Splines (2)

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Review:
An n-degree parametric curve

$$T = [t^n \quad \dots \quad t^1 \quad 1];$$

$$C = \begin{bmatrix} c_n^x & c_n^y & c_n^z \\ c_{n-1}^x & c_{n-1}^y & c_{n-1}^z \\ \dots & \dots & \dots \\ c_0^x & c_0^y & c_0^z \end{bmatrix};$$

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = T \cdot C = \sum_{i=0}^{n} \vec{c}_i t^i$$

the Degree-3 example:
A Cubic curve segment:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z,$$
$$0 \le t \le 1$$

Given 4 geometric constraint vectors: we can solve all unknown coefficients

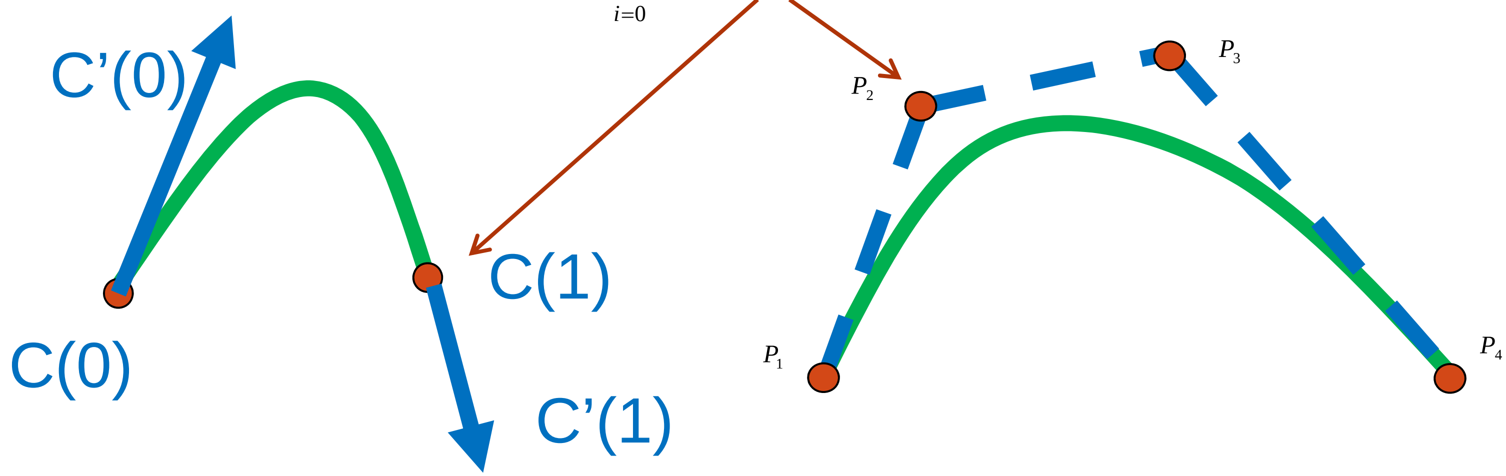
Different schemes: Hermite, Bezier…

- Are mathematically equivalent, one can convert to another
- allow different constraint vectors,
- convey different geometric insights

Review (cont.):
Cubic Hermite & Bezier Curves

Aforementioned two ways to formulate the cubic curve by weighted sum:

$$C(t) = \sum_{i=0}^{i=3} B_i^n(t) G_i$$



Given 4 geometric constraint vectors: we can solve all unknown coefficients

Different schemes: Hermite, Bezier…

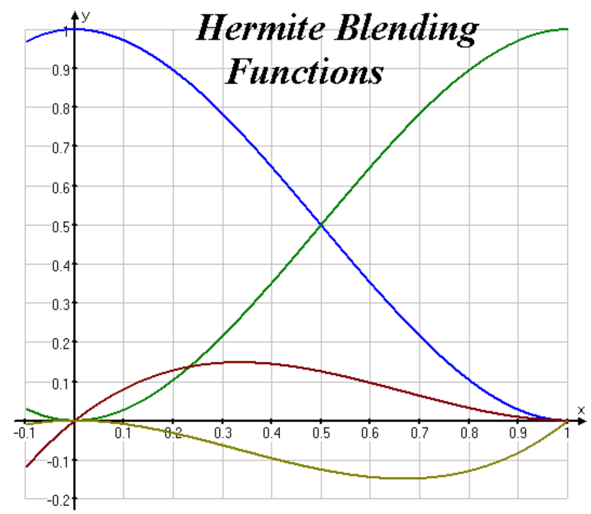
- Hermite: 2 endpoints position, two tangent vectors
- Bezier: 2 endpoints position, two control points

Review (cont.): Cubic Hermite & Bezier Curves

$$C(t) = \sum_{i=0}^{i=3} \underline{B_i^n}(t) G_i$$

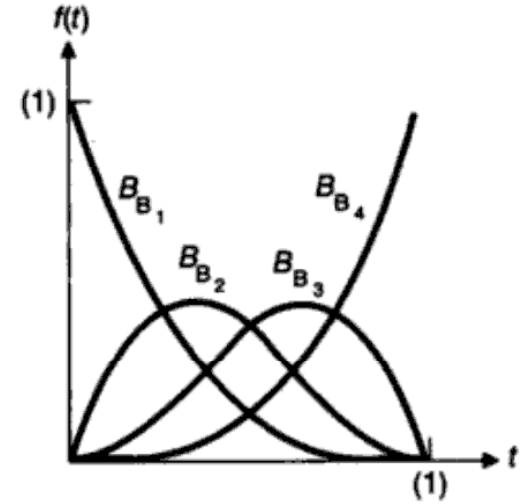
- Hermite Curves:

$$f_1(t) = 2t^3 - 3t^2 + 1$$
$$f_2(t) = -2t^3 + 3t^2$$
$$f_3(t) = t^3 - 2t^2 + t$$
$$f_4(t) = t^3 - t^2$$



- Bezier Curves:

$$B_0^3(t) = (1-t)^3$$
$$B_1^3(t) = 3t(1-t)^2$$
$$B_2^3(t) = 3t^2(1-t)$$
$$B_3^3(t) = t^3$$



Review (cont.):
Parametric and Geometric Continuity

- Parametric continuity
  - Depends on parameterization
- Geometric continuity
  - Can become parametric continuity after reparameterization

Bezier Curves

- Bezier curves of degree n

$$\mathbf{c}(t) = \sum_{i=0}^{n} B_i^n(t) P_i$$

- Basis functions (Bernstein polynomials of degree n):

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$$

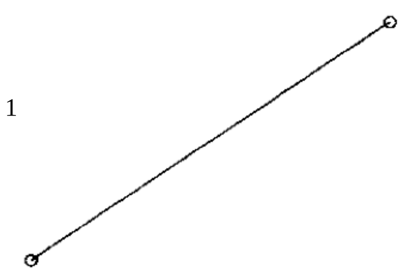
$$\binom{n}{i} = \frac{n!}{(n-i)!\, i!}$$

Some Bezier curve examples

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}; C(t) = \sum_{i=0}^{i=n} B_i^n(t) P_i$$

- n=1 : linear interpolation

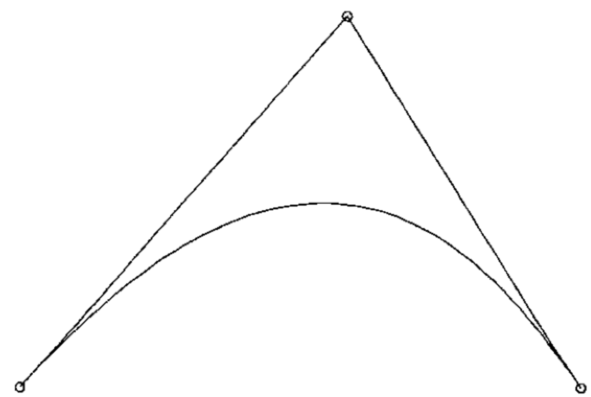
$$B_0^1(t) = 1 - t; B_1^1(t) = t : C(t) = (1-t)P_0 + tP_1$$



- n=2 : recursive linear interpolation

$$C(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$$

  - $\{P_0, P_1, P_2\}$ → control polygon
  - $P_0=C(0)$ and $P_2=C(1)$
  - Tangent directions at endpoints are parallel to P_1-P_0 and P_2-P_1
  - Curve contained in triangle $P_0P_1P_2$



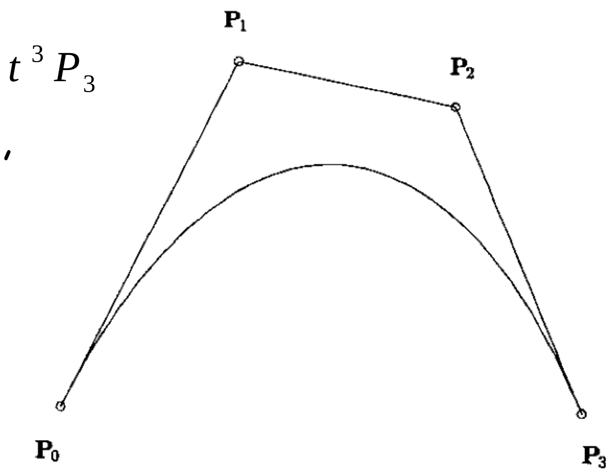
Some Bezier curve examples

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}; C(t) = \sum_{i=0}^{i=n} B_i^n(t) P_i$$

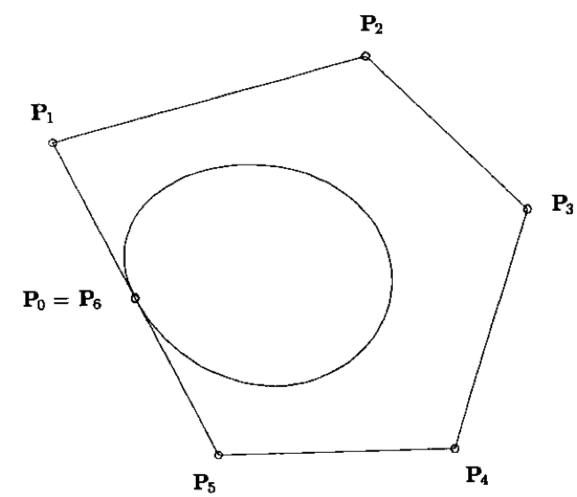
- n=3 : cubic Bezier curve

$$C(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$

  - Control polygon (CP) approximates the curve shape, curve contained in this convex hull
  - Interpolate endpoints
  - Tangent at endpoints are parallel to P_1-P_0 and P_2-P_1
  - Variation diminishing property: no straight plane intersects the curve more times than it intersects the CP (curve doesn't wiggle more than CP)



- n=6 : a degree-6 closed Bezier curve
  - G1 continuous at C(0)=C(1)



Basic Properties of Bezier Cubic Curves

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}; C(t) = \sum_{i=0}^{i=n} B_i^n(t) P_i$$

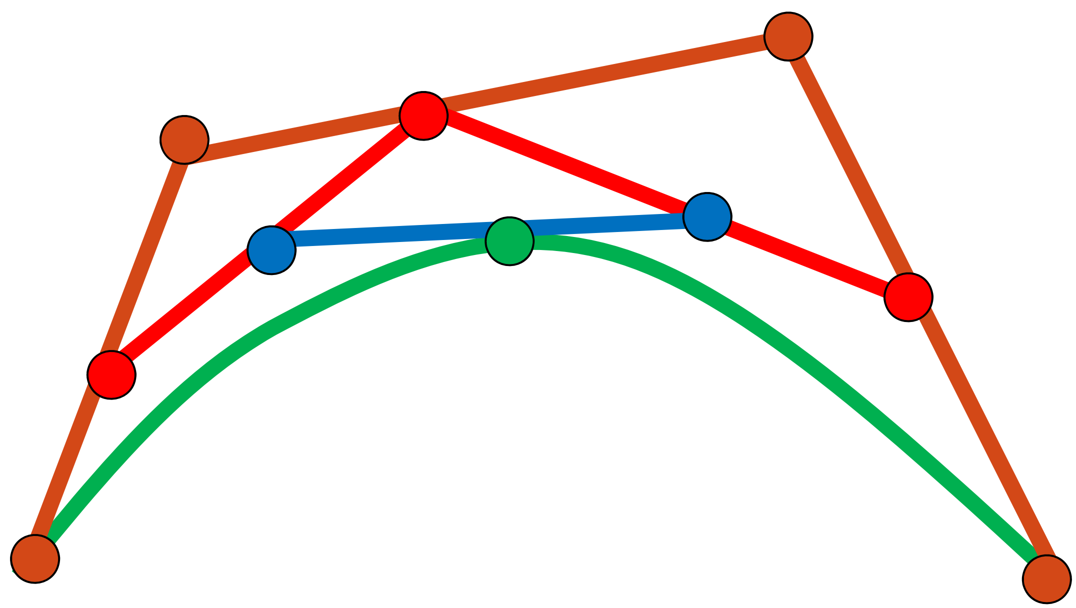
- The curve passes through the first and the last points (end-point interpolation)
- The curve is a linear combination of control points and basis functions
- Basis functions are all polynomials
- Basis functions sum to one (partition of unity)
- Basis functions are non-negative
- Convex hull
- Predictability

Recursive Computation

$$\mathbf{p}_i^0 = \mathbf{p}_i, i = 0,1,2,\ldots n$$

$$\mathbf{p}_i^j = (1-t)\mathbf{p}_i^{j-1} + t\mathbf{p}_{i+1}^{j-1}$$

$$\mathbf{c}(t) = \mathbf{p}_0^n(t)$$

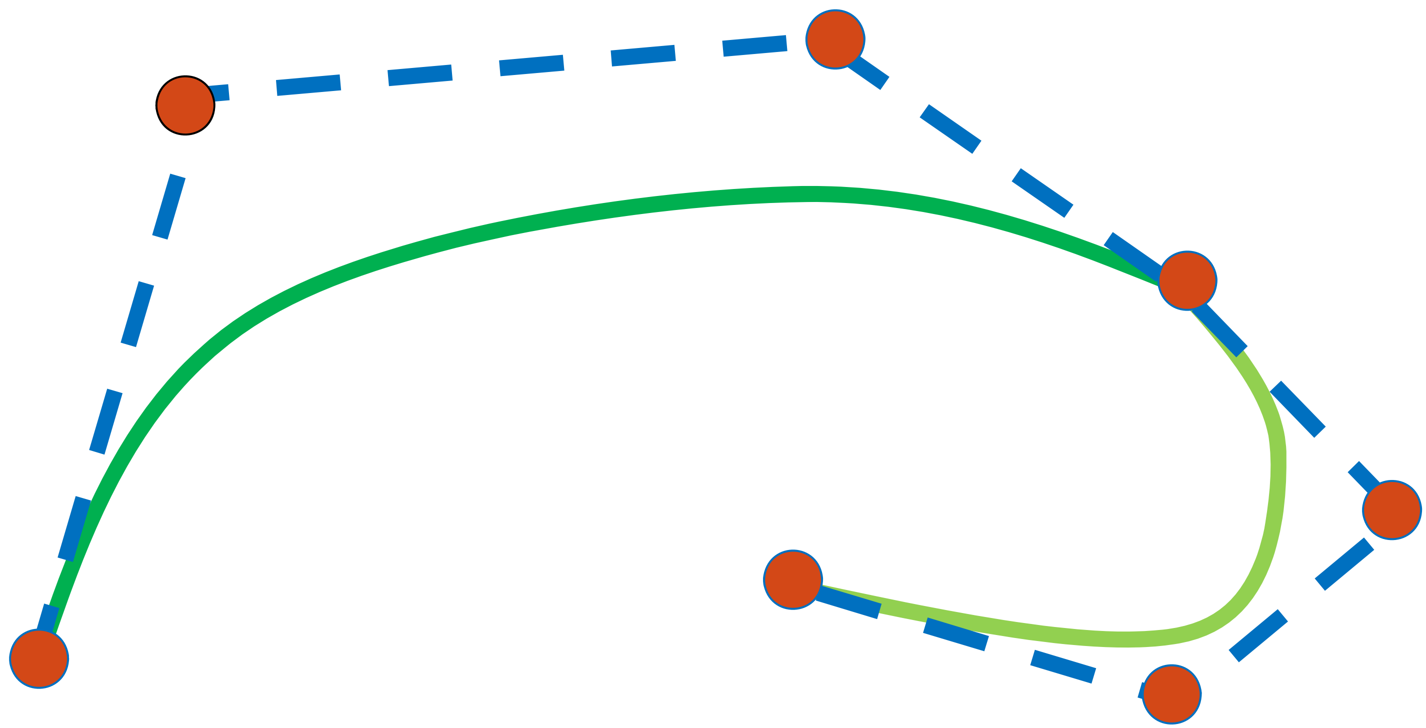


Recursive Computation

- N+1 levels

$$
\begin{array}{ccccccc}
 & (1-t) & & & (t) & & \\
\mathbf{p}_0^0 & & \cdots & & \cdots & & \mathbf{p}_n^0 \\
 & \mathbf{p}_0^1 & & \cdots & & \mathbf{p}_{n-1}^1 & \\
 & \cdots & & \cdots & & \cdots & \\
 & & \mathbf{p}_0^{n-1} & & \mathbf{p}_1^{n-1} & & \\
 & & & \mathbf{p}_0^n = \mathbf{c}(t) & & &
\end{array}
$$

Piecewise Bezier Curves



Piecewise Bezier Curves

- C0 continuity

$$\mathbf{p}_3 = \mathbf{q}_0$$

- C1 continuity

$$\mathbf{p}_3 = \mathbf{q}_0$$
$$(\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{q}_1 - \mathbf{q}_0)$$

- G1 continuity

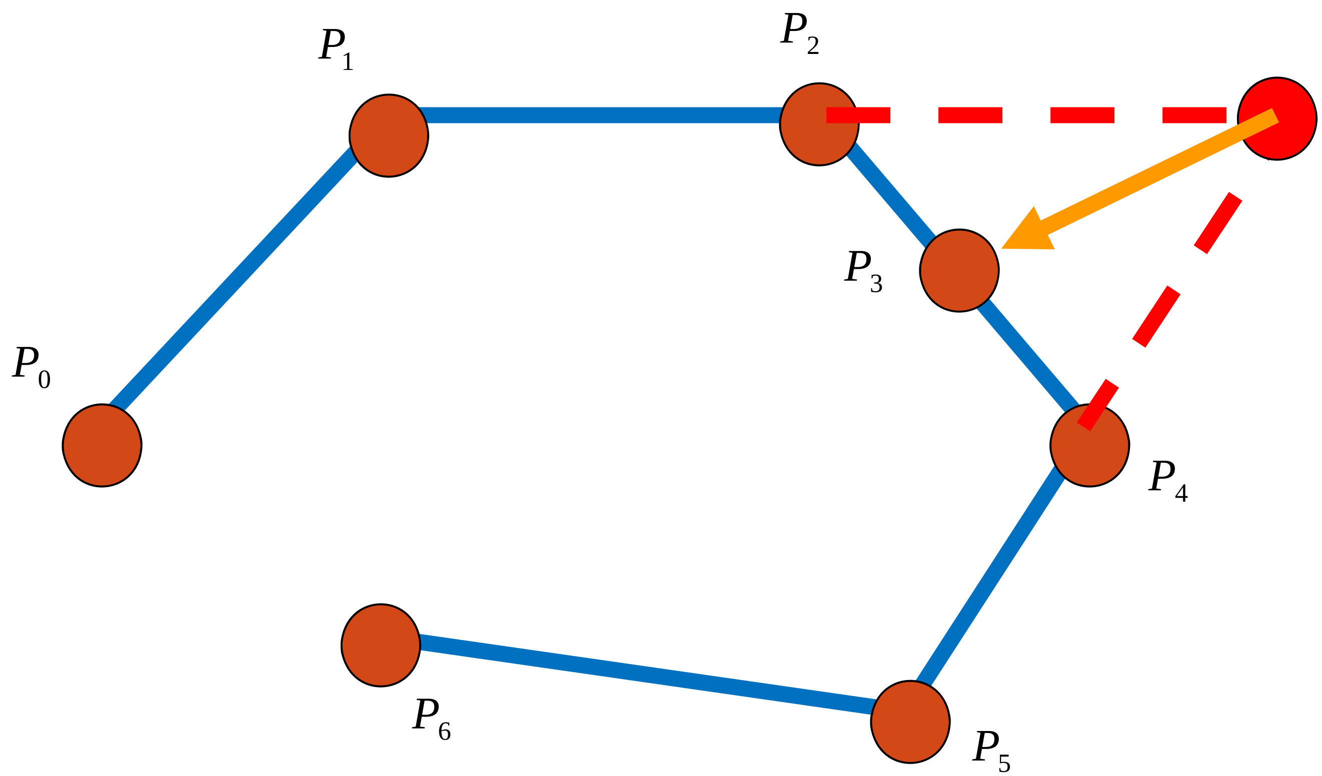
$$\mathbf{p}_3 = \mathbf{q}_0$$
$$(\mathbf{p}_3 - \mathbf{p}_2) = \alpha(\mathbf{q}_1 - \mathbf{q}_0)$$

- C2 continuity

$$\mathbf{p}_3 = \mathbf{q}_0$$
$$(\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{p}_3 - 2\mathbf{p}_2 + \mathbf{p}_1 = \mathbf{q}_2 - 2\mathbf{q}_1 + \mathbf{q}_0$$

- G2 continuity

Piecewise C2 Bezier Curves



From Bezier curves to Splines

- To design a long curve with many undulations
  - One approach → a high-degree Bezier curve
    - Global influence
  - Piecewise Bezier
    - Need to match endpoints and tangents
      ($p_2^j, p_3^j = p_0^{j+1}, p_1^{j+1}$ to be on the same line)
    - No C2 continuity
- Three commonly desirable properties of cubic curves:
  1. C2 continuity
  2. Interpolation
  3. Local control
- Piecewise-Bezier curves → 2,3 or 1,2
- Natural cube spline → 1,2
- B-Spline → 1,3

Splines

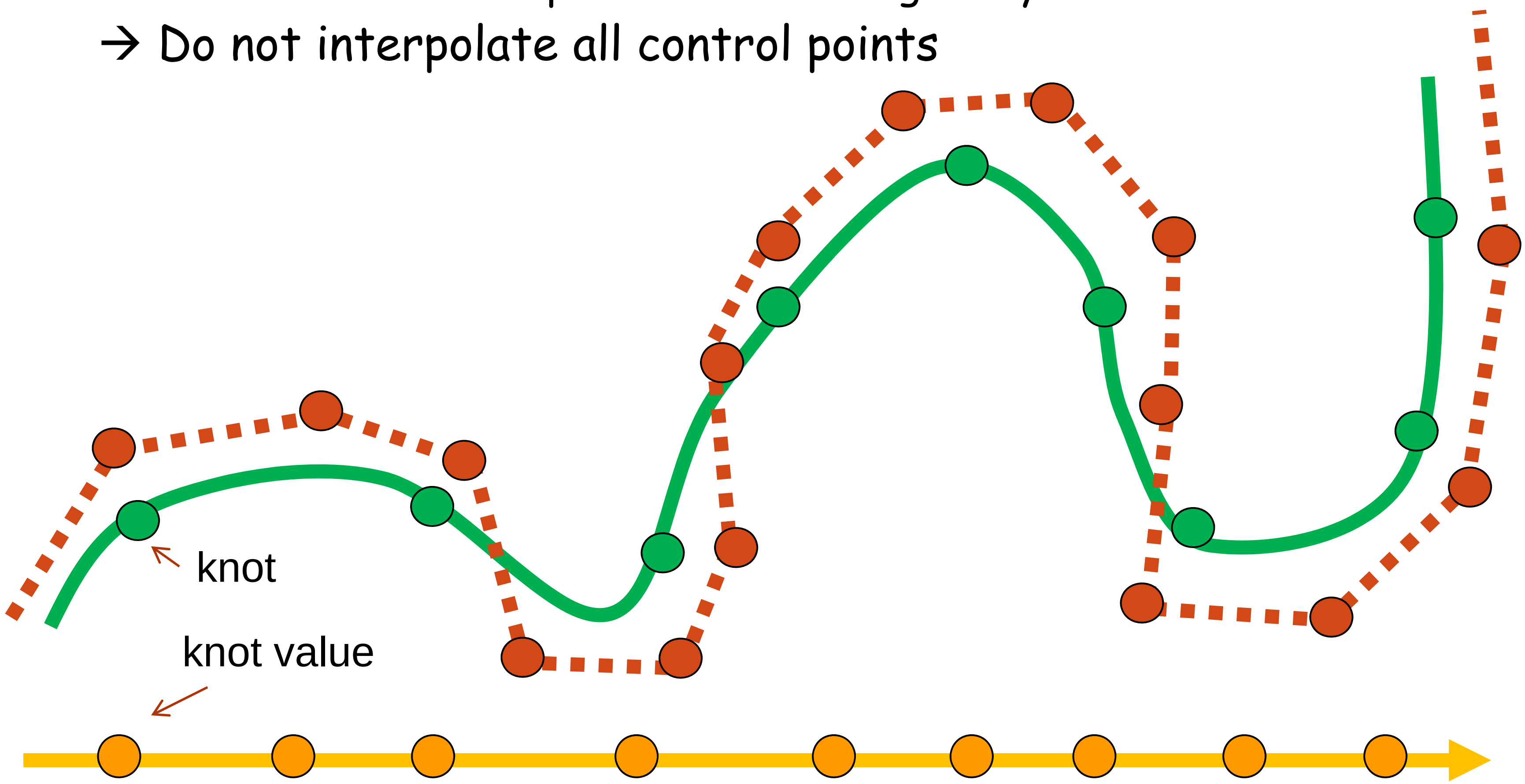
- Spline = long flexible strips of metal used by draftspersons to lay out the surfaces of airplanes, cars, and ships
- The metal splines, unless severely stressed, had second-order continuity
- The mathematical equivalent of these strips, the natural cubic spline, is C2 continuous cubic polynomial that interpolates the control points (1 more degree of continuity than Hermite and Bezier forms discussed previously)
- Problem of natural cubic spline:
  - Global control: dependent on all n control points
  - Computational time: inverting an n+1 by n+1 matrix

R. Bartels, J. Beatty, and B. Barsky, “An Introduction to Splines for Use in Computer Graphics and Geometric Modeling”, Morgan Kaufmann, 1987

B-Spline Motivation

- ❑ Local control:
  - ❑ Moving a control point only affects a small part
  - ❑ Coefficient computational time is greatly reduced

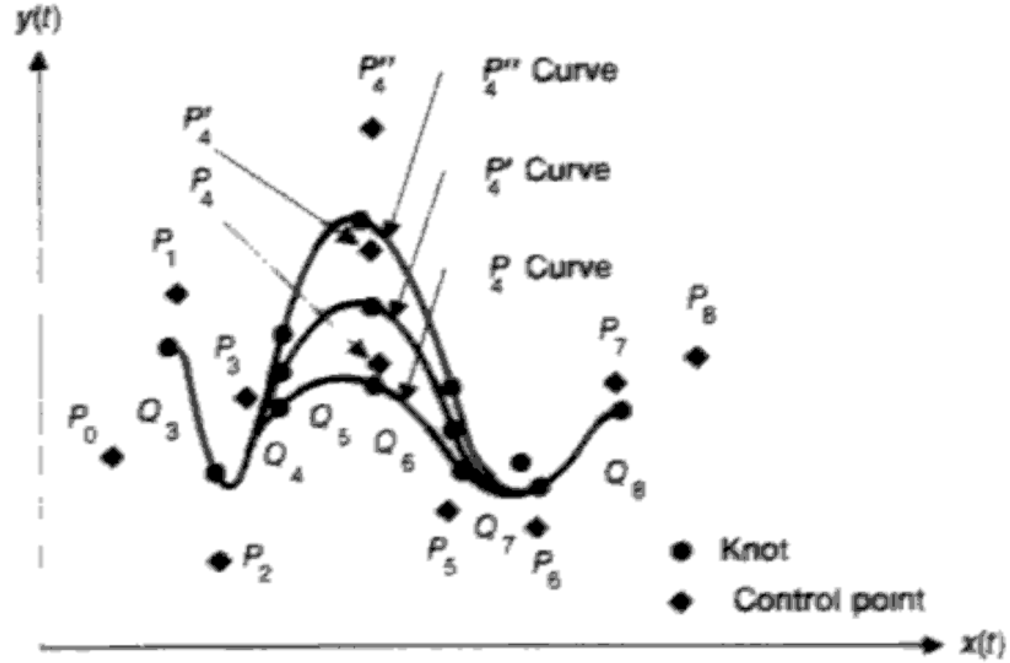
→ Do not interpolate all control points



B-Spline Definition

- Definition: $\mathbf{S}(t) = \sum_{i=0}^{m-n} \mathbf{P}_i b_{i,n}(t)$, $t \in [t_{n-1}, t_{m-n}]$
  - A B-spline of degree n is a parametric curve
    - defined on $t_0 \leq t_1 \leq \cdots \leq t_{m-1}$ (decided by given m knots)
    - composed by a linear combination of degree-n basis functions
- e.g. : a degree-3 parametric curve defined by:
  - m+1 control points: $P_0, P_1, \ldots P_m, m \geq 3$
  - m-2 cubic polynomial curve segments: $Q_3, Q_4, \ldots Q_m$
  - Each cubic curve segment can be defined on $0 \leq t < 1$, but we translate them to be sequential: $t_i \leq t < t_{i+1}$
  - There is a join point or **knot** between each Q_{i-1} and Q_i at the parameter value t_i ; the parameter value at such a point is called a **knot value**.
  - Totally: m-1 knots (including t_3 and t_{m+1})

B-Spline Definition



- Definitions:
  - Uniform B-spline: knots are spaced at equal intervals of the parameter t (e.g. $t_3=0$, $t_{i+1}-t_i=1$)
  - Nonuniform B-spline: …
  - Nonrational vs rational (later)
  - B-spline: →basis : splines are weighted sums of polynomial basis functions (in contrast to the natural splines)

B-Spline Basis Functions

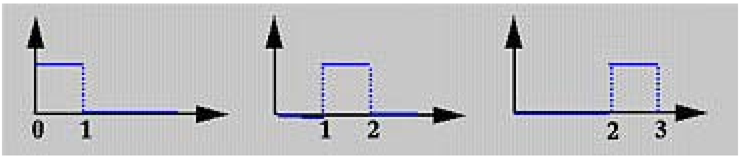
$$B_{i,1}(t) = \begin{cases} 1 & t_i <= t < t_{i+1} \\ 0 & otherwise \end{cases}$$

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(t)$$

Bezier curve is a special case of it

B-Spline Basis Functions

- Degree-1:



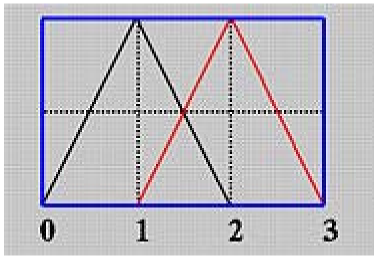
$$B_{0,1}(t) = \begin{cases} 1 & t \in [0,1] \\ 0 & o/w \end{cases}$$

- Degree-2:

$$B_{0,2}(t) = \begin{cases} t & t \in [0,1] \\ 2 - t & t \in [1,2] \end{cases}$$

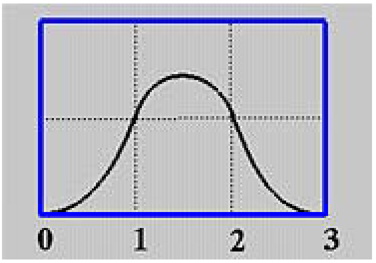
$$B_{1,2}(t) = \begin{cases} t - 1 & t \in [1,2] \\ 3 - t & t \in [2,3] \end{cases}$$

$$B_{2,2}(t) = \begin{cases} t - 2 & t \in [2,3] \\ 4 - t & t \in [3,4] \end{cases}$$



B-Spline Basis Functions

- Degree-3: Quadratic example (knot vector is [0,1,2,3,4,5,6])



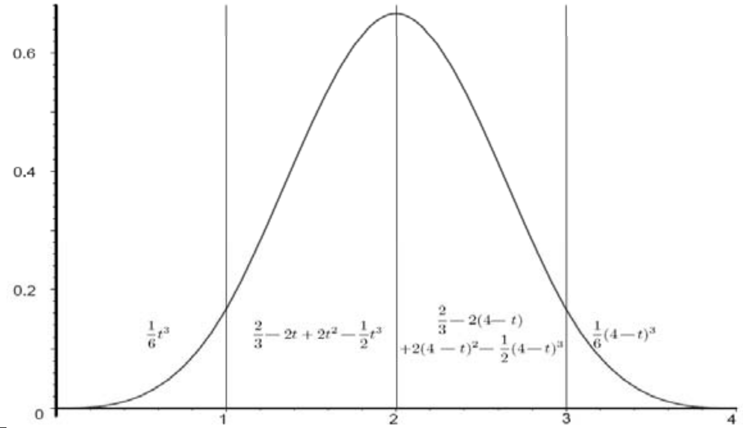
$$B_{0,3}(t) = \begin{cases} \frac{1}{2}t^2, & 0 <= t < 1 \\ \frac{1}{2}t(2-t) + \frac{1}{2}(t-1)(3-t), & 1 <= t < 2 \\ \frac{1}{2}(3-t)^2, & 2 <= t < 3 \end{cases}$$

$$B_{1,3}(t) = \begin{cases} \frac{1}{2}(t-1)^2, & 1 <= t < 2 \\ \frac{1}{2}(t-1)(3-t) + \frac{1}{2}(t-2)(4-t), & 2 <= t < 3 \\ \frac{1}{2}(4-t)^2, & 3 <= t < 4 \end{cases}$$

$$B_{2,3}(t) = \ldots\ldots$$

$$B_{3,3}(t) = \ldots\ldots$$

- Degree-4:



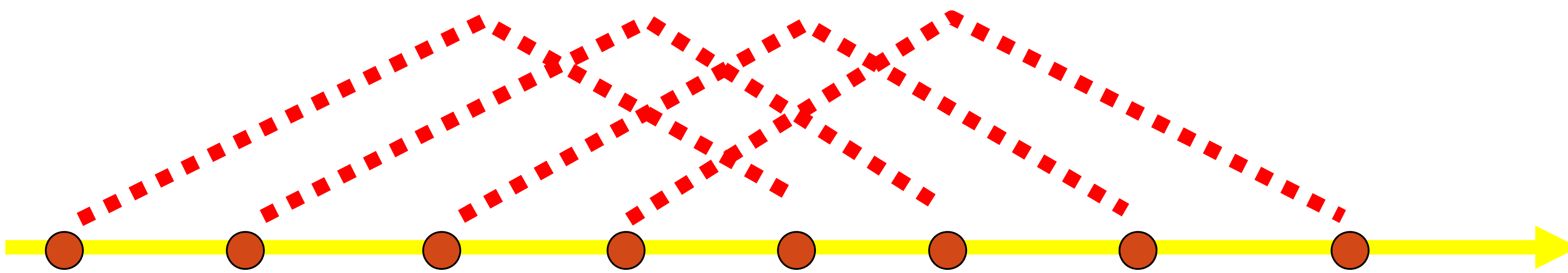
B-Spline Basis Functions

$B_{0,1} \quad B_{1,1} \quad B_{2,1} \quad B_{3,1} \quad B_{4,1} \quad B_{5,1} \quad B_{6,1}$

$B_{0,2} \quad B_{1,2} \quad B_{2,2} \quad B_{3,2} \quad B_{4,2} \quad B_{5,2}$

$B_{0,3} \quad B_{1,3} \quad B_{2,3} \quad B_{3,3} \quad B_{4,3}$

$B_{0,4} \quad B_{1,4} \quad B_{2,4} \quad B_{3,4}$

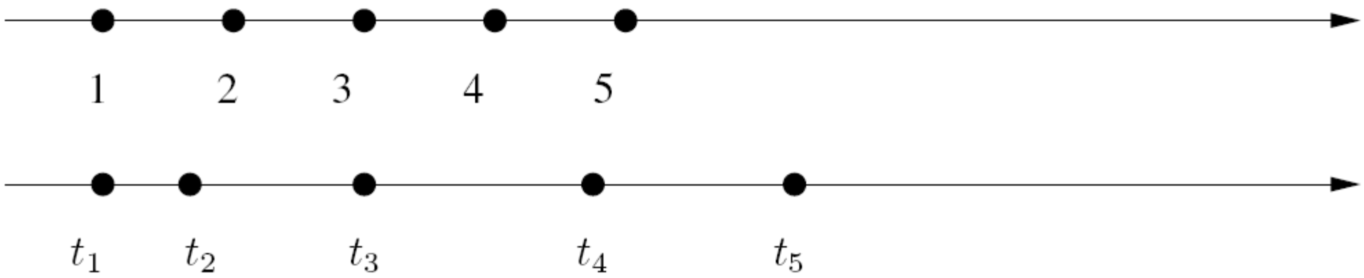


B-Spline Properties

- $B_{i,n}(t)$ is a piecewise polynomial of degree n, and with C^{n-1} continuity
- $B_{i,n}(t)$ has a support of length n+1
- Each curve segment is defined by n+1 control points, and each control point affects at most n+1 curve segments
- The degree of basis functions is independent of the number of control points
- Convex hull, local control
- Positivity, partition of unity, recursive evaluation

Uniform B-Spline

- Uniform vs Nonuniform:

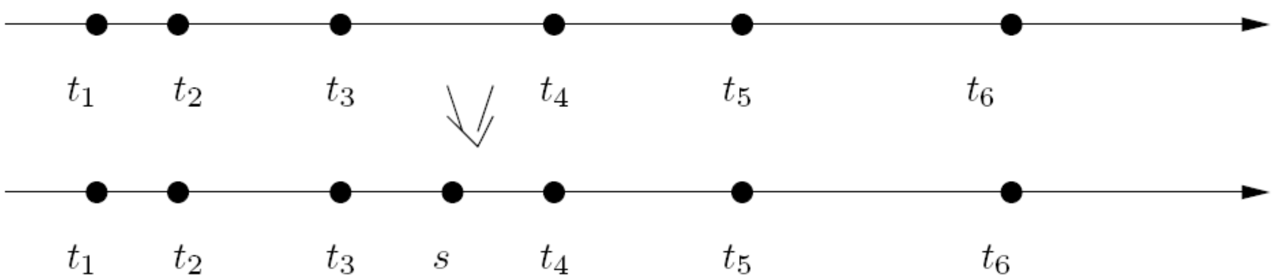


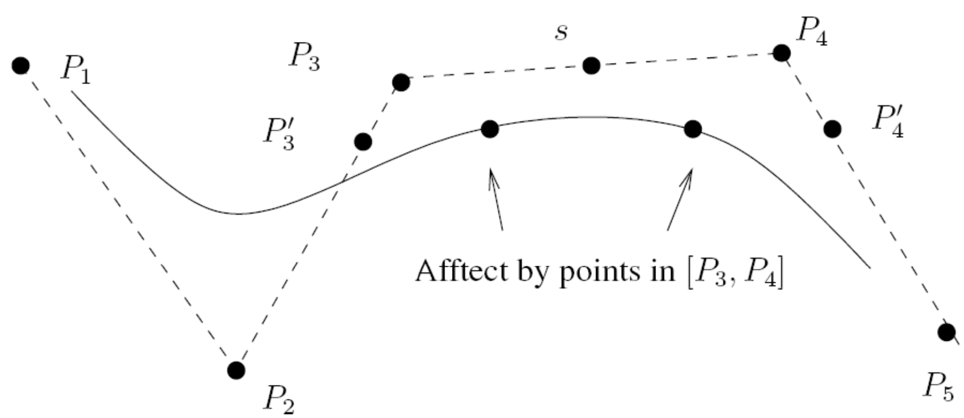
- Uniform Cubic B-spline (represented as Bezier control points)

$$\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \\ \mathbf{p}_{i+3} \end{bmatrix}$$

Non-uniform B-Spline

- One of the most important advantage:
  - Knot insertion (locally adding a control point without changing the curve, for feature adjustment later)
    - Insert a new knot
    - Add a new control point, and update two control points





NURBS

- ❑NURBS = Non Uniform Rational B-Splines
- ❑Rational Functions → ratios of two polynomials
- ❑Why: the need to represent some analytic shapes, for example, conic sections (e.g., circles, ellipses, parabolas)

→A non-uniform and rational extension of B-splines,

→A unified representation for polynomials, conic sections, etc.

→The industry standard representation

- ❑Intuitively, rational representation adds weights to the control points, so that some control points are more important.

B-Spline ⇒

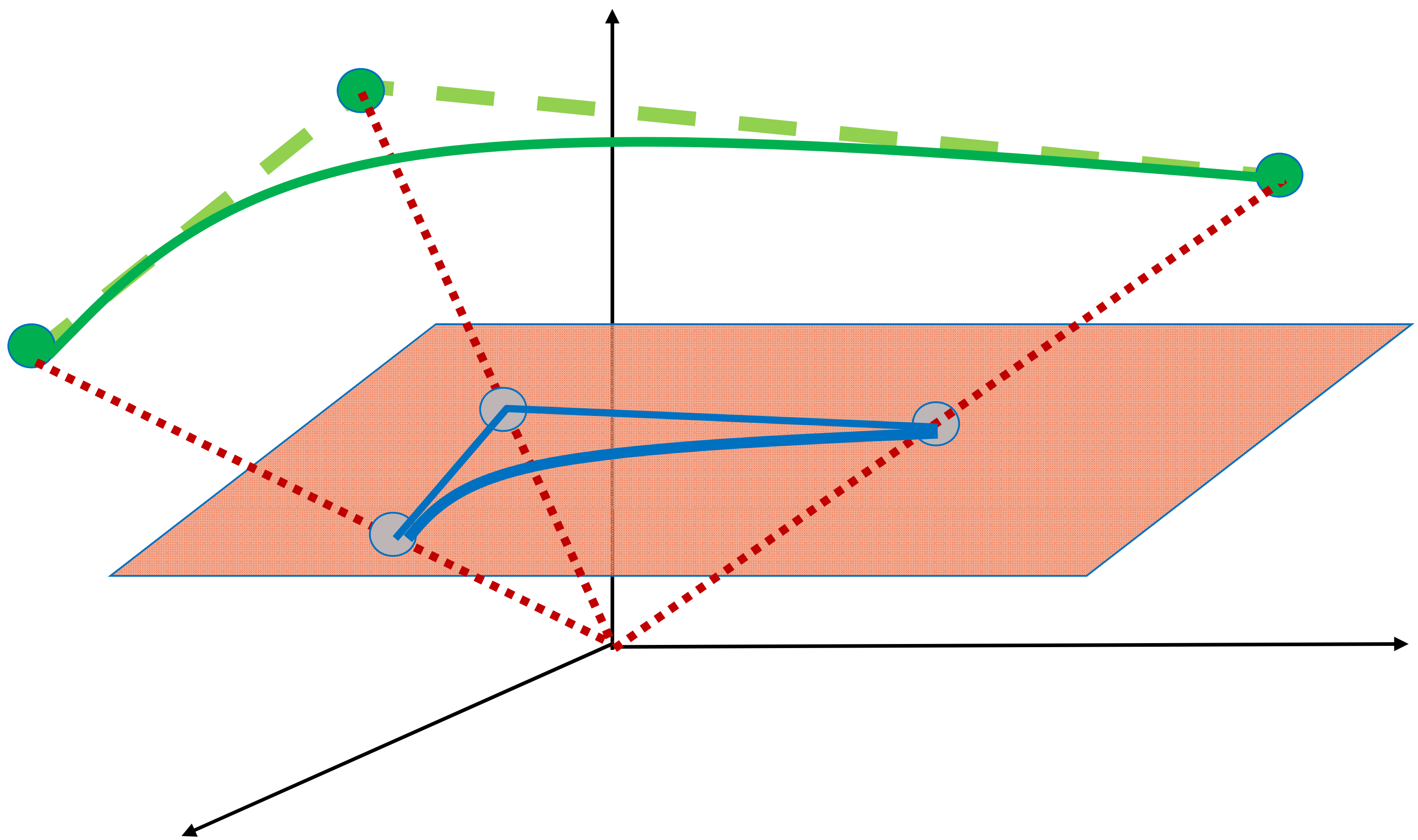
$$\mathbf{c}(u) = \sum_{i=0}^{n} \begin{bmatrix} \mathbf{p}_{i,x} w_i \\ \mathbf{p}_{i,y} w_i \\ \mathbf{p}_{i,z} w_i \\ w_i \end{bmatrix} B_{i,k}(u)$$

NURBS ⇒

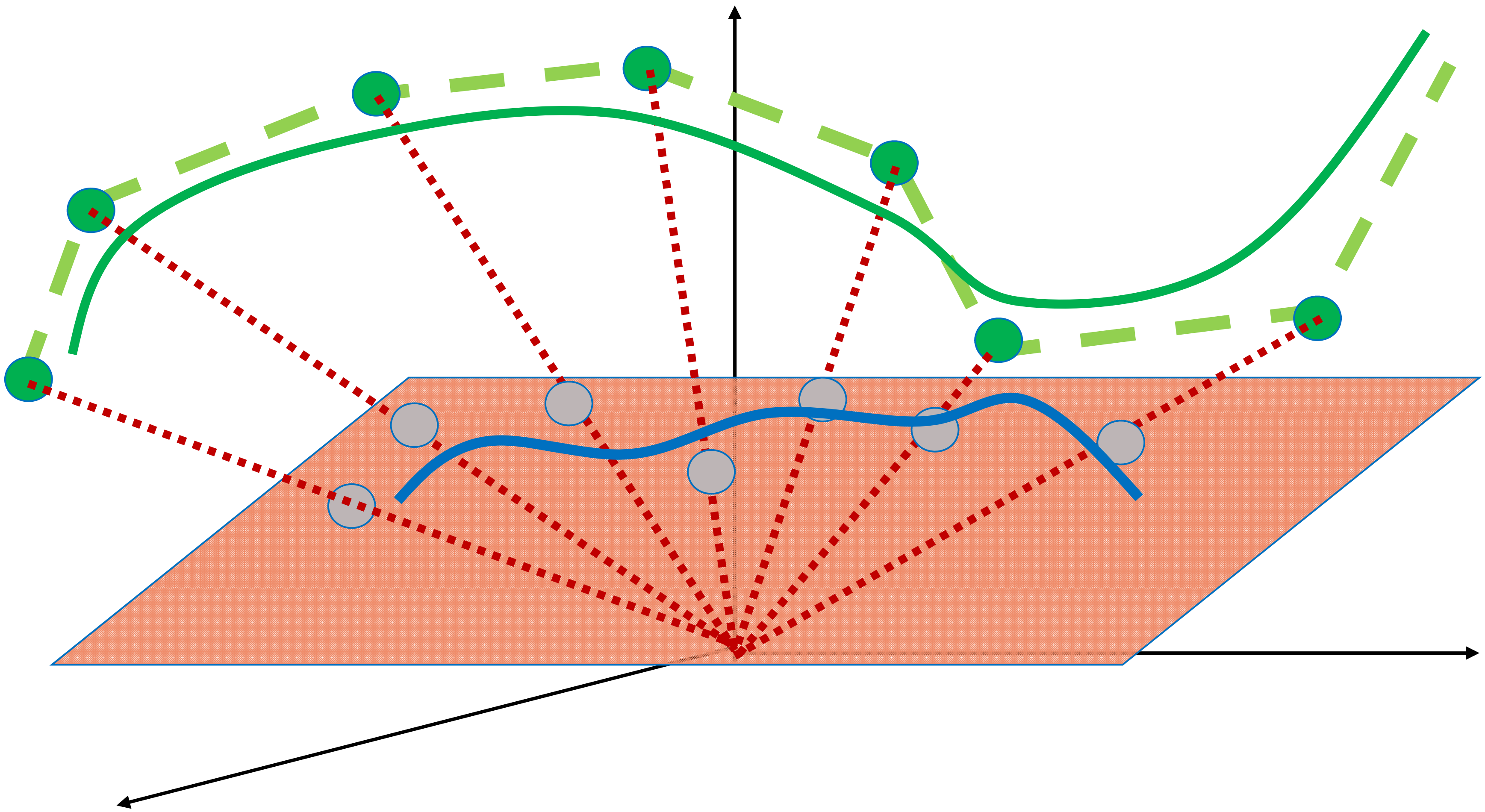
$$\mathbf{c}(u) = \frac{\sum_{i=0}^{n} \mathbf{p}_i w_i B_{i,k}(u)}{\sum_{i=0}^{n} w_i B_{i,k}(u)}$$

Rational Bezier Curve

- Projecting a Bezier curve onto w=1 plane



From B-Splines to NURBS



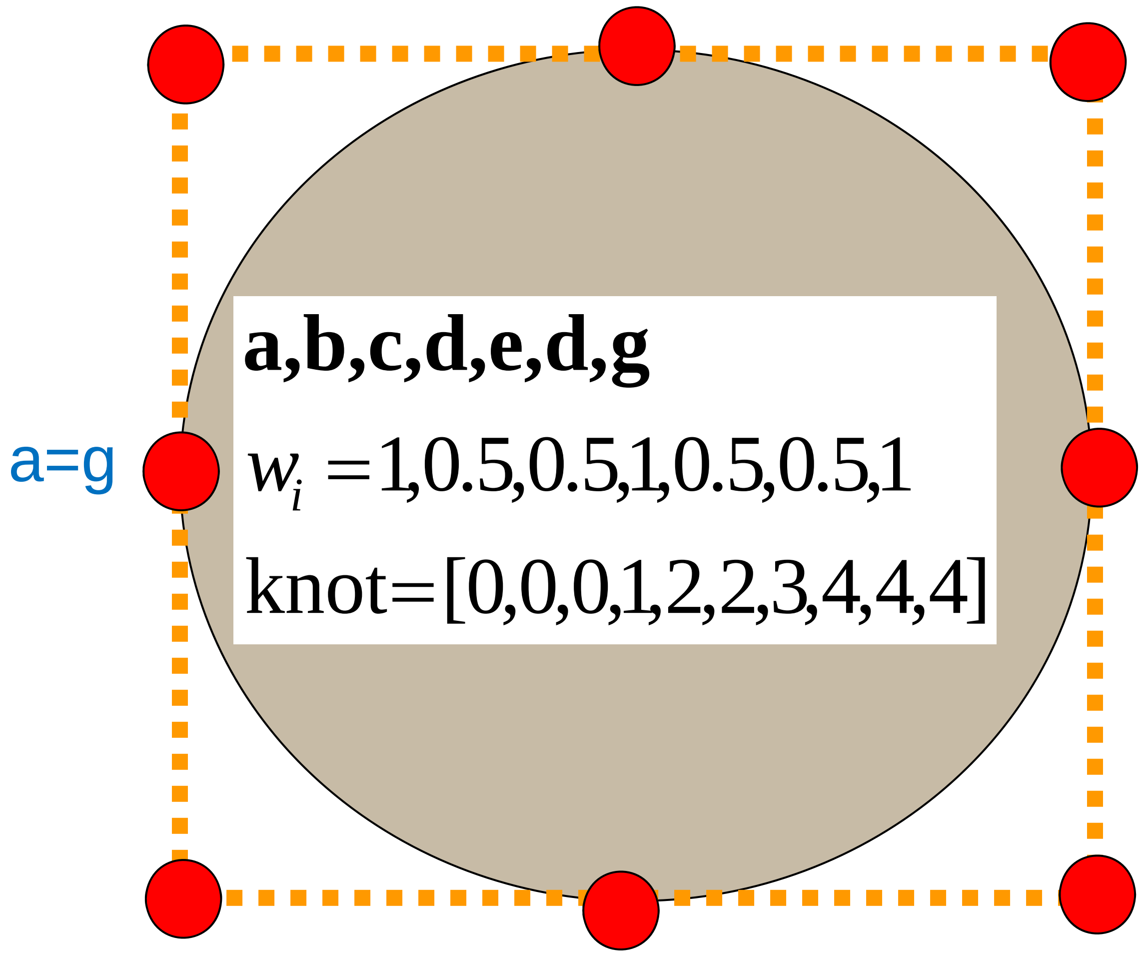
NURBS Weights

- Weight increase "attracts" the curve towards the associated control point
- Weight decrease "pushes away" the curve from the associated control point

NURBS for Analytic Shapes

- Conic sections
- Natural quadrics
- Extruded surfaces
- Ruled surfaces
- Surfaces of revolution

NURBS Circle



a,b,c,d,e,d,g

$w_i = 1, 0.5, 0.5, 1, 0.5, 0.5, 1$

knot=[0,0,0,1,2,2,3,4,4,4]

… Will be explained in the next class…

NURBS Curve

- Geometric components
  - Control points, parametric domain, weights, knots
- Homogeneous representation of B-splines
- Geometric meaning --- obtained from projection
- Properties of NURBS
  - Represent standard shapes, invariant under perspective projection, B-spline is a special case, weights as extra degrees of freedom, common analytic shapes such as circles, clear geometric meaning of weights