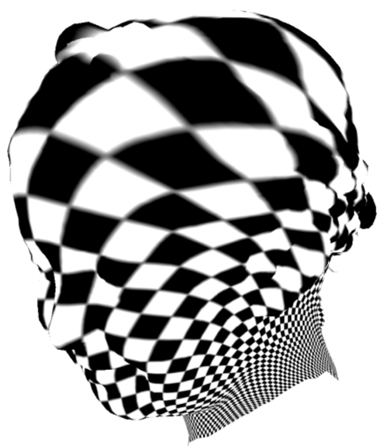
Three different popular formula

- On another surface:

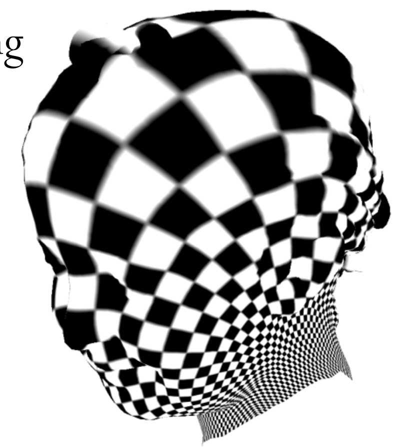
Bimba Surface



Graph Embedding



Mean Value



Harmonic

- Visually, we can tell the difference.
- But how to judge them numerically? And where do these mapping formula come from?
  - E.g. why the harmonic mapping looks conformal?
- How do we design (or choose to use) a mapping technique?
  - E.g. shall we always use harmonic?

- Purely Conformal or a Balance?
  - Applications needs angle-preserving
  - Applications that also needs area-preserving
- How about more general surfaces?
  - Closed Genus-0 surfaces → spherical mapping
  - Higher genus surfaces → global parameterization
  - Surface to surface → inter-surface mapping

Differential Geom. Background

- A surface $S \subset \mathbb{R}^3$ (2-manifold), has the parametric representation:

$$\mathbf{x}(u^1, u^2) = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2))$$

 for points (u^1, u^2) in some domains in $\mathbb{R}^2$

- A representation is regular if
  i. The functions x_1, x_2, x_3 are smooth (differentiable when we need)
  ii. The vectors $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}$ are linearly independent

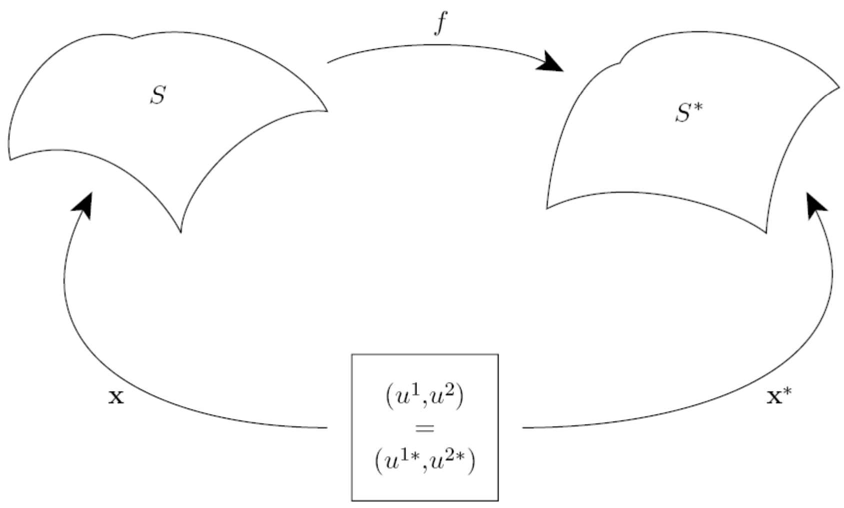
- 1st fundamental form (quadratic inner product on the tangent space) :
  → permits the calculation of surface metric

$$ds^2 = \mathbf{x}_1 \cdot \mathbf{x}_1 \, (du^1)^2 + 2\, \mathbf{x}_1 \cdot \mathbf{x}_2 \, du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 \, (du^2)^2$$

denoting $g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \qquad \alpha = 1, 2, \quad \beta = 1, 2,$

We have $ds^2 = (du^1 \; du^2)\, \mathbf{I} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix}$, where $\mathbf{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$

Differential Geom. Background (cont.)



f is allowable if the parameterizations x and x* are both regular.

Isometric mappings

Isometric ⇔ length-preserving

(e.g. cylinder → plane (cylindrical coordinates → Cartesian coordinates))

Theorem 1. *An allowable mapping from S to S^* is isometric if and only if the coefficients of the first fundamental forms are the same, i.e.,*

$$\mathbf{I} = \mathbf{I}^*.$$

Under an isometry:

- Curve-lengths don't change
- Angles don't change
- Areas don't change
- Gaussian curvatures don't change

Conformal mappings

Conformal ⇔ angle-preserving

(e.g. stereographic and Mercator projections)

Theorem 2. *An allowable mapping from S to S^* is conformal or angle-preserving if and only if the coefficients of the first fundamental forms are proportional, i.e.,*

$$\mathbf{I} = \eta(u^1, u^2)\,\mathbf{I}^*, \tag{1}$$

for some scalar function $\eta \neq 0$.

Under an conformal map:

- Angles don't change
- Circle → another circle (only scaling allowed)

Equiareal mappings

Equiareal ⇔ area-preserving
(e.g. Lambert projections)

Theorem 3. *An allowable mapping from S to S^* is equiareal if and only if the discriminants of the first fundamental forms are equal, i.e.,*

$$g = g^*. \tag{2}$$

(Note that: $g = \det \mathbf{I} = g_{11}g_{22} - g_{12}^2$)

Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\textbf{isometric} \Leftrightarrow \textbf{conformal} + \textbf{equiareal}.$$

An example: planar mappings

A planar mapping is a special type of the surface mapping:

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (u(x, y), v(x, y))$$

its 1st fundamental form: $\mathbf{I} = J^T J$

where $J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is the Jacobian of f.

Proposition 1. *For a planar mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ the following equivalencies hold:*

1. f is isometric $\Leftrightarrow$ $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Leftrightarrow$ $\lambda_1 = \lambda_2 = 1$
2. f is conformal $\Leftrightarrow$ $\mathbf{I} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$ $\Leftrightarrow$ $\lambda_1 / \lambda_2 = 1$
3. f is equiareal $\Leftrightarrow$ $\det \mathbf{I} = 1$ $\Leftrightarrow$ $\lambda_1 \lambda_2 = 1$

eigenvalues of $\mathbf{I}$

Conformal→Harmonic

A conformal mapping

– a complex function satisfies the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Delta u = 0, \qquad \Delta v = 0,$$ where $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

A harmonic mapping

– a complex function satisfies these two Laplace equations

Isometric ➔ Conformal ➔ Harmonic

Harmonic Mapping

- Easy to compute, easy to approximate
- Guaranteed existence (when suitable boundary mapping is provided)
- Minimizing deformation (minimizing the Dirichlet energy)

Theorem 5 (RKC). *If $f : S \to \mathbb{R}^2$ is harmonic and maps the boundary ∂S homeomorphically into the boundary ∂S^* of some convex region $S^* \subset \mathbb{R}^2$, then f is one-to-one;*

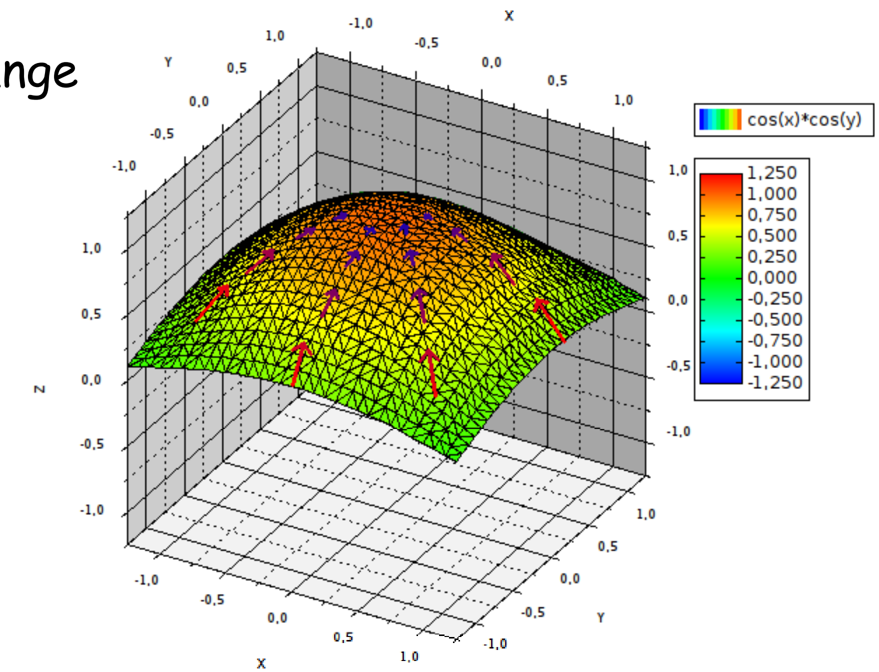
- Conformality depends on the boundary condition
- One-sidedness

Harmonic Map & its Intuition

- Minimizing deformation

$$E_D(f) = \frac{1}{2}\int_S \|\mathrm{grad} f\|^2 = \frac{1}{2}\int_S (\|\nabla u\|^2 + \|\nabla v\|^2)$$

-- minimize the magnitude of the change



- Intuitive explanation
  - 1D
  - 2D
  - 3D