Splines (3)

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Review

- Polynomial representation:
  - Curve as

$$x(t) = a_0 + a_1 t + \ldots + a_n t^n$$
$$y(t) = b_0 + b_1 t + \ldots + b_n t^n$$
$$z(t) = c_0 + c_1 t + \ldots + c_n t^n$$

$$T = [t^n \quad \ldots \quad t^1 \quad 1];$$

$$C = \begin{bmatrix} c_n^x & c_n^y & c_n^z \\ c_{n-1}^x & c_{n-1}^y & c_{n-1}^z \\ \ldots & \ldots & \ldots \\ c_0^x & c_0^y & c_0^z \end{bmatrix};$$

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = T \cdot C = \sum_{i=0}^{n} \vec{c}_i t^i$$

Or the weighted sum format with different types of basis functions:

For example, Bezier curve:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$C(t) = \sum_{i=0}^{i=n} B_i^n(t) P_i$$

Rational Bezier Curves

- Although polynomials offer many advantages, there exist a number of important curve and surface types which cannot be represented precisely using polynomials (e.g. circles, ellipses, hyperbolas, cylinders, cones, spheres, etc.)
- Example: unit circle in the xy plane can't be represented using polynomial functions
  - If it has a parametric representation : $x(t) = a_0 + a_1 t + \ldots + a_n t^n$
    $y(t) = b_0 + b_1 t + \ldots + b_n t^n$
  - Then $x^2+y^2-1=0$ implies:

$$
\begin{aligned}
0 &= (a_0 + a_1 t + \ldots + a_n t^n)^2 + (b_0 + b_1 t + \ldots + b_n t^n)^2 - 1 \\
&= (a_0{}^2 + b_0{}^2 - 1) + 2(a_0 a_1 + b_0 b_1)t + (a_1{}^2 + 2a_0 a_2 + b_1{}^2 + 2b_0 b_2)t^2 \qquad (1) \\
&+ \ldots + 2(a_n a_{n-1} + b_n b_{n-1})t^{2n} - 1 + (a_n{}^2 + b_n{}^2)t^{2n}
\end{aligned}
$$

  - Equation (1) should hold for all t, which implies that all coefficients are zero

Rational Bezier Curves (cont.)

- Example: unit circle in the xy plane can't be represented using polynomial functions

$$0 \equiv (a_0 + a_1 t + ... + a_n t^n)^2 + (b_0 + b_1 t + ... + b_n t^n)^2 - 1$$
$$= (a_0^2 + b_0^2 - 1) + 2(a_0 a_1 + b_0 b_1)t + (a_1^2 + 2a_0 a_2 + b_1^2 + 2b_0 b_2)t^2$$
$$+ ... + 2(a_n a_{n-1} + b_n b_{n-1})t^{2n} - 1 + (a_n^2 + b_n^2)t^{2n}$$

  - (cont.)
  - (1) $a_n^2 + b_n^2 = 0 \Rightarrow a_n = b_n = 0$
  - (2) $a_{n-1}^2 + 2a_{n-2}a_n + b_{n-1}^2 + 2b_{n-2}b_n = 0 \Rightarrow a_{n-1} = b_{n-1} = 0$
  - (3) …
  - (n) $a_1^2 + 2a_0 a_2 + b_1^2 + 2b_0 b_2 = 0 \Rightarrow a_0 = b_0 = 0$

    But this implies 0=(0+0-1)=-1

    This proves a circle can't be represented by a polynomial form.

- Conic sections can be represented by rational functions:
  - Unit circle: $x(t) = \frac{1-t^2}{1+t^2}; y(t) = \frac{2t}{1+t^2}$
  - Ellipse (major radius 2 on y-axis, and minor radius 1 on x-axis): $x(t) = \frac{1-t^2}{1+t^2}; y(t) = \frac{4t}{1+t^2}$
  - Hyperbola, center at (0, 4/3), with y-axis the transverse axis:

$$x(t) = \frac{-1+2t}{1+2t-2t^2}; y(t) = \frac{4t(1-t)}{1+2t-2t^2}$$

Rational Bezier curve

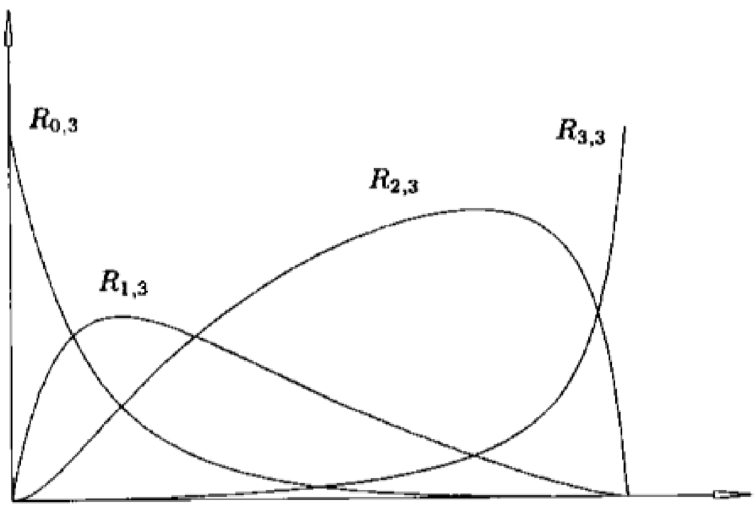
- nth-degree rational Bezier curve:

$$C(t) = \frac{\sum_{i=0}^{n} B_{i,n}(t) w_i P_i}{\sum_{i=0}^{n} B_{i,n}(t) w_i}, 0 \le t \le 1$$

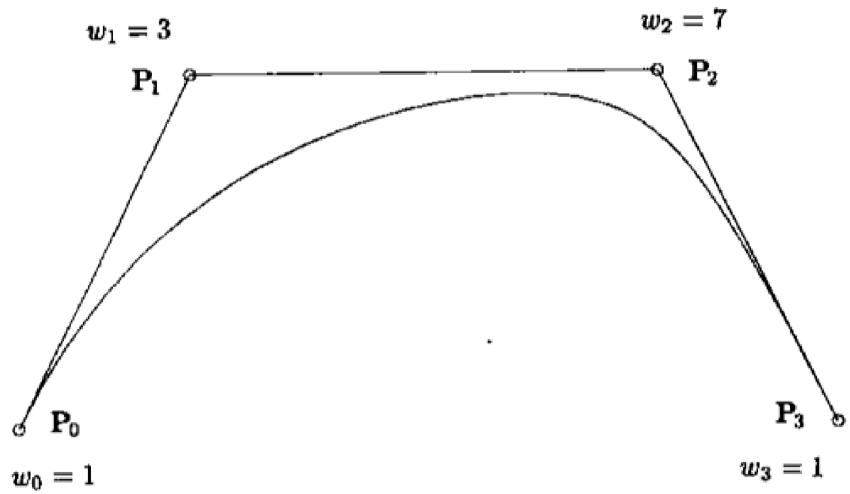
or $C(t) = \sum_{i=0}^{n} R_{i,n}(t) P_i, 0 \le t \le 1$ where $R_{i,n}(t) = \frac{B_{i,n}(t) w_i}{\sum_{i=0}^{n} B_{i,n}(t) w_i}, 0 \le t \le 1$

- Properties:
  - Nonnegativity, partition of unity, endpoints interpolation
  - $B_{i,n}(t)$ are a special case of the $R_{i,n}(t)$
  - Convex hull property, affine transformation invariance, variation diminishing property
  - The k^{th} derivative at $t=0$ ($t=1$) depends on the first (last) $k+1$ control points and weights, in particular $C'(0)$ and $C'(1)$ are parallel to P_1-P_0 and P_n-P_{n-1} respectively.

Rational Bezier curve example



(a) Basis functions;



(b) Bézier curve.

Using Homogeneous Coordinates

A 2D curve example:

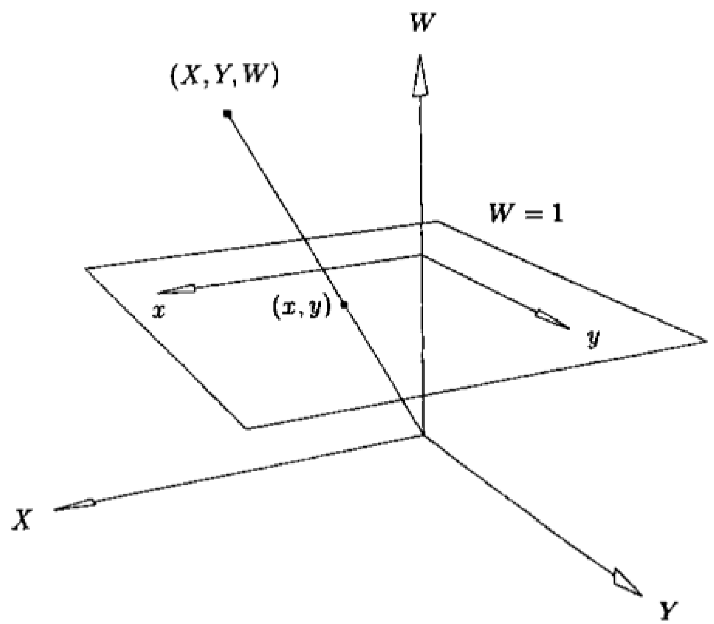
- Given a set of control points $\{P_i\}$, and weights $\{w_i\}$
- Construct the weighted control points Q_i $(w_i x_i, w_i y_i, w_i)$
- In 3D:

$$x^H(t) = \sum_{i=0}^{n} B_{i,n}(t) w_i x_i$$

$$y^H(t) = \sum_{i=0}^{n} B_{i,n}(t) w_i y_i$$

$$w(t) = \sum_{i=0}^{n} B_{i,n}(t) w_i$$

- Project it back onto the w=1 plane:



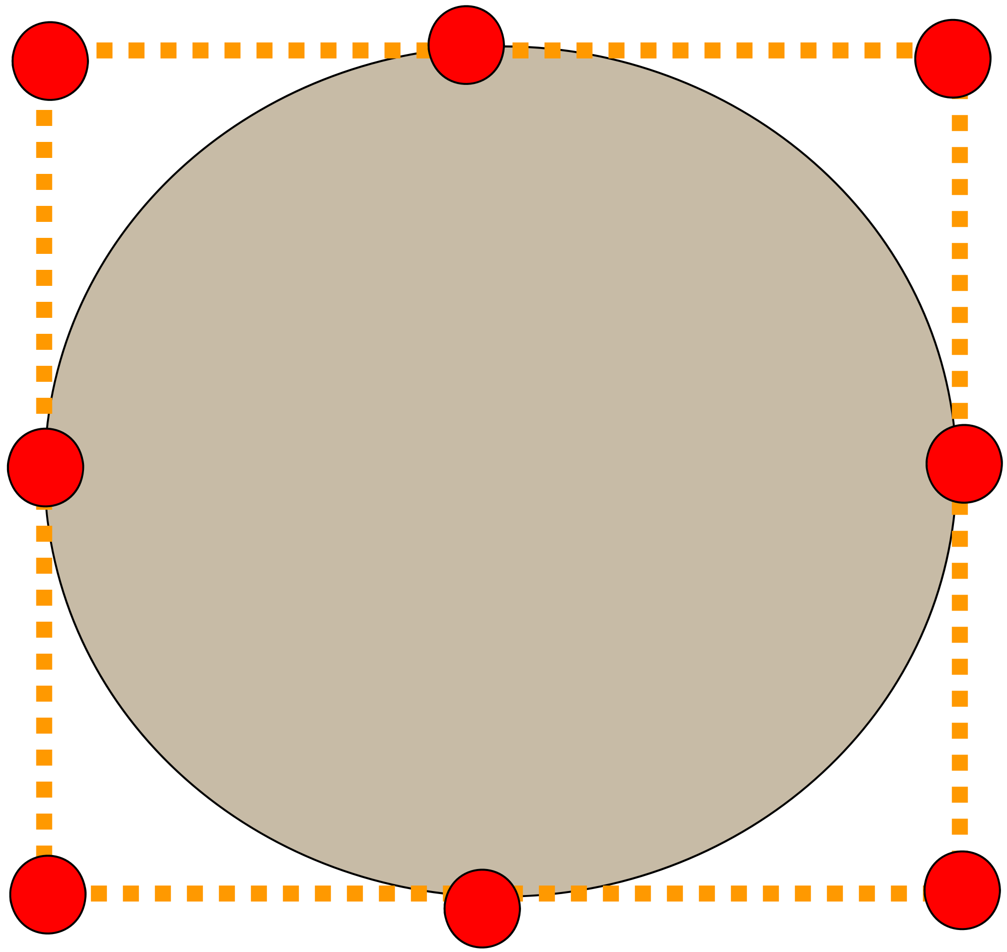
$$x(t) = \frac{x^H(t)}{w(t)} = \frac{\sum_{i=0}^{n} B_{i,n}(t) w_i x_i}{\sum_{i=0}^{n} B_{i,n}(t) w_i}, \quad y(t) = \frac{y^H(t)}{w(t)} = \frac{\sum_{i=0}^{n} B_{i,n}(t) w_i y_i}{\sum_{i=0}^{n} B_{i,n}(t) w_i}$$

$$C(t) = \frac{\sum_{i=0}^{n} B_{i,n}(t) w_i P_i}{\sum_{i=0}^{n} B_{i,n}(t) w_i}$$

Example:

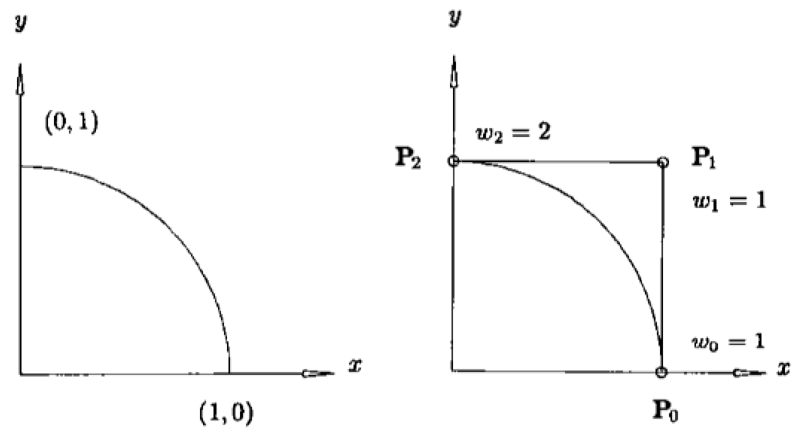
Rational functions for unit circle

Where to put control points, and how to set their weights?



Suppose you know: a parametric representation of a circle is $x(t) = \frac{1-t^2}{1+t^2}; y(t) = \frac{2t}{1+t^2}$

The unit circle example



Look at one quadrant of the unit circle: $\left(x(t) = \frac{1-t^2}{1+t^2}, y(t) = \frac{2t}{1+t^2} \right), 0 \le t \le 1$

- The quadric curve should have P_0, P_1, and P_2 placed as shown in the right figure. (Why?)
- For the weights, we have: $w(t) = 1 + t^2 = \sum_{i=0}^{n} B_{i,2}(t) w_i = (1-t)^2 w_0 + 2t(1-t) w_1 + t^2 w_2$

  with t=0,0.5,1 → $w_0, w_1, w_2 = 1,1,2$
- Get homogeneous coordinates

$$\begin{aligned} P_0 &= (1,0) & & Q_0 = (1,0,1) \\ P_1 &= (1,1) & \Rightarrow & \; Q_1 = (1,1,1) \\ P_2 &= (0,1) & & Q_2 = (0,2,2) \end{aligned}$$

The unit circle example

The 3D parametric curve is a parabolic arc:

$$C^H(t) = \sum_{i=0}^{n} B_{i,2}(t) Q_i$$

Which projects onto a circular arc on the W=1 plane

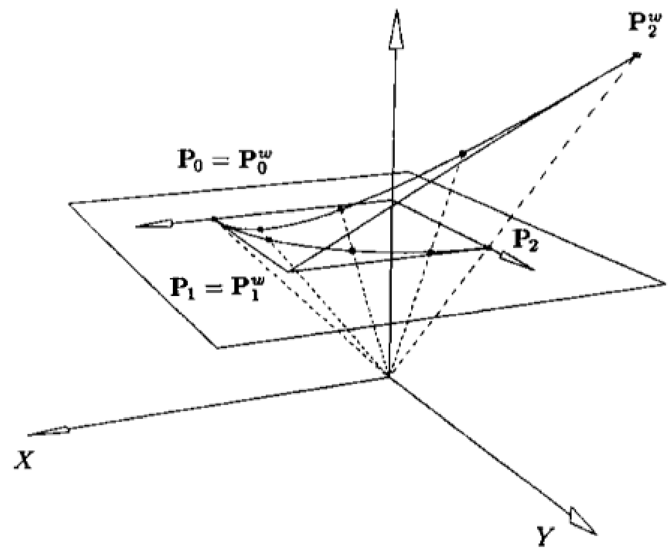
On any given t, for example, $t=1/2$

$$C^H(\frac{1}{2}) = \sum_{i=0}^{2} B_{i,2}(\frac{1}{2}) Q_i$$

$$= \left(1-\left(\frac{1}{2}\right)\right)^2 (1,0,1) + 2\left(1-\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)(1,1,1) + \left(\frac{1}{2}\right)^2 (0,2,2)$$

$$= (\frac{3}{4}, 1, \frac{5}{4})$$

➡ $C(\frac{1}{2}) = (3/5, 4/5)$



NURBS Curves

An p-degree NURBS Curve:

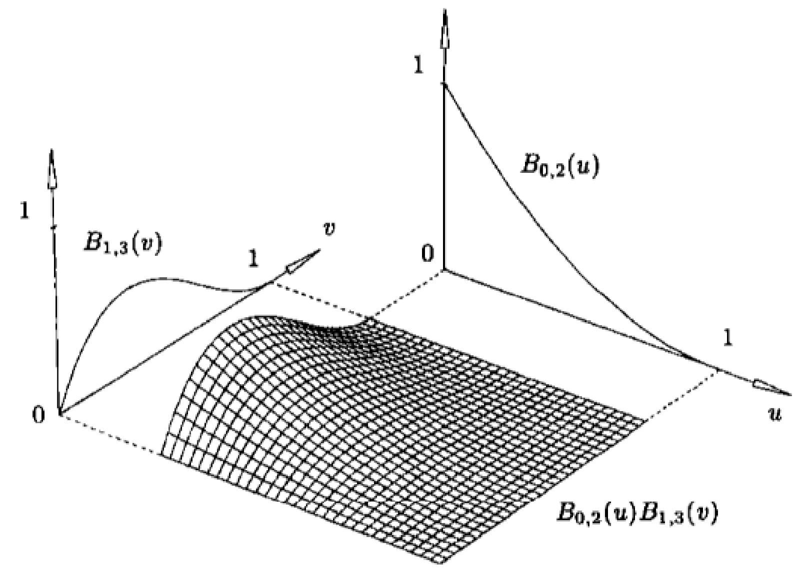
$$\mathbf{c}(u) = \frac{\sum_{i=0}^{n} \mathbf{p}_i w_i B_{i,k}(u)}{\sum_{i=0}^{n} w_i B_{i,k}(u)} \quad \text{where} \quad B_{i,0}(u) = \begin{cases} 1 & u_i \le u < u_{i+1} \\ 0 & o/w \end{cases}$$

$$B_{i,k}(u) = \frac{u - u_i}{u_{i+k} - u_i} B_{i,k-1}(u) + \frac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+1}} B_{i+1,k-1}(u)$$

Note:

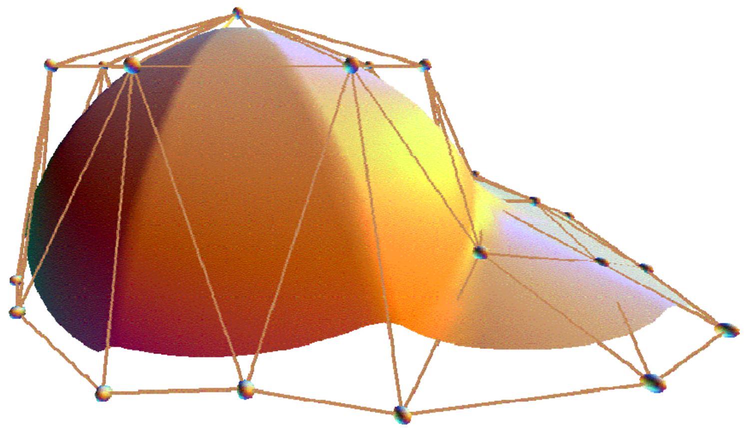
- Computation of a set of basis functions requires specification of a knot vector U and the degree k
- It may yield the quotient $0/0$, we define it to be zero
- $B_{i,p}(u)$ defined on the entire real line, but only the $[u_0, u_m]$ is of interest.
- The interval $[u_i, u_{i+1})$ is called the ith knot span, and can have zero length
- The computation of pth-degree functions generates a trucated triangular table
- Exercise: Curve with $U = \{\underbrace{0, \ldots, 0}_{k+1}, \underbrace{1, \ldots, 1}_{k+1}\}$ is a generalized p-degree Bezier representation.

Spline Surface



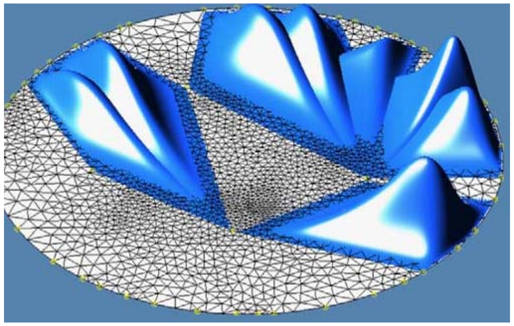
A curve → vector function of one parameter, mapping of a straight line segment into Euclidean 3D space

A surface → a vector-valued function of two parameters, mapping of a region into Euclidean 3D space



Spline Surface Categories (classified by domain schemes):

- ❑Tensor product patches
- ❑Triangular patches
- ❑…



Tensor Product Surfaces

Basis functions:
bivariate functions of u and v
(constructed as products of univariate basis functions)

A tensor product surface

$$S^T(u,v) = (x(u,v), y(u,v), z(u,v)) = \sum_{i=0}^{n}\sum_{j=0}^{m} f_i(u) g_j(v) b_{i,j};$$

$$where \begin{cases} b_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j}) \\ 0 \le u, v \le 1 \end{cases}$$

The (u,v) domain of this mapping is a square (rectangle)

$$S^T(u,v) = [f_i(u)]^T [b_{i,j}][g_j(u)]$$

(n+1)*(m+1) matrix of 3D points

An example

$$S^T(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} f_i(u) g_j(v) b_{i,j}$$

A general parametric surface:

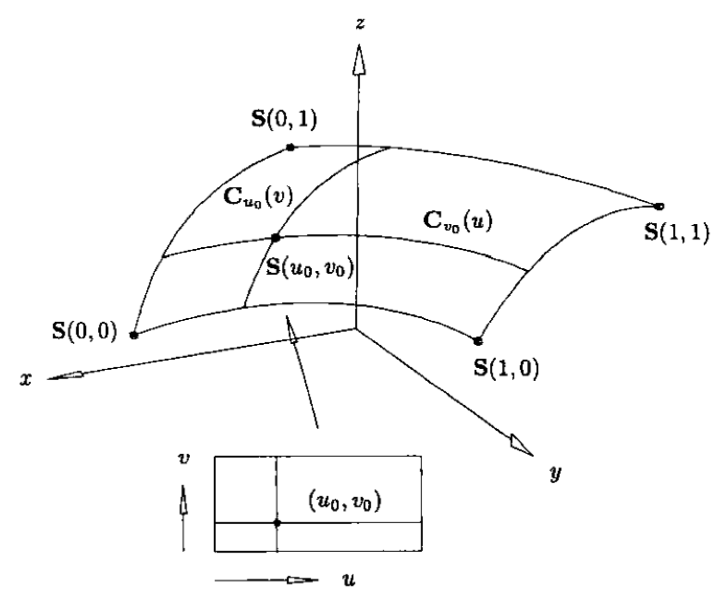
$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{a}_{i,j} u^i v^j = [u^i]^T [\mathbf{a}_{i,j}] [v^j] \qquad \begin{cases} \mathbf{a}_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j}) \\ 0 \le u, v \le 1 \end{cases}$$

In tensor product representation: $f^i \rightarrow u_i$ and $g_j \rightarrow v^j$, basis functions make the products $\{u^i v^j\}$.

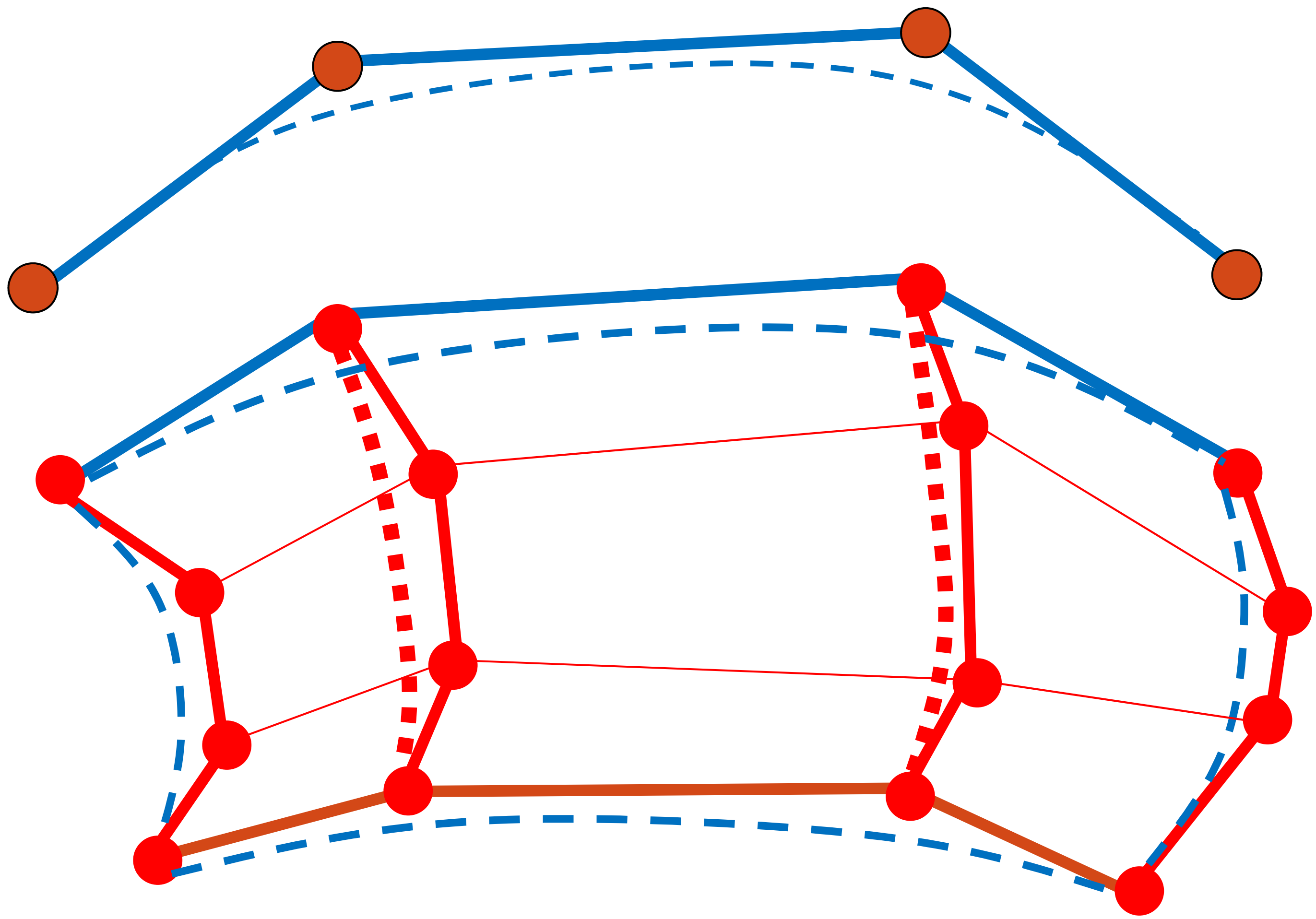
If we fix $u=u_0$ then we get an iso-curve:

$$\mathbf{C}_{u_0}(v) = \mathbf{S}(u_0, v) = \sum_{j=0}^{m} \left(\sum_{i=0}^{n} \mathbf{a}_{i,j} u_0^i \right) v^j = \sum_{j=0}^{m} \mathbf{b}_j(u_0) v^j$$

$$\mathbf{b}_j(u_0) = \sum_{i=0}^{n} \mathbf{a}_{i,j} u_0^i$$



Nonrational Bezier Surfaces



Nonrational Bezier Surfaces

A bidirectional net of control points and products of the univariate Bernstein polynomials:

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v)\mathbf{P}_{i,j} \qquad 0 \le u, v \le 1$$

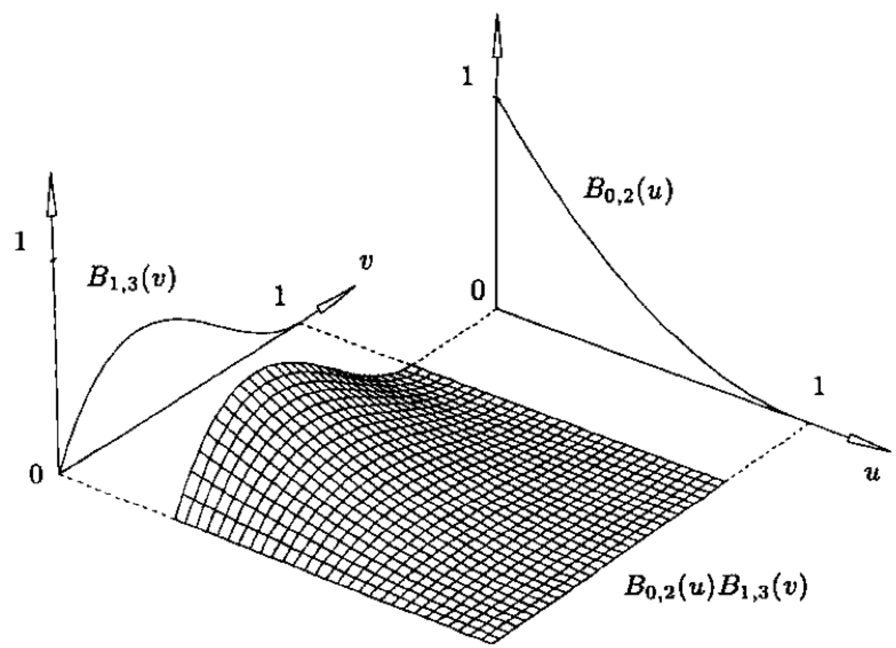
For fixed $u=u_0$: we get a Bezier curve

$$\mathbf{C}_{u_0}(v) = \mathbf{S}(u_0,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u_0)B_{j,m}(v)\mathbf{P}_{i,j}$$

$$= \sum_{j=0}^{m} B_{j,m}(v)\left(\sum_{i=0}^{n} B_{i,n}(u_0)\mathbf{P}_{i,j}\right)$$

$$= \sum_{j=0}^{m} B_{j,m}(v)\mathbf{Q}_j(u_0)$$

$$\mathbf{Q}_j(u_0) = \sum_{i=0}^{n} B_{i,n}(u_0)\mathbf{P}_{i,j} \qquad j = 0, \ldots, m$$



Properties of Nonrational Bezier Surfaces

- Non-negativity.

$$B_{i,n}(u)B_{j,m}(v) \geq 0 \text{ for all } i, j, u, v$$

- Partition of Unity.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v) = 1 \text{ for all } u \text{ and } v$$

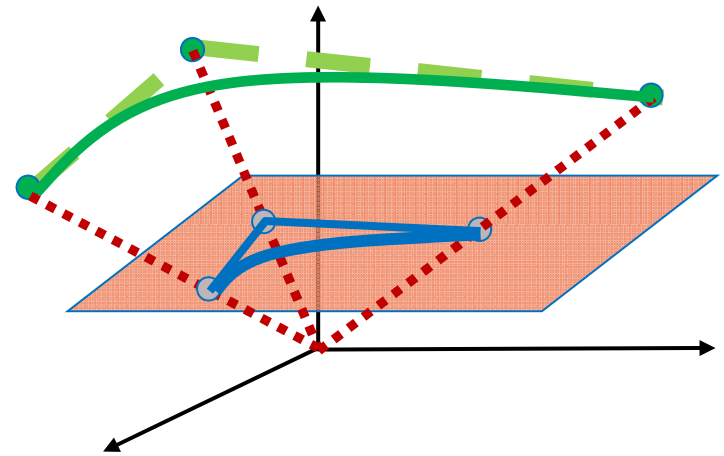
- S(u,v) is contained in the convex hull of its control points.
- Affine Transformation Invariance.
- The surface interpolates the four corner control points.

NURBS Curves → Surfaces

- NURBS curves.
- Tensor product?
- Question: can we get NURBS surface this way?
- Answer: NO!

  → NURBS are not tensor-product surfaces
- Can we have NURBS surface?
- YES.

NURBS Curves



$$c(u) = \frac{\sum_{i=1}^{n} p_i w_i B_{i,k}(u)}{\sum_{i=1}^{n} w_i B_{i,k}(u)}$$

$$\begin{bmatrix} c_x / c_w \\ c_y / c_w \\ c_z / c_w \end{bmatrix} \Leftarrow \begin{bmatrix} c_x(u) \\ c_y(u) \\ c_z(u) \\ c_w(u) \end{bmatrix} = \sum_{i=1}^{n} B_{i,k}(u) \begin{bmatrix} w_i x_i \\ w_i y_i \\ w_i z_i \\ w_i \end{bmatrix}$$

NURBS Surface

- NURBS surface definition:
  - A NURBS surface of degree k in u direction and degree l in the v direction is:

$$S(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{p}_{i,j} w_{i,j} B_{i,k}(u) B_{j,l}(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{i,j} B_{i,k}(u) B_{j,l}(v)}$$

- Geometric interpolation
- Not the tensor-product formulation. (Compare it with Bezier and B-spline construction)

NURBS Surface

$$s(u) = \frac{\sum_{i,j=1}^{n} p_{ij} w_{ij} B_{i,k}(u) B_{j,l}(v)}{\sum_{i,j=1}^{n} w_{ij} B_{i,k}(u) B_{j,l}(v)}$$

$$\begin{bmatrix} s_x / s_w \\ s_y / s_w \\ s_z / s_w \end{bmatrix} \Leftarrow \begin{bmatrix} s_x(u) \\ s_y(u) \\ s_z(u) \\ s_w(u) \end{bmatrix} = \sum_{i,j=1}^{n} B_{i,k}(u) B_{j,l}(v) \begin{bmatrix} w_{ij} x_{ij} \\ w_{ij} y_{ij} \\ w_{ij} z_{ij} \\ w_{ij} \end{bmatrix}$$

NURBS Surface

- Parametric variables: u and v
- Control points and their associated weights: (m+1)(n+1)
- Degrees of basis functions: (k-1) and (l-1)
- Knot sequence:

$$u_0 <= u_1 <= \ldots\ldots <= u_{m+k}$$

$$v_0 <= v_1 <= \ldots\ldots <= v_{n+l}$$

- Parametric domain:

$$u_{k-1} <= u <= u_{m+1}$$

$$v_{l-1} <= v <= v_{n+1}$$

NURBS Surface Property

Nonnegativity: $R_{i,j}(u,v) \geq 0$ for all i, j, u, and v;

Partition of unity: $\sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,j}(u,v) = 1$ for all $(u,v) \in [0,1] \times [0,1]$;

Local support: $R_{i,j}(u,v) = 0$ if (u,v) is outside the rectangle given by $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$;

In any given rectangle of the form $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, at most $(p+1)(q+1)$ basis functions are nonzero, in particular the $R_{i,j}(u,v)$ for $i_0 - p \leq i \leq i_0$ and $j_0 - q \leq j \leq j_0$ are nonzero;

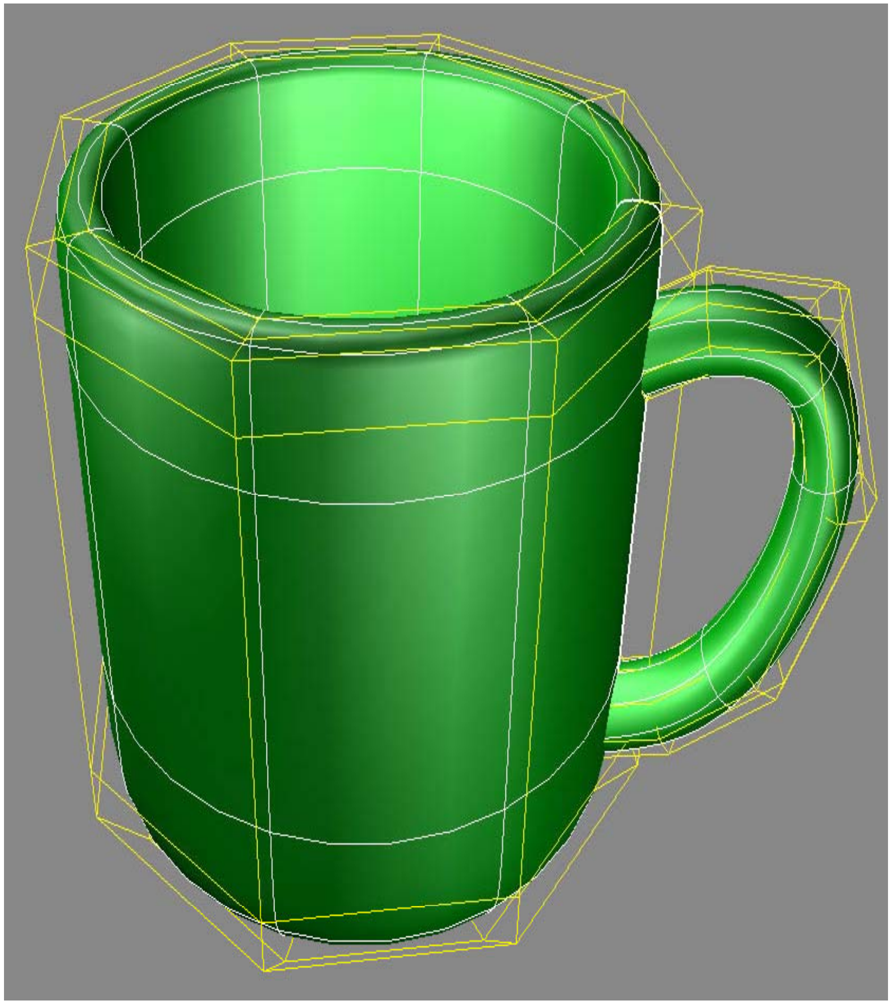
Corner point interpolation: $\mathbf{S}(0,0) = \mathbf{P}_{0,0}$, $\mathbf{S}(1,0) = \mathbf{P}_{n,0}$, $\mathbf{S}(0,1) = \mathbf{P}_{0,m}$, and $\mathbf{S}(1,1) = \mathbf{P}_{n,m}$;

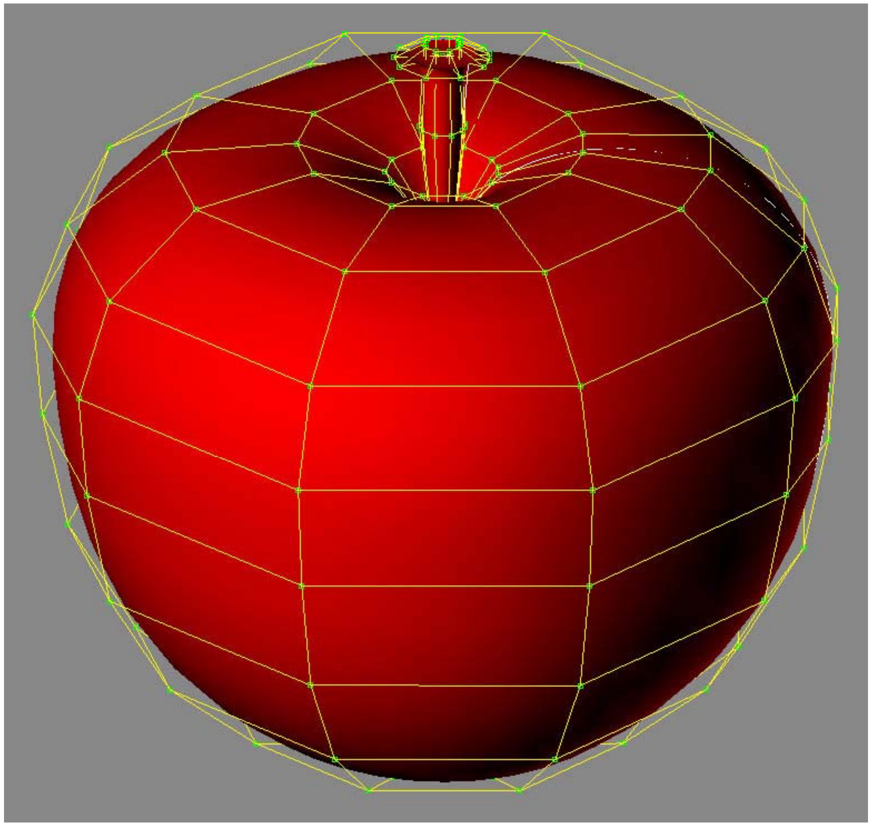
Affine invariance: an affine transformation is applied to the surface by applying it to the control points;

Strong convex hull property: assume $w_{i,j} \geq 0$ for all i, j. If $(u,v) \in [u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, then $\mathbf{S}(u,v)$ is in the convex hull of the control points $\mathbf{P}_{i,j}$, $i_0 - p \leq i \leq i_0$ and $j_0 - q \leq j \leq j_0$;

Local modification: if $\mathbf{P}_{i,j}$ is moved, or $w_{i,j}$ is changed, it affects the surface shape only in the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$;

NURBS Surface Examples





NURBS Surfaces

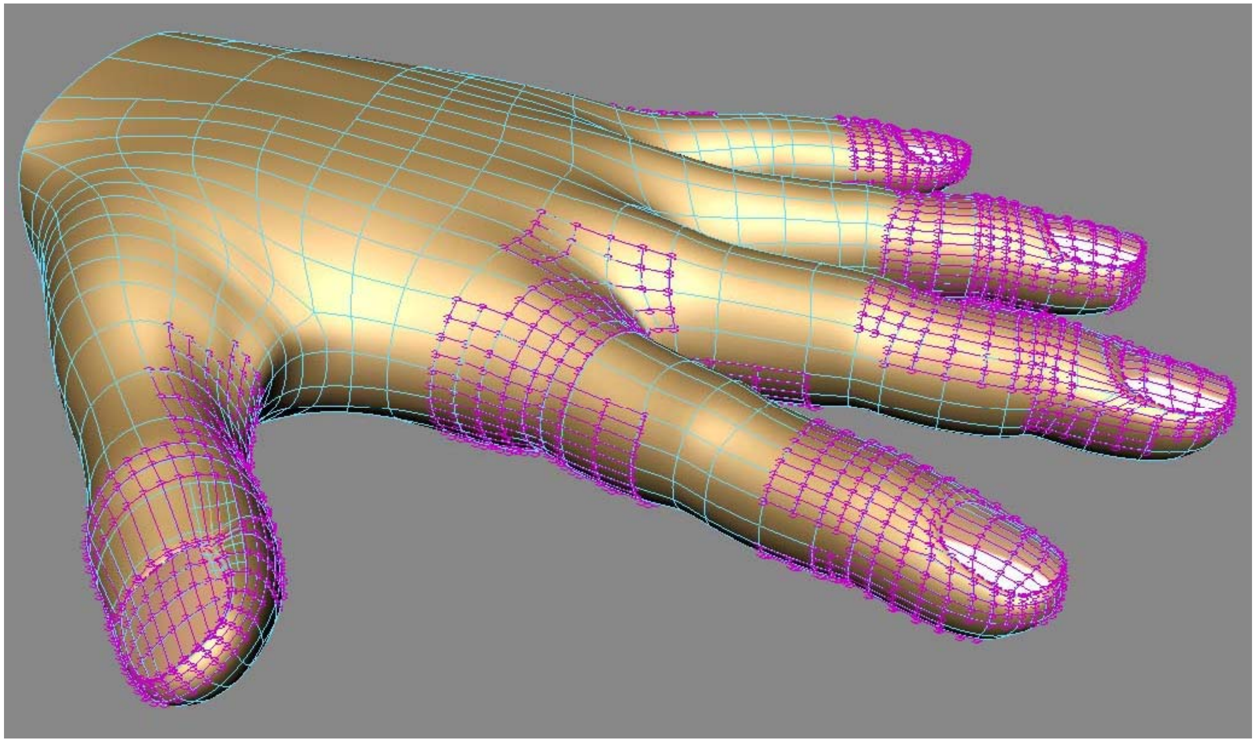
- Good for
  - Mechanical, manufactured parts
  - Smooth free-form surface representation
- Bad for
  - Non-genus-0 surfaces
  - Interactive design of free-form surfaces

Why NURBS

- Support free-form curves/surfaces modeling.
- Represent standard analytic shapes precisely.
- Local support.
- Convex hull.
- Affine transformation invariant.
- Strict analytic form for evaluation (important in CAD/CAM/CAE).

Why NOT NURBS

- Hard to model arbitrary topology.
- Regularity of tensor-product control polygon poses difficulty for level of detail.



Allow T-junctions
→ T-splines
(details in
EE7000 course)