Simplicial Complex and Barycentric Coordinates

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Barycentric Coordinates

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- A triangle mesh can be rigorously defined as a simplicial complex, unlike the simple grid structure which is also widely used.
- Later, splines can be defined on simplicial complex structures (e.g. on triangle meshes), or the tensor-product structures (e.g. on grids).

Simplicial Complex - Simplex

- Intuition: to define a triangle mesh, you start from defining vertices, edges, and faces.
- These elements are called *Simplexes*.

Definition (k-dimensional Simplex)

Suppose $k+1$ points $\{v_0, v_1, \ldots, v_k\}$ are *non-degenerate* in $\mathbb{R}^n$, $n \geq k+1$, the standard simplex $[v_0, v_1, \ldots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \ldots, v_k] = \{x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

where non-degenerate means linearly independent, e.g. any 3 points are not collinear, any 4 points are not coplanar, *ldots*

Simplicial Complex - Simplex (cont.)

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- we call $v_0, v_1, \ldots, v_k$ the vertices of the simplex σ.
- If $\tau \subset \sigma$ is also a simplex, then we say τ is a facet of σ.

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A simplicial complex Σ is a union of simplices, such that:

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2. If $\sigma_1, \sigma_2 \subset K, \sigma_1 \bigcap \sigma_2 \neq \emptyset$, then the intersection of σ_1 and σ_2 is also a common facet.

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- Triangular meshes are simplicial complexes (vertex, oriented edges, and oriented faces are 0-simplexes, 1-simplexes, and 2-simplexes respectively).

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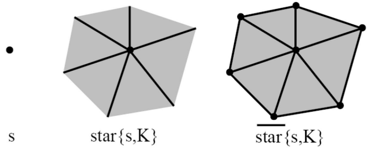
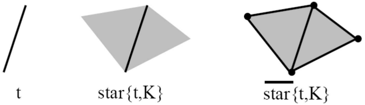


(a) on a vertex;



(b) on an edge

Barycentric Coordinates

- A simplex is the smallest convex set containing its vertices.
- These $\{\lambda_i\}$, satisfying

$$\lambda_i > 0, \sum_{i=0}^{n} \lambda_i = 1,$$

  are called barycentric coordinates.
- Each set $(\lambda_0, \ldots, \lambda_n)$ determines a unique point in the n dimensional simplex.
  - interior of a simplex $\rightarrow \forall \lambda_i > 0$;
  - boundary of a simplex $\rightarrow \exists \lambda_i = 0$;

Computing Barycentric Coordinates

- Given a point v inside a simplex s, how do we compute its barycentric coordinates?
- If s is an edge $[v_1, v_2]$, easy.
- If s is a face $[v_1, v_2, v_3]$:
  - $v = \sum_{i=1}^{3} \lambda_i v_i$, $\sum_{i=1}^{3} \lambda_i = 1$, $\lambda_1, \lambda_2, \lambda_3 > 0$,
  - then we have $\lambda_3 = 1 - \lambda_1 - \lambda_2$;
  - look at a local 2D coordinate system defined on the face $[v_1, v_2, v_3]$:

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + (1 - \lambda_1 - \lambda_2) x_3 \\ y = \lambda_1 y_1 + \lambda_2 y_2 + (1 - \lambda_1 - \lambda_2) y_3 \end{cases} \quad (1)$$

Computing Barycentric Coordinates (cont.)

$$\Rightarrow \begin{cases} \lambda_1(x_1 - x_3) + \lambda_2(x_2 - x_3) + (x_3 - x) = 0 \\ \lambda_1(y_1 - y_3) + \lambda_2(y_2 - y_3) + (y_3 - y) = 0 \end{cases} \tag{2}$$

$$\Rightarrow \begin{bmatrix} (x_1 - x_3) & (x_2 - x_3) \\ (y_1 - y_3) & (y_2 - y_3) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} x - x_3 \\ y - y_3 \end{bmatrix} \tag{3}$$

- the matrix has the inverse (since $v_1 - v_3$ and $v_2 - v_3$ are linear independent) $M^{-1} =$

$$\frac{1}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)} \begin{bmatrix} (y_2 - y_3) & -(y_1 - y_3) \\ -(x_2 - x_3) & (x_1 - x_3) \end{bmatrix} \tag{4}$$

Computing Barycentric Coordinates (cont.)

$$\Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = M^{-1} \begin{bmatrix} (x - x_3) \\ (y - y_3) \end{bmatrix} \tag{5}$$

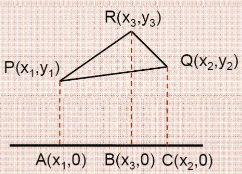
$$\Rightarrow \begin{cases} \lambda_1 = \frac{(x-x_3)(y_2-y_3)-(x_2-x_3)(y-y_3)}{(x_1-x_3)(y_2-y_3)-(x_2-x_3)(y_1-y_3)} \\ \lambda_2 = \cdots \\ \lambda_3 = \cdots \end{cases} \tag{6}$$

Computing Barycentric Coordinates (cont.)

The denominator:

$$(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) = \boxed{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}$$

Twice the signed area of triangle $((x_1,y_1), (x_2, y_2), (x_3,y_3))$



Area(PQR)=Area(PABR)+Area(RBCQ)-Area(PACQ)

Area(PABR)=$(x_3-x_1)(y_1+y_3)/2$

Area(RBCQ)=$(x_2-x_3)(y_2+y_3)/2$

Area(PACQ)=$(x_2-x_1)(y_1+y_2)/2$

→ Area(PQR)=$(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))/2$

Similarly, the numerator $(x - x_3)(y_2 - y_3) - (x_2 - x_3)(y - y_3)$

is twice the signed area of triangle $((x,y), (x_2, y_2), (x_3,y_3))$

→ λ_1 = Area(v,v_2,v_3)/Area(v_1,v_2,v_3)

Computing Barycentric Coordinates (cont.)

- Now back to 3D, these computations are easy.
- Given three vertices v_1, v_2, v_3, and a vertex v, we can compute $Area(\Delta(v_1, v_2, v_3))$ and $Area(\Delta(v, v_2, v_3)$ by vector cross-products.
- Therefore, every point p on the triangle mesh K can now be uniquely represented. p can be a vertex, a point on an edge, or a point inside a triangle.