Surface Parameterization

Problem Definition

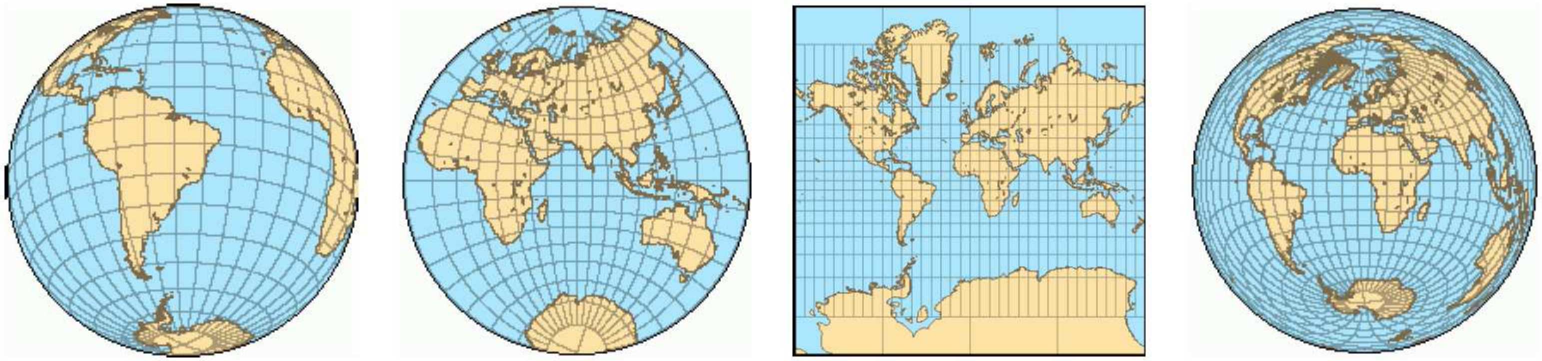
- Recall the Texture Mapping that wrap an Image onto a mesh
  - A one-to-one map from geometry shape S to a texture image (2D domain) D
  - D here is a rectangular domain, e.g. $D = [0,1] \times [0,1]$
  - The mapping: a vector function $\vec{f} : S \rightarrow D \subset \mathbb{R}^2$,

    composed by two scalar function f_u and f_v.
  - ⬄Define a "u-v" coordinates over the surface S.
  - Infinite mapping ways, which one is good?

Motivations

- Texture mapping: generating enhanced effects over simple geometry shapes
  - From geometry space to the texture (image) space
    → surface mapping
  - Quality of the mapping: dictates the effect of texture mapping (low distortion preferred)
- Spline Representation: a compact representation, good for precise computer aided design, and scientific computing
  - For spline fitting : a good parameterization is important for generating smooth spline with small # of control points

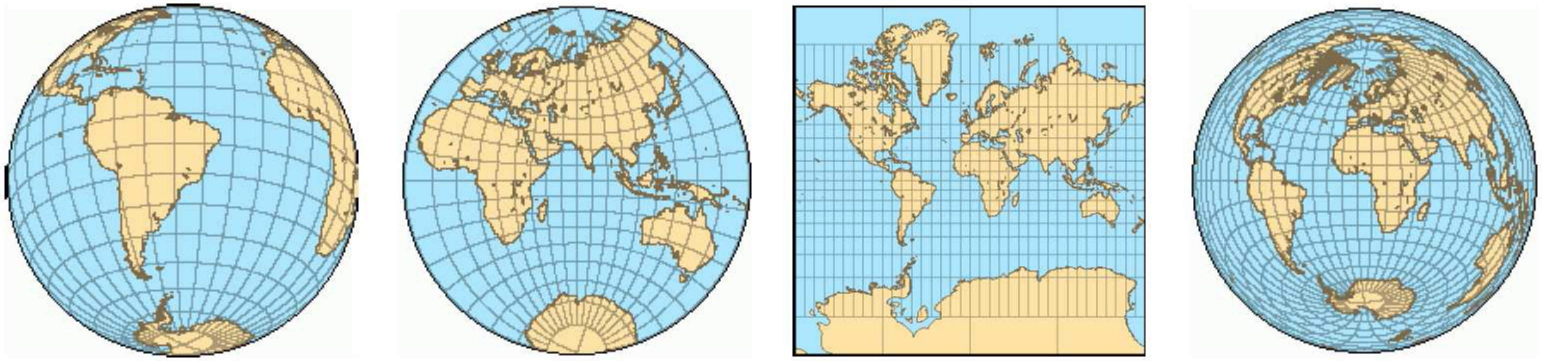
Historical Background



(a) Orthographic; (b) stereographic; (c) Mercator; and (d) Lambert

- Cartography
- Distortion: angles and areas distortion
  - Isometry: no distortion
  - Not all surfaces has the isometry to a planar region
  - Peeling oranges → can't be of no distortion
- Ptolemy was the first known to produce the data for creating a map showing the world (100-150AD)
  - [Geography] → project a sphere by longitude and latitude

Historical Background (cont.)



(a) Orthographic; (b) stereographic; (c) Mercator; and (d) Lambert

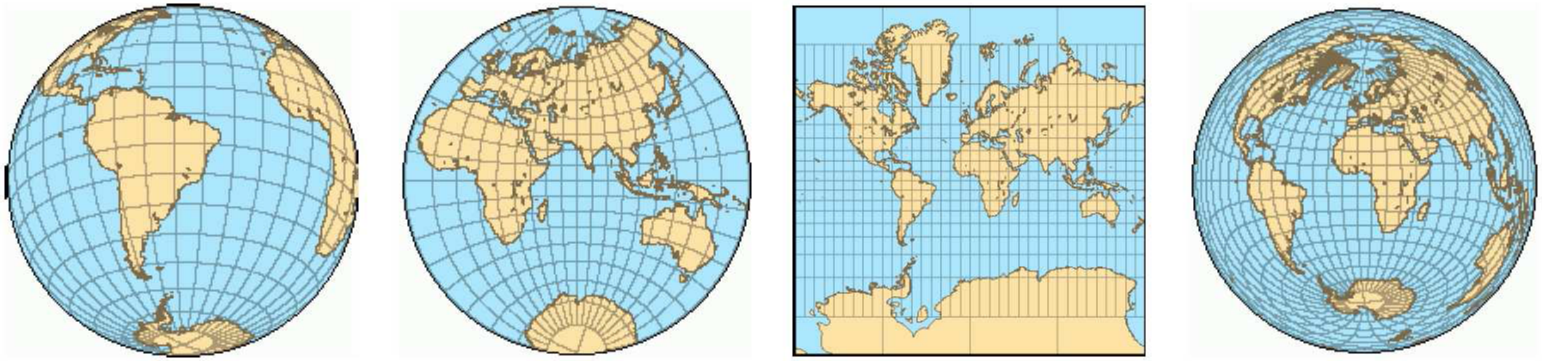
(a) The orthographic projection (Egyptians and Greeks, > 2000 years ago) → modifies both angles and areas

(b) Stereographic projection (Hipparchus, 190-120B.C.) → preserves angles, not areas

(c) Mercator projection (Mercator 1569) → preserves angles, not areas

(d) Lambert projection (Lambert 1772) → preserves areas, not angles

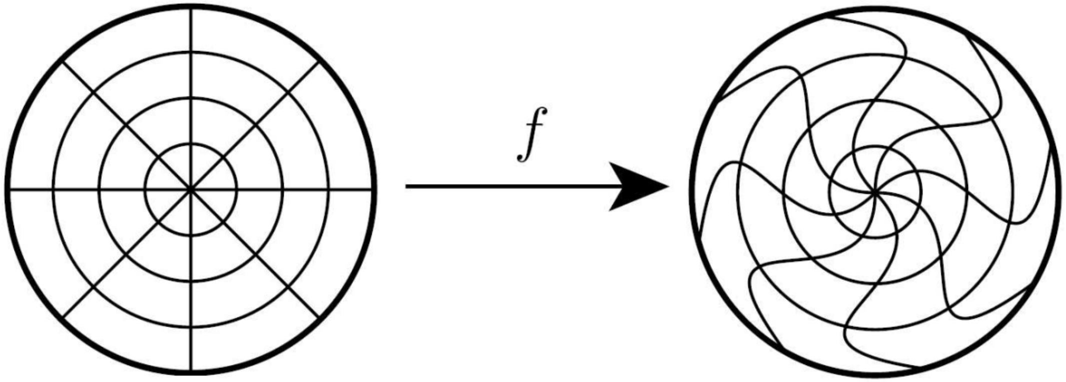
Good "UV" versus bad "UV"?



- What do we look for? What do we preserve?
- Should we map it onto a rectangle? Or a disk? Or something awkward? What do we choose?
- If the target shape is fixed (e.g. a rectangle, or a disk…), what is the best mapping then?
- At this beginning stage:
  - Source: a genus-zero open surface (a topological disk)
  - Target: planar square

Mapping Criteria

- Angle Distortion: change of the local angles
  - Conformal mapping: no angle distortion (locally, a right angle → a right angle, or a circle → a circle), preserving shape information
- Area Distortion: change of the local area
  - Equiareal mapping: no area change
- Isometric Mapping: neither angles nor area distortion
- Isometric ⇔ conformal + equiareal
- Isometry exists between a given surface and a planar domain, only if this surface is "developable"
- Purely Equiareal Mapping is infinitely dimensional and not necessarily useful

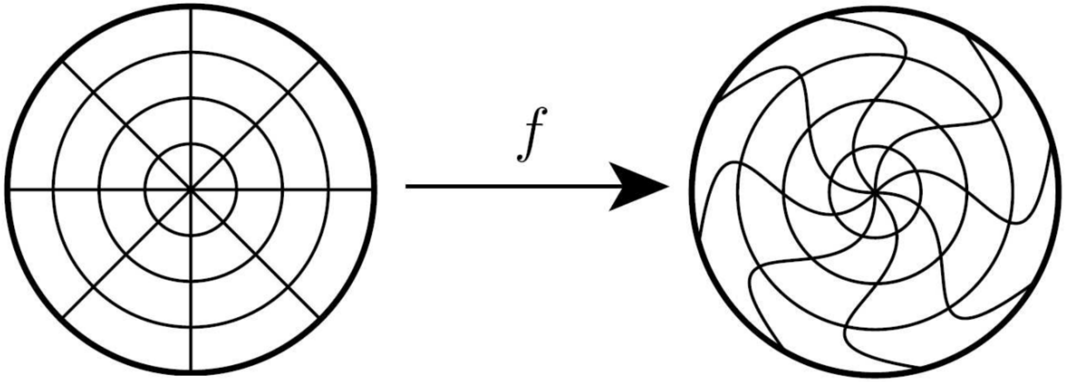


Mapping Criteria

- Therefore:

Given an arbitrary topological disk surface and a planar domain

  - Isometric mapping rarely exists
  - Conformal mapping always exists (Riemann Mapping Theorem)
  - Infinitely many equiareal mapping, as a pure criterion, not easy to control and design



Flattening Triangle Mesh

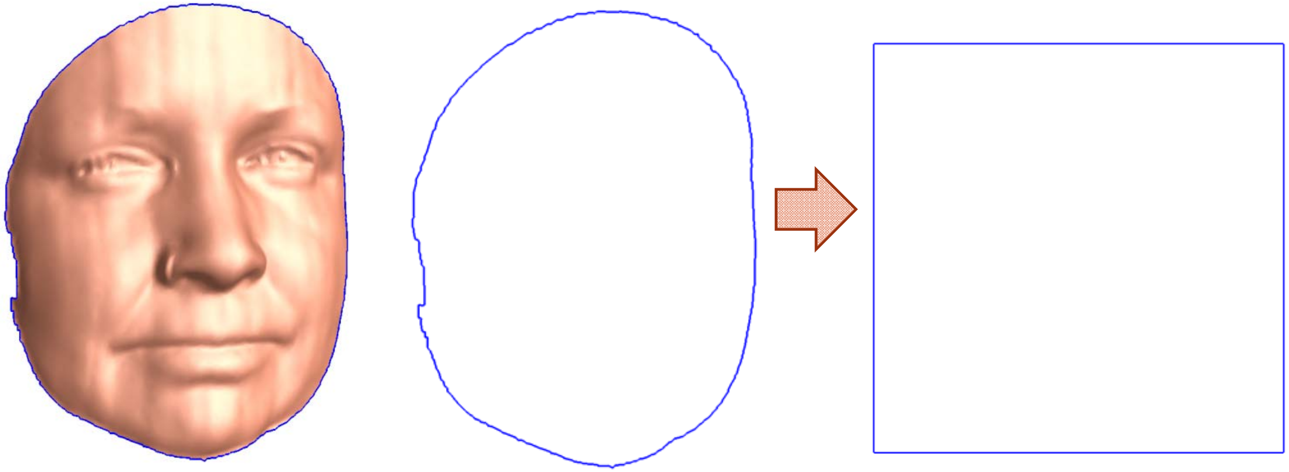
- An intuitive way : considering that you are flattening a triangle mesh (deforming it and make it flat)
  1) Pin vertices on the boundary loop on a planar rectangle boundary
  2) Move the interior vertices into the rectangle properly

Algorithm Pipeline:

computing two harmonic functions $f_u: (x,y,z) \rightarrow u$, and $f_v: (x,y,z) \rightarrow v$

1) For boundary vertices, map them to one of the following four segments

a) $u=0, 0<v<1;$
b) $0<u<1, v=0;$
c) $u=1, 0<v<1;$
d) $0<u<1, v=1.$



Flattening Triangle Mesh

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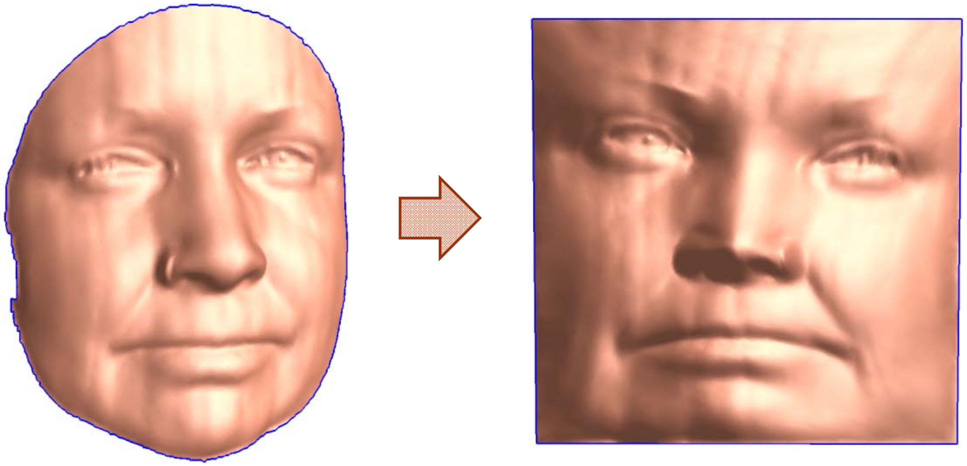
Algorithm Pipeline:

computing two harmonic functions $f_u: (x,y,z) \rightarrow u$, and $f_v: (x,y,z) \rightarrow v$

1) For each interior vertex, map it to $0<u<1$, $0<v<1$

there should not be flip-over, every vertex v_i should be mapped into the interior region of its one ring vertices v_j.



Flatten 3D Mesh by Harmonic Map

- Flattening = Finding the smoothest function that minimizes its variance (minimize the magnitude of the change)

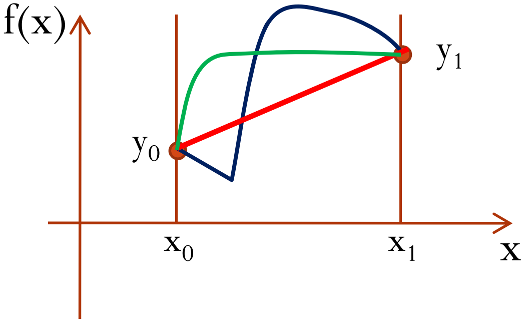
$$E(f) = \frac{1}{2}\int_S \| \nabla f \|^2 dx \qquad (1)$$

→ Called the harmonic energy

- A function that minimizes this energy is called a harmonic function
  - It satisfies $\Delta f(x) = 0, \forall x \in S \qquad (2)$
  - It is uniquely determined by the boundary condition
- Harmonic Function Examples:
  - 1D Curve:

Given: $f(x_0)=y_0$, $f(x_1)=y_1$

The harmonic function $f(x)$ is uniquely defined, and can be computed by minimizing E in (1)



Harmonic Function (1D)

- Harmonic Function Examples:
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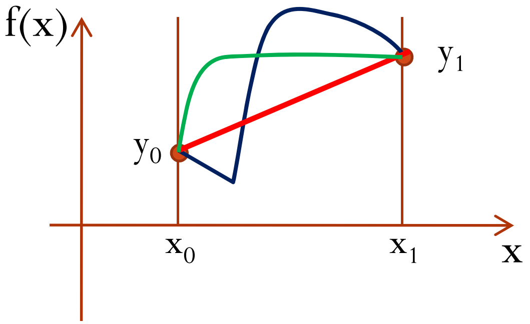


- Property of a harmonic function $f(x)$, (the red curve)
  - **Mean-value principle :**

    $$f(x) = \frac{1}{2\varepsilon}\int_{|y-x|<\varepsilon} f(y)dy, \forall x, y \in S$$

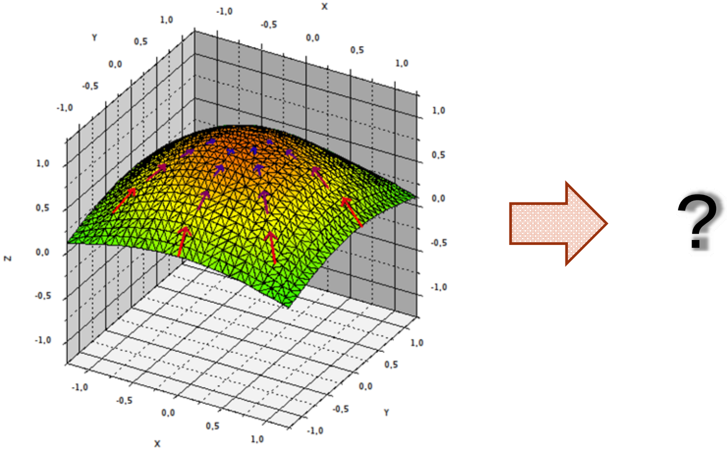
    function value on a point is the average of values of it surrounding points

    → we use this to numerically compute the function (later)
  - **Maximal principle :**

    Maximal/minimal function values only exist on the boundary

Flatten 3D Mesh by Harmonic Map

- Flatten a 2D variable function f(u,v) , similarly minimize the harmonic energy
$$E(f)=\frac{1}{2}\int_{(u,v)\in S}\|\nabla f\|^2 dudv$$
- It is equivalent to solving
$$\Delta f(u,v)=(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2})f(u,v)=0,\forall u,v\in S$$
- If the boundary conditions:
  (1) If $f(u,v)|_{\partial S}=C \rightarrow f(u,v)=C,\forall(u,v)\in S$
  (2) If $f(u,v)|_{\partial S}=\partial D \rightarrow f(u,v)\in D,\forall(u,v)\in S$



Mapping Mesh To Square

- A physical model:
  - Edges of the triangle mesh are springs (spring network)
  - Fix the boundary on the plane
  - Relax the interior of this network
  - Physical law being the only rule
  - Stabilized position → mapping for the interior vertices
- A mesh with n+b (interior: 1.. n, boundary: n+1…n+b) vertices:
  - The rest string length → 0
  - Potential energy → $(Ds^2)/2$, (D-constant, s-final string length)
  - Boundary vertices p_i → u_i (2d-vector u_i)
  - Minimize spring energy:

$$E = \frac{1}{2}\sum_{i=1}^{n+b}\sum_{j\in N_i}\frac{1}{2}D_{ij}||\boldsymbol{u}_i - \boldsymbol{u}_j||^2,$$

where $D_{ij} = D_{ji}$ is the spring constant of the spring between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$

Mesh Mapping (cont.)

- To find the minimized solution:

$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j \in N_i} D_{ij}(\boldsymbol{u}_i - \boldsymbol{u}_j) = 0 \quad \Longrightarrow \quad \sum_{j \in N_i} D_{ij}\boldsymbol{u}_i = \sum_{j \in N_i} D_{ij}\boldsymbol{u}_j$$

(for any interior vertex i=1…n)

- Remove boundary points from the left to right hand side:

$$\boldsymbol{u}_i - \sum_{\underline{j \in N_i, j \le n}} \lambda_{ij}\boldsymbol{u}_j = \sum_{\underline{j \in N_i, j > n}} \lambda_{ij}\boldsymbol{u}_j, \qquad \lambda_{ij} = \frac{D_{ij}}{\sum_{j \in N_i} D_{ij}}$$

- Lead to two sparse linear systems (in two axis directions):

$$AU = \bar{U} \quad \text{and} \quad AV = \bar{V},$$

$$\bar{u}_i = \sum_{j \in N_i, j > n} \lambda_{ij} u_j \quad \text{and} \quad \bar{v}_i = \sum_{j \in N_i, j > n} \lambda_{ij} v_j \tag{3}$$

$$A = (a_{ij})_{i,j=1,\ldots,n} \; : \; a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\lambda_{ij} & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Boundary Mapping

- No fold-over → direct projection may not work
- Flatten a curve:
  a) Choosing the shape of the planar domain boundary
  b) Choosing the distribution of the points on the boundary

a) Boundary Shape: Usually rectangle, circle, etc.
  - Convex shape → bijectivity guarantees for many weights
  - Larger distortion when surface is highly concave
  - Choose square here

b) Distribution: Usually uniform length, chord length, …
  - Uniform distribution: works for well (uniformly) sampled data
  - Chord length: working well in most cases

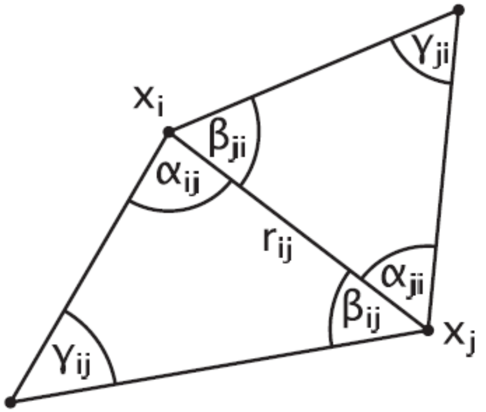
(2) Interior Mapping
– different weights

- Different D_{ij} :
  - Wachspress coordinates:
    - Earliest generalization of barycentric coordinates
    - Mainly used in finite element methods
  - Harmonic coordinates:
    - Standard piecewise linear approximation to Laplace equation
    - Minimizing deformation energy
  - Mean value coordinates:
    - Discretizing mean value theorem of harmonic function
    - Positive weights guaranteed, stable parameterization

$$w_{ij} = \frac{\cot \alpha_{ji} + \cot \beta_{ij}}{{r_{ij}}^2}$$

$$w_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji}$$

$$w_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}}$$

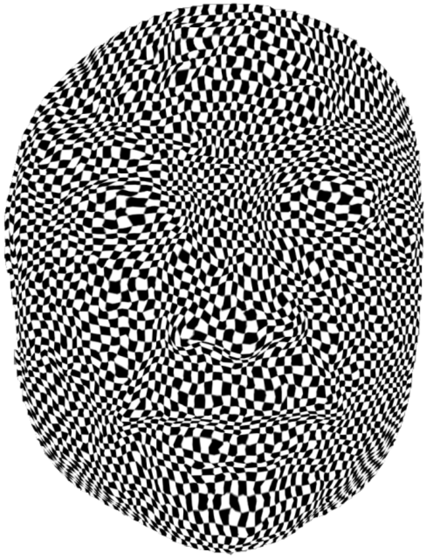


It has been proved that:
Any symmetric weights ($w_{ij}=w_{ji}$) minimizes a spring energy.

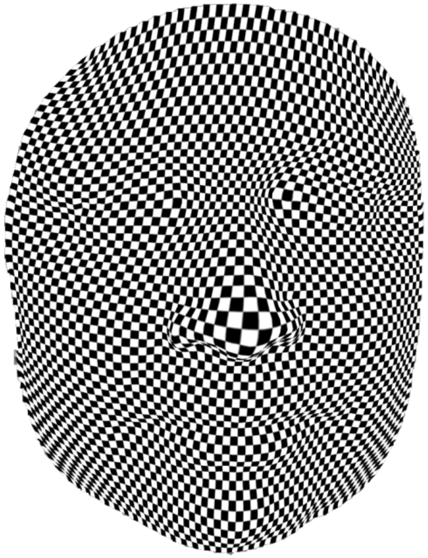
Three different popular formula

- Graph Embedding: [Tutte 1963]
- Discrete Harmonic Mapping: [Eck 1995]
- Meanvalue Coordinates: [Floater 1997]

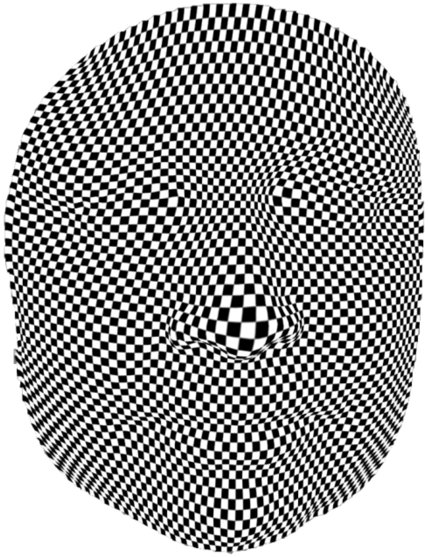
Susan Surface



Graph Embedding



Harmonic



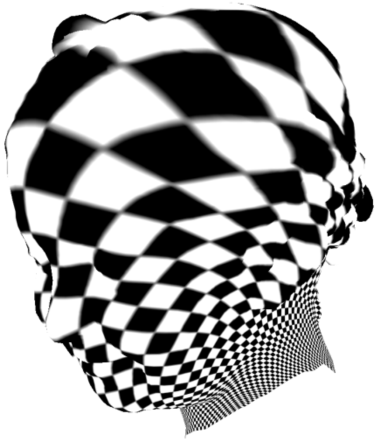
Mean Value

Three different popular formula

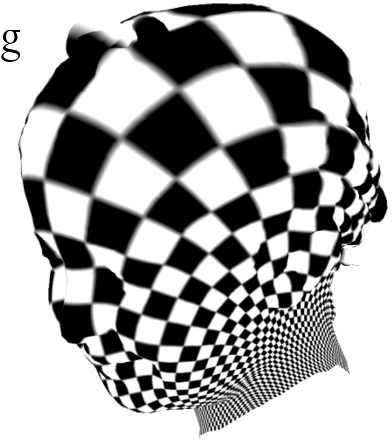
- On another surface:

Bimba Surface

Graph Embedding



Mean Value



Harmonic

❑Carefully Read & Understand Previous Slides
❑The following materials/slides are optional

- Visually, we can tell the difference.
- But how do we measure the distortion numerically? And where do these weight formula come from?
  - E.g. why the harmonic mapping looks conformal?
- How do we design (or choose to use) a mapping technique?
  - E.g. shall we always use harmonic?
- Purely Conformal or a Balance?
  - Applications needs angle-preserving
  - Applications that also needs area-preserving
- How about more general surfaces?
  - Closed Genus-0 surfaces → spherical mapping
  - Higher genus surfaces → global parameterization
  - Surface to surface → inter-surface mapping

Differential Geom. Background

- A surface $S \subset \mathbb{R}^3$ (2-manifold), has the parametric representation:

$$\mathbf{x}(u^1, u^2) = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2))$$

for points (u^1, u^2) in some domains in $\mathbb{R}^2$

- A representation is regular if
  - i. The functions x_1, x_2, x_3 are smooth (differentiable when we need)
  - ii. The vectors $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}$ are linearly independent

- 1st fundamental form (quadratic inner product on the tangent space) :

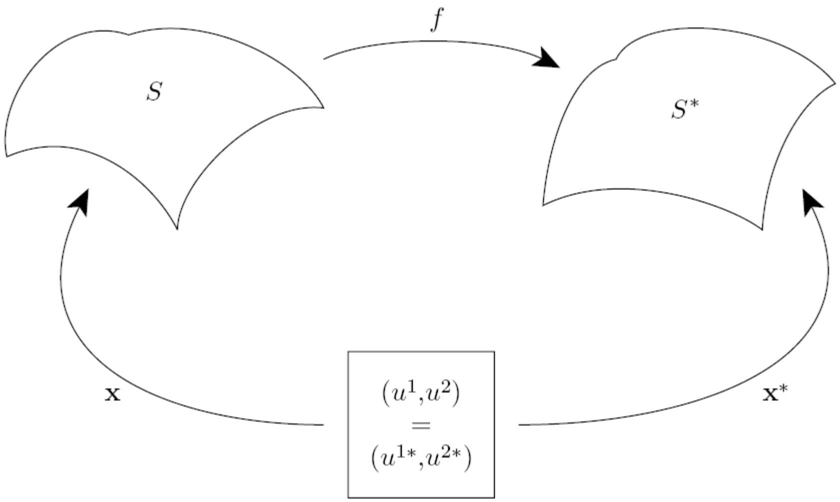
→ permits the calculation of surface metric

$$ds^2 = \mathbf{x}_1 \cdot \mathbf{x}_1 \, (du^1)^2 + 2\, \mathbf{x}_1 \cdot \mathbf{x}_2 \, du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 \, (du^2)^2$$

denoting $g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \qquad \alpha = 1, 2, \quad \beta = 1, 2,$

We have $ds^2 = (du^1 \; du^2)\, \mathbf{I} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix}$, where $\mathbf{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$

Differential Geom. Background (cont.)

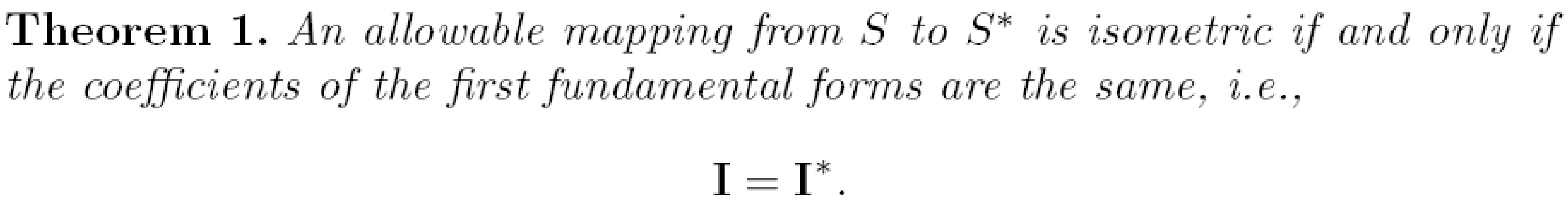


f is allowable if the parameterizations x and x* are both regular.

Isometric mappings

Isometric ⇔ length-preserving

(e.g. cylinder → plane (cylindrical coordinates → Cartesian coordinates))



Theorem 1. *An allowable mapping from S to S^* is isometric if and only if the coefficients of the first fundamental forms are the same, i.e.,*

$$\mathbf{I} = \mathbf{I}^*.$$

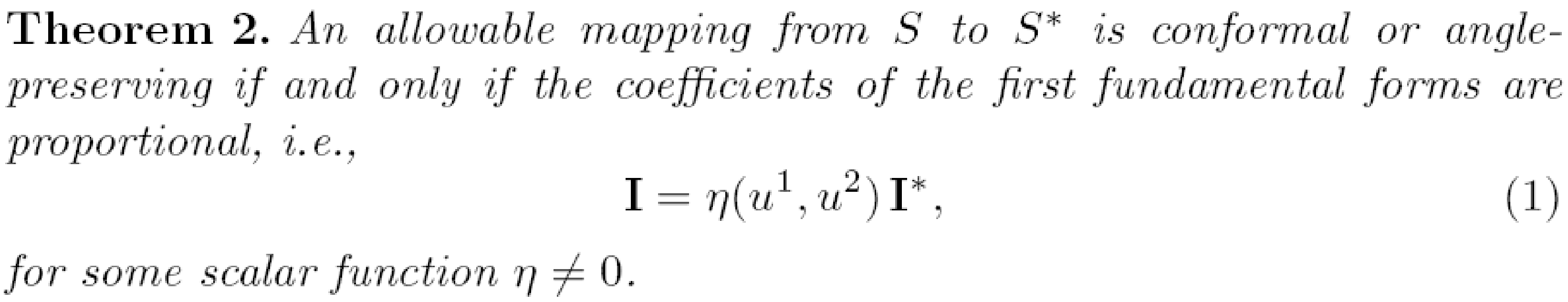
Under an isometry:

- Curve-lengths don't change
- Angles don't change
- Areas don't change
- Gaussian curvatures don't change

Conformal mappings

Conformal ⇔ angle-preserving

(e.g. stereographic and Mercator projections)



Theorem 2. *An allowable mapping from S to S^* is conformal or angle-preserving if and only if the coefficients of the first fundamental forms are proportional, i.e.,*

$$\mathbf{I} = \eta(u^1, u^2)\,\mathbf{I}^*, \tag{1}$$

for some scalar function $\eta \neq 0$.

Under an conformal map:

- Angles don’t change
- Circle → another circle (only scaling allowed)

Equiareal mappings

Equiareal ⇔ area-preserving

(e.g. Lambert projections)

Theorem 3. *An allowable mapping from S to S^* is equiareal if and only if the discriminants of the first fundamental forms are equal, i.e.,*

$$g = g^*. \tag{2}$$

(Note that: $g = \det \mathbf{I} = g_{11}g_{22} - g_{12}^2$)

Theorem 4. *Every isometric mapping is conformal and equiareal, and every conformal and equiareal mapping is isometric, i.e.,*

$$\textbf{isometric} \Leftrightarrow \textbf{conformal} + \textbf{equiareal}.$$

An example: planar mappings

A planar mapping is a special type of the surface mapping:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2,\ f(x,y) = (u(x,y), v(x,y))$$

its 1st fundamental form: $\mathbf{I} = J^T J$

where $J = \left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)$ is the Jacobian of f.

Proposition 1. *For a planar mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the following equivalencies hold:*

$$\begin{array}{lllllll}
1. & f \text{ is isometric} & \Leftrightarrow & \mathbf{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \Leftrightarrow & \lambda_1 = \lambda_2 = 1 \\
2. & f \text{ is conformal} & \Leftrightarrow & \mathbf{I} = \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta \end{smallmatrix}\right) & \Leftrightarrow & \lambda_1 / \lambda_2 = 1 \\
3. & f \text{ is equiareal} & \Leftrightarrow & \det \mathbf{I} = 1 & \Leftrightarrow & \lambda_1 \lambda_2 = 1
\end{array}$$

eigenvalues of **I**

Conformal→Harmonic

A conformal mapping

- a complex function satisfies the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Downarrow$$

$\Delta u = 0, \qquad \Delta v = 0,$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

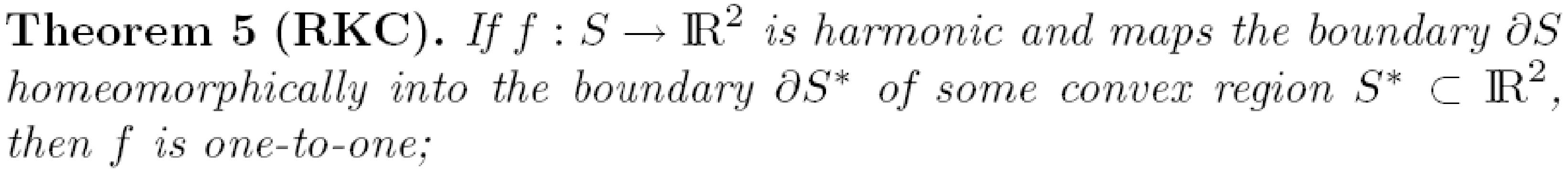
A harmonic mapping

- a complex function satisfies these two Laplace equations

Isometric → Conformal → Harmonic

Harmonic Mapping

- Easy to compute, easy to approximate
- Guaranteed existence (when suitable boundary mapping is provided)
- Minimizing deformation (minimizing the Dirichlet energy)



Theorem 5 (RKC). *If* $f : S \rightarrow \mathbb{R}^2$ *is harmonic and maps the boundary* ∂S *homeomorphically into the boundary* ∂S^* *of some convex region* $S^* \subset \mathbb{R}^2$, *then* f *is one-to-one;*

- Conformality depends on the boundary condition
- One-sidedness

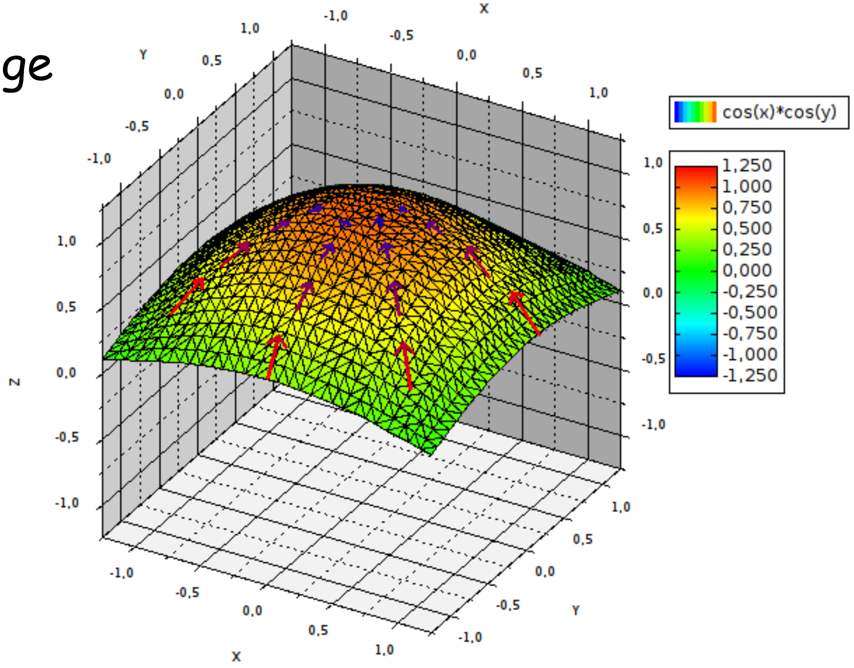
Harmonic Map & its Intuition

- Minimizing deformation

$$E_D(f) = \frac{1}{2}\int_S \|\mathrm{grad} f\|^2 = \frac{1}{2}\int_S \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)$$

-- minimize the magnitude of the change

- Intuitive explanation
  - 1D
  - 2D
  - 3D



Harmonic Map on Mesh

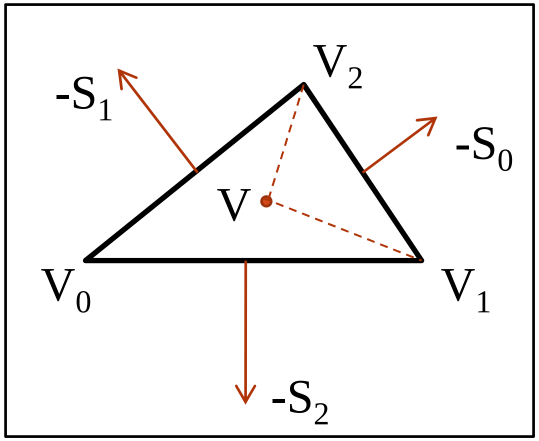
- Following the smooth case definition → discrete setting:

$$E(f) = \int_S \| \nabla f \|^2 \, ds = \sum_{\Delta \in F} < \nabla f_\Delta, \nabla f_\Delta > A_\Delta$$

- Look at one triangle (V_1, V_2, V_3):
  - Define: $S_i = \underline{n} \times (\underline{V_{i+2}} - \underline{V_{i+1}})$

    Normalized normal index mod 3
  - We have: $S_0 + S_1 + S_2 = n \times (V_2 - V_1 + V_0 - V_2 + V_1 - V_0) = 0$

    ➔ $< S_i, S_i > = < S_i, -\sum_{j \neq i} S_j > = -\sum_{j \neq i} < S_i, S_j >$



- An interior point V can be represented by barycentric coordinates:

$$V = \sum_i \lambda_i V_i , \quad \lambda_i = A_i / A \quad \text{and} \quad A_i = \frac{1}{2} | VV_{i+1} \| V_{i+1} V_{i+2} | \sin(\angle V V_{i+1} V_{i+2}) = < -S_i, V_{i+1} - V >$$

Linear function: $f(V) = \sum_i f(\lambda_i V_i) = \sum_i \lambda_i f(V_i) = \sum_i \frac{f(V_i)}{2A} < S_i, V > \underline{- \sum_i \frac{f(V_i)}{2A} < S_i, V_{i+1} >}$

$$\nabla f(V) = \sum_i \frac{1}{2A} f_i S_i \quad , f_i \leftarrow f(V_i)$$

Harmonic Map on Mesh (cont.)

- The local energy: $<\nabla f_\Delta, \nabla f_\Delta > A = \frac{1}{4A} < \sum_i f_i S_i, \sum_j f_j S_j >$

$$= \frac{1}{4A}(\sum_i f_i^2 < S_i, S_i > + 2\sum_{i<j} f_i f_j < S_i, S_j >)$$

(because $< S_i, S_i >= -\sum_{j \neq i} < S_i, S_j >$)

$$= \frac{1}{4A}(-f_0^2(< S_0, S_1 > + < S_0, S_2 >)\ldots + 2\sum_{i<j} f_i f_j < S_i, S_j >)$$

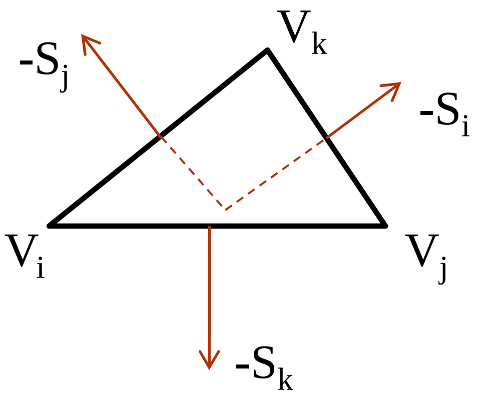
$$= \frac{-1}{4A}((f_0 - f_1)^2 < S_0, S_1 > + \ldots)$$

$$= \frac{-1}{4A}\sum_{i<j}(f_i - f_j)^2 < S_i, S_j >$$

Therefore : $E_\Delta(f) = \frac{1}{2}\sum_{i<j} w_{ij}(f_j - f_i)^2$

where $w_{ij} = -\frac{< S_i, S_j >}{2A}$

$$= -\frac{e_i e_j \cos(\pi - \theta_k)}{e_i e_j \sin\theta_k} = ctg(\theta_k)$$



Harmonic Map on Mesh (cont.)

- Total discrete harmonic energy:

$$E(f) = \frac{1}{2} \sum_{halfedge(i,j)} w_{ij} (f_j - f_i)^2$$

- It is minimized when

$$\frac{\partial E(f)}{\partial f_i} = \sum_{halfedge(i,j)} w_{ij} (f_j - f_i) = 0$$

$$\boxed{f_i = \frac{\sum (ctg\theta_{ij} + ctg\theta_{ji}) f_j}{\sum (ctg\theta_{ij} + ctg\theta_{ji})}}$$

Cotangent Weights of Discrete Harmonic Map

Mean Value Coordinates

- ❑ A problem of the cotangent weight

$$w_{ij} = \cot\gamma_{ij} + \cot\gamma_{ji}$$

Need remeshing? or

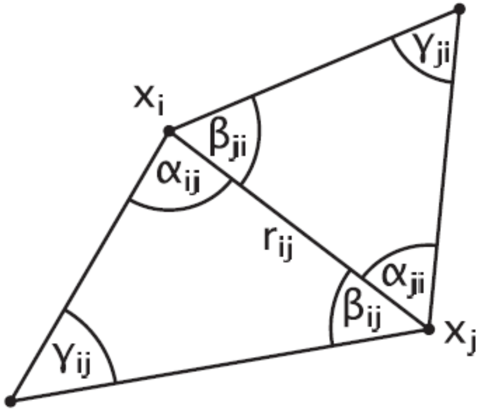
- ❑ Weights with “barycentric” property:

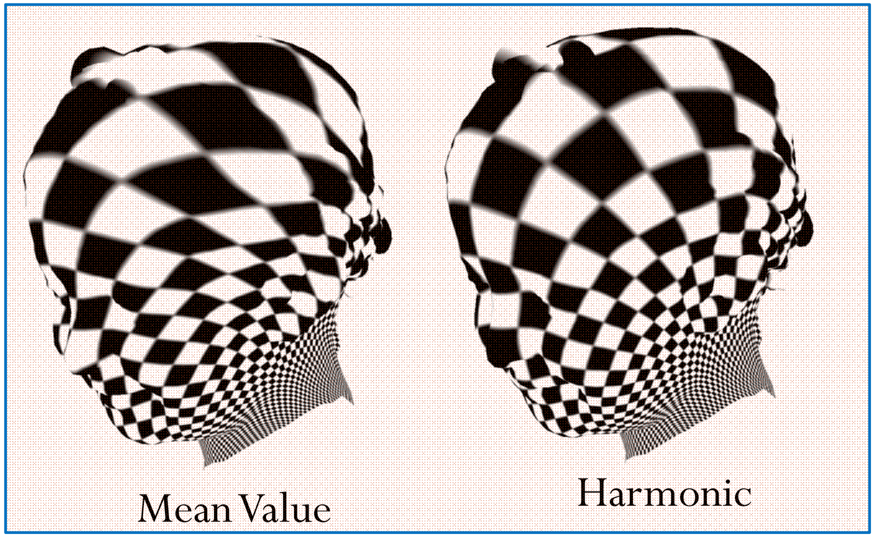
$$\begin{cases} V = \sum \lambda_i V_i \\ \sum \lambda_i = 1, \forall \lambda_i > 0 \end{cases}$$

Using Mean Value Property of the Harmonic Function

Mean Value Weights

$$w_{ij} = \frac{\tan\frac{\alpha_{ij}}{2} + \tan\frac{\beta_{ji}}{2}}{r_{ij}}$$





The definition review

- Simply connected domain $\Omega \subset \mathbb{R}^2$

$$\text{the } \textit{unit square}: \quad \Omega = \{(u,v) \in \mathbb{R}^2 : u,v \in [0,1]\}, \quad \text{or}$$
$$\text{the } \textit{unit disk}: \quad \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\},$$

- A continuous injection (no 2 distinct points → same point) $f : \Omega \to \mathbb{R}^3$
- The image S of Ω under f ➔ a surface

$$S = f(\Omega) = \{f(u,v) : (u,v) \in \Omega\},$$

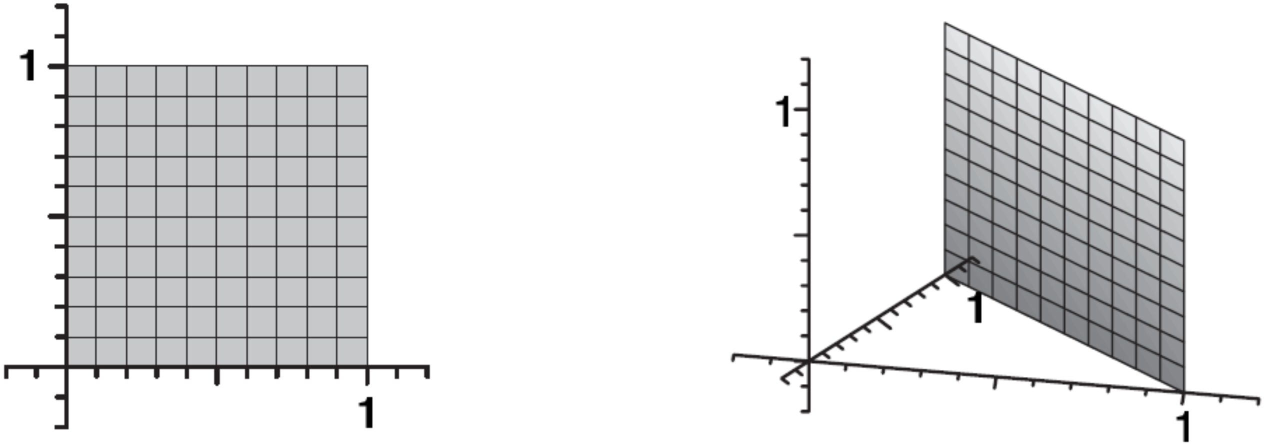
f is a parameterization of S over the parameter domain Ω

- →f is a bijection between Ω and S → $f^{-1} : S \to \Omega$

Surface Examples (1)

- Simple linear function:

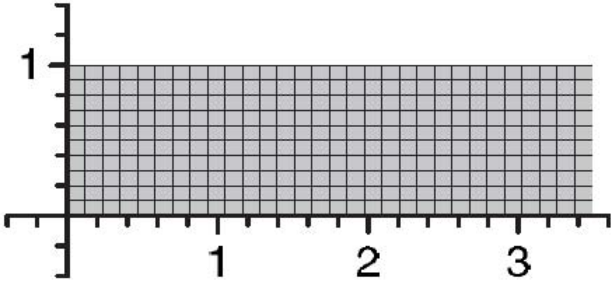
$$\begin{aligned} \textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u,v \in [0,1]\} \\ \textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \in [0,1], x+y=1\} \\ \textit{parameterization:} \quad & f(u,v) = (u, 1-u, v) \\ \textit{inverse:} \quad & f^{-1}(x,y,z) = (x,z) \end{aligned}$$

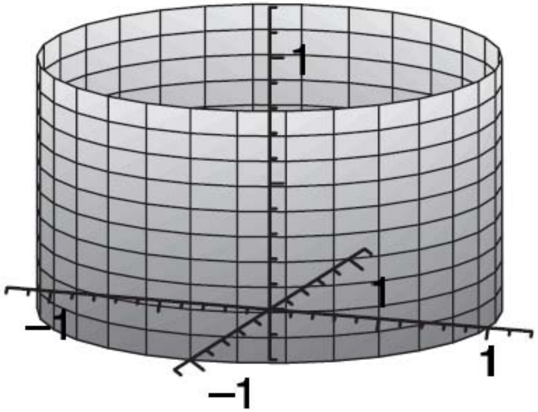


Surface Examples (2)

- Cylinder:

$$
\begin{aligned}
\textit{parameter domain:} \quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u \in [0, 2\pi), v \in [0,1]\} \\
\textit{surface:} \quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0,1]\} \\
\textit{parameterization:} \quad & f(u,v) = (\cos u, \sin u, v) \\
\textit{inverse:} \quad & f^{-1}(x,y,z) = (\arccos x, z)
\end{aligned}
$$

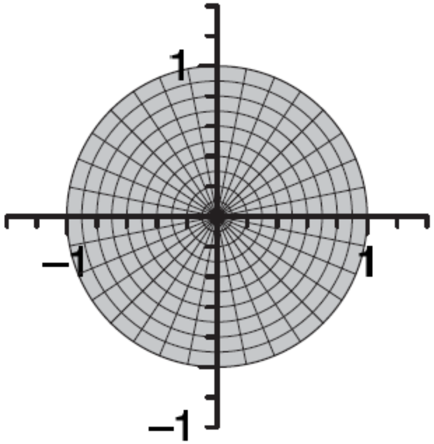


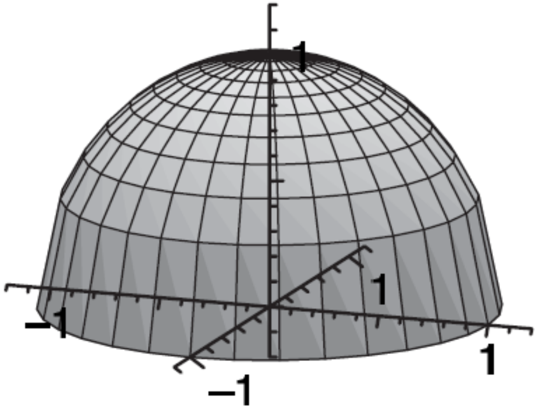


Surface Examples (3)

- Hemisphere (orthographic definition) :

$$\begin{aligned} \textit{parameter domain:}\quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\ \textit{surface:}\quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\ \textit{parameterization:}\quad & f(u,v) = (u, v, \sqrt{1 - u^2 - v^2}) \\ \textit{inverse:}\quad & f^{-1}(x,y,z) = (x,y) \end{aligned}$$

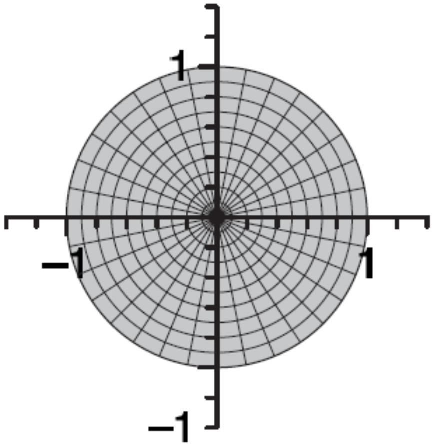


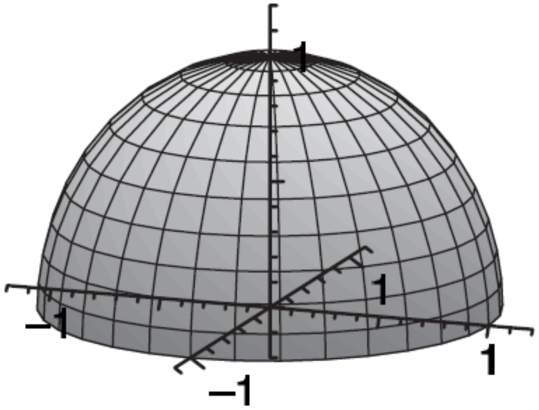


Surface Examples (4)

- Hemisphere (stereographic definition) :

$$
\begin{aligned}
\textit{parameter domain:}\quad & \Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \\
\textit{surface:}\quad & S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\
\textit{parameterization:}\quad & f(u,v) = (\tfrac{2u}{1+u^2+v^2}, \tfrac{2v}{1+u^2+v^2}, \tfrac{1-u^2-v^2}{1+u^2+v^2}) \\
\textit{inverse:}\quad & f^{-1}(x,y,z) = (\tfrac{x}{1+z}, \tfrac{y}{1+z})
\end{aligned}
$$





Reparameterization

- Example (3) and (4):

  → There can be more than one parameterizations of S over Ω

- Any bijection $\varphi : \Omega \to \Omega$

  induces a reparameterization: $g = f \circ \varphi$

- Exercise: write the reparameterization $\varphi(u, v)$ between (3) and (4)

(3) $f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$

⬇

$\varphi(u, v)$ =?

(4) $f(u, v) = (\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2})$

- Surface Mapping Optimization Procedure = Reparameterization Procedure

Intrinsic Surface Properties

- Intrinsic and extrinsic
  - → intrinsic: about the shape itself, not about its representation and location
- Intrinsic property examples: curvature (Gaussian, mean), normal
- Tangent Plane: spanned by $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$
- Surface Normal: $n_f = \frac{f_u \times f_v}{\|f_u \times f_v\|}$
- Example (orthographic hemisphere):

$$f(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

$$f_u(u,v) = (1,0,\frac{-u}{\sqrt{1-u^2-v^2}})$$

$$f_v(u,v) = (0,1,\frac{-v}{\sqrt{1-u^2-v^2}})$$

$$n_f(u,v) = (u, v, \sqrt{1-u^2-v^2}) = (x, y, z)$$

(same with stereographics)

→ Following our intuition: normal is independent of the parameterization

(intrinsic property)

1st Fundamental Form and Surface Area

- Area of a surface is intrinsic too
- The first fundamental form $\mathbf{I}_f = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$
- Area element: $dA = | f_u \times f_v | dudv = \sqrt{(f_u \cdot f_u)(f_v \cdot f_v) - (f_u \cdot f_v)^2} dudv = \sqrt{EG - F^2} dudv$
- Example: Area of a unit hemisphere (orthographic parameterization)

$$f(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

$$EG - F^2 = \frac{1}{1 - u^2 - v^2}$$

$$\begin{aligned} A(S) &= \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{1}{\sqrt{1 - u^2 - v^2}} du\, dv \\ &= \int_{-1}^{1} \left[\arcsin \frac{u}{\sqrt{1 - v^2}} \right]_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} dv \\ &= \int_{-1}^{1} \pi\, dv \\ &= 2\pi, \end{aligned}$$

- Exercise:

Area under stereographic parameterization

- Intrinsic property: Area is independent of the parameterization

Metric Distortion

- Look at surface point f(u,v), move a little away from (u,v):

  Displacement: $(\Delta u, \Delta v)$ → new point: $f(u + \Delta u, v + \Delta v)$

  approximated by 1st order Taylor expansion:

$$\tilde{f}(u + \Delta u, v + \Delta v) = f(u, v) + f_u(u, v)\Delta u + f_v(u, v)\Delta v$$

Planar local region: the vicinity of $\boldsymbol{u} = (u, v)$

↓

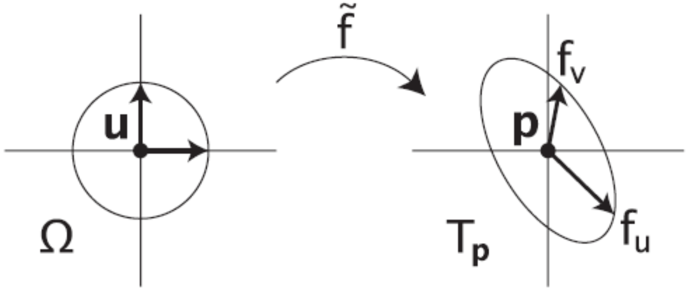
Region on tangent plane T_p at $p = f(u, v) \in S$

Circles around **u**

↓

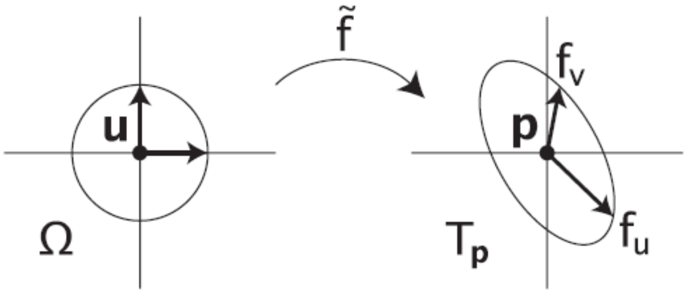
ellipses around **p**

$$\tilde{f}(u + \Delta u, v + \Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v} \quad \text{where } J_f = (f_u \;\; f_v) \text{ is the } \mathit{Jacobian} \text{ of } f$$



Metric Distortion (cont.)

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v}$$



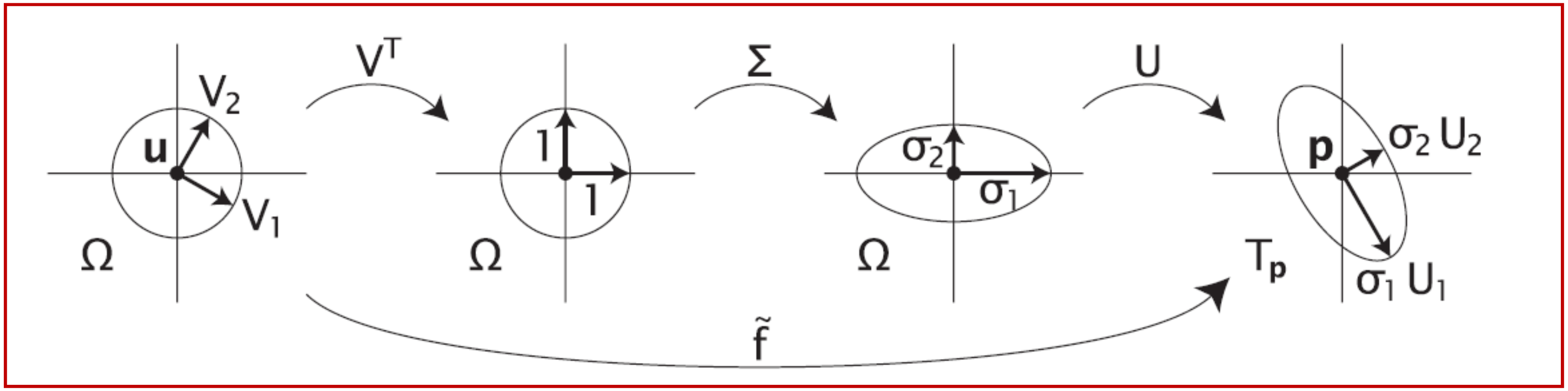
Decompose the Jacobian (3*2) matrix by SVD:

$$J_f = U\Sigma V^T = U\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

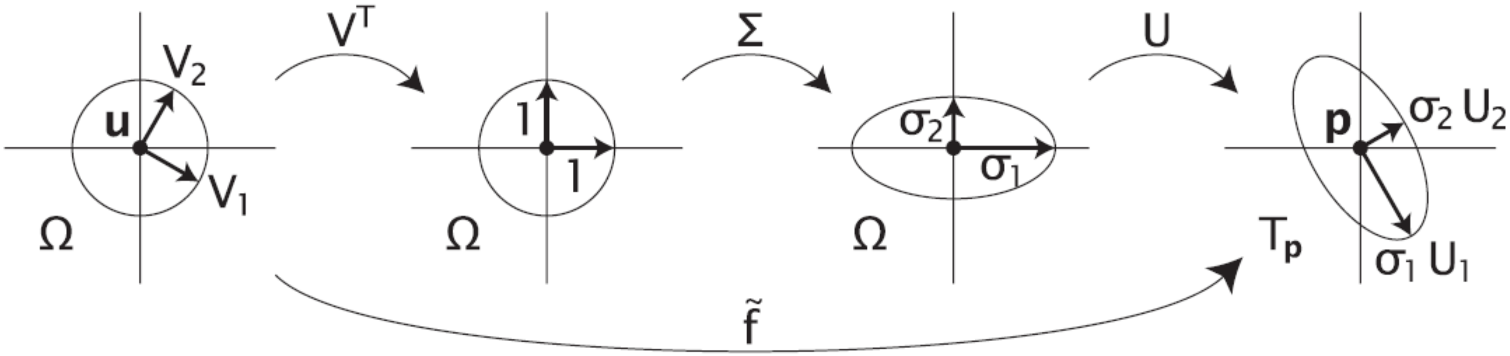
unitary, orthonormal $U \in \mathbb{R}^{3\times 3}$

$V \in \mathbb{R}^{2\times 2}$

$singular\ values\ \sigma_1 \geq \sigma_2 > 0$



Metric Distortion (cont.)



$$J_f = U\Sigma V^T = U\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T$$

---

(1) 2D Rotation V → planar rotation around **u**;

(2) Stretching matrix $\sum$ → stretches by factor σ_1 and σ_2 in the u and v directions;

(3) 3D rotation U → map the planar region onto the tangent plane

Tiny sphere with radius-r → ellipse with semi-axes of length $r\sigma_1$ and $r\sigma_2$

$\sigma_1 = \sigma_2 \longrightarrow$ Local scaling, circles to circles : **Confomal**

$\sigma_1\sigma_2 = 1 \longrightarrow$ Area preserved : **Equiareal**

Metric Distortion (cont.)

Singular values of any matrix A are the square roots of the eigenvalues of the matrix A^TA

Look at ${J_f}^T J_f$

$$ {J_f}^T J_f = \begin{pmatrix} {f_u}^T \\ {f_v}^T \end{pmatrix} (f_u \; f_v) = \mathbf{I}_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix} $$

The symmetric 2*2 matrix's eigenvalues:

$$ \lambda_{1,2} = \frac{1}{2}\left((E+G) \pm \sqrt{4F^2 + (E-G)^2}\right) $$

$$ \begin{aligned} f \text{ is } \textit{isometric} \text{ or } \textit{length-preserving} \quad &\Longleftrightarrow \quad \sigma_1 = \sigma_2 = 1 \quad \Longleftrightarrow \quad \lambda_1 = \lambda_2 = 1, \\ f \text{ is } \textit{conformal} \text{ or } \textit{angle-preserving} \quad &\Longleftrightarrow \quad \sigma_1 = \sigma_2 \quad \Longleftrightarrow \quad \lambda_1 = \lambda_2, \\ f \text{ is } \textit{equiareal} \text{ or } \textit{area-preserving} \quad &\Longleftrightarrow \quad \sigma_1 \sigma_2 = 1 \quad \Longleftrightarrow \quad \lambda_1 \lambda_2 = 1. \end{aligned} $$

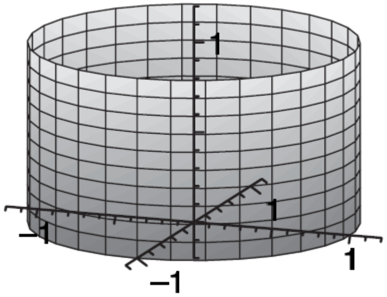
$$ \text{isometric} \quad \Longleftrightarrow \quad \text{conformal} + \text{equiareal} $$

Metric Distortion Example

(1) Cylinder

- *parameterization:* $f(u,v) = (\cos u, \sin u, v)$
- *Jacobian:* $J_f = \begin{pmatrix} \cos u & 0 \\ -\sin u & 0 \\ 0 & 1 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = 1$

Isometry

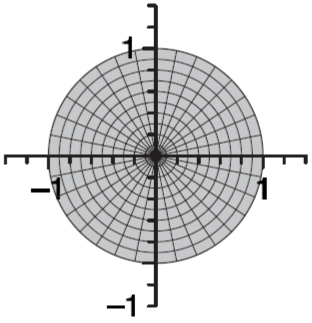


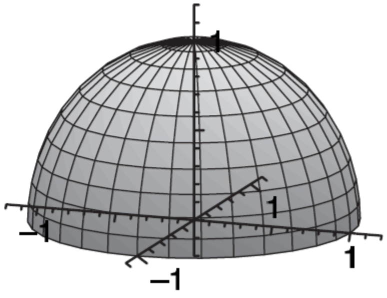
Metric Distortion Example

(2) Hemisphere (stereographic)

- *parameterization:* $f(u,v) = (2ud, 2vd, (1-u^2-v^2)d)$ where $d = \frac{1}{1+u^2+v^2}$
- *Jacobian:* $J_f = \begin{pmatrix} 2d-4u^2d^2 & -4uvd^2 \\ -4uvd^2 & 2d-4v^2d^2 \\ -4ud^2 & -4vd^2 \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 4d^2 & 0 \\ 0 & 4d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 4d^2, \quad \lambda_2 = 4d^2$

Conformal



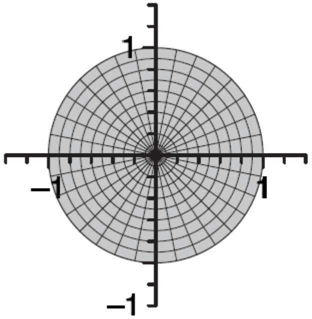


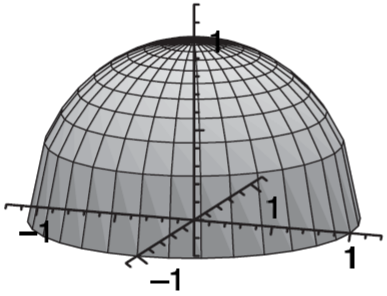
Metric Distortion Example

(3) Hemisphere (orthographic)

- *parameterization:* $f(u,v) = (u, v, \frac{1}{d})$ where $d = \frac{1}{\sqrt{1-u^2-v^2}}$
- *Jacobian:* $J_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -ud & -vd \end{pmatrix}$
- *first fundamental form:* $\mathbf{I}_f = \begin{pmatrix} 1+u^2d^2 & uvd^2 \\ uvd^2 & 1+v^2d^2 \end{pmatrix}$
- *eigenvalues:* $\lambda_1 = 1, \quad \lambda_2 = d^2$

Not conformal, not equiareal





Minimizing Metric Distortion

Overall distortion of a parameterization f can be generally defined by:

$$\bar{E}(f) = \int_{\Omega} E(\sigma_1(u,v), \sigma_2(u,v))\, du\, dv \Big/ A(\Omega)$$

Minimizing $\bar{E}(f)$ over the space of all admissible parameterizations → best parameterization

Discretely, we look at linear function f:

from parameter triangles $t \in \Omega$ to surface triangles $T \in \mathcal{T}$

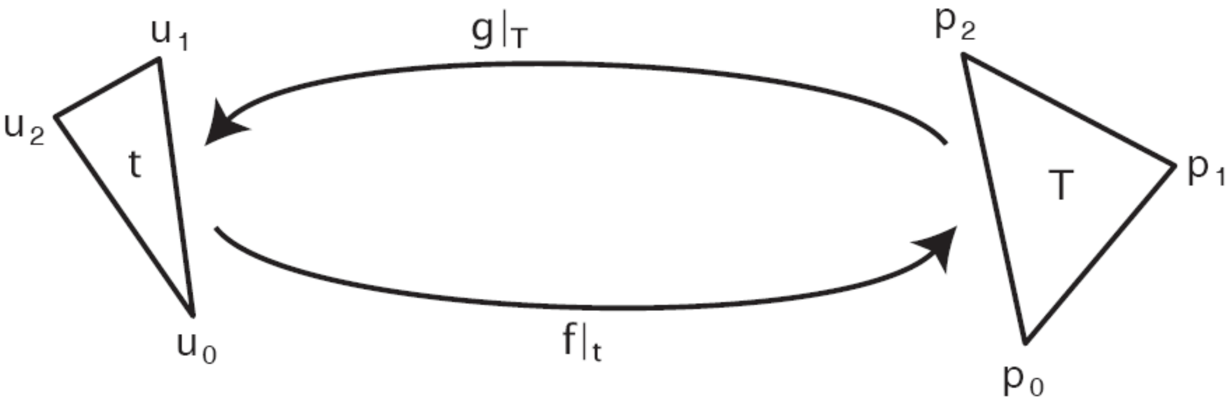
$$\bar{E}(f) = \sum_{t \in \Omega} E(\sigma_1^t, \sigma_2^t) A(t) \Big/ \sum_{t \in \Omega} A(t)$$

Or we can look at inverse function $g=f^{-1}$:

$$\sigma_1^T = 1/\sigma_2^t \text{ and } \sigma_2^T = 1/\sigma_1^t$$

$$\bar{E}(g) = \sum_{T \in \mathcal{T}} E(\sigma_1^T, \sigma_2^T) A(T) \Big/ \sum_{T \in \mathcal{T}} A(T)$$

Minimizing Metric Distortion (cont.)



$$A(t) = \frac{1}{2}\det(\boldsymbol{u}_1 - \boldsymbol{u}_0, \boldsymbol{u}_2 - \boldsymbol{u}_0) \qquad A(T) = \frac{1}{2}\|(\boldsymbol{p}_1 - \boldsymbol{p}_0) \times (\boldsymbol{p}_2 - \boldsymbol{p}_0)\|$$

$$(\sigma_1^t)^2 + (\sigma_2^t)^2 = \frac{1}{A(t)^2}\sum_{i=0}^{2}\|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2\left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^t \sigma_2^t = \frac{A(T)}{A(t)}$$

$$(\sigma_1^T)^2 + (\sigma_2^T)^2 = \frac{1}{A(T)^2}\sum_{i=0}^{2}\|\boldsymbol{u}_{i+2} - \boldsymbol{u}_{i+1}\|^2\left[(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{p}_{i+2} - \boldsymbol{p}_i)\right]$$

$$\sigma_1^T \sigma_2^T = \frac{A(t)}{A(T)},$$

Minimizing Metric Distortion (cont.)

Discrete Harmonic Map
[Pinkall EM'93] [Eck SIG'95]:

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) = \tfrac{1}{2}({\sigma_1}^2 + {\sigma_2}^2)$$

Least Square Conformal Map
[Desbrun SIG'02] [Levy SIG'02]:

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \tfrac{1}{2}(\sigma_1 - \sigma_2)^2$$

$$E_{\mathrm{D}}(\sigma_1, \sigma_2) - E_{\mathrm{C}}(\sigma_1, \sigma_2) = \sigma_1 \sigma_2$$

→

$$\bar{E}_{\mathrm{D}}(g) - \bar{E}_{\mathrm{C}}(g) = \sum_{t \in \Omega} A(t) \Big/ \sum_{T \in \mathcal{T}} A(T) = \frac{A(\Omega)}{A(S_{\mathcal{T}})}$$

Therefore, if we take a conformal map, fix its boundary and thus the area of the parameter domain Ω, and then compute the harmonic map with this boundary, then we get the same mapping, which illustrates the well-known fact that any conformal mapping is harmonic, too.

Minimizing Metric Distortion (cont.)

Conformal Mapping: → try to make $\sigma_1 = \sigma_2$

$$E_{\mathrm{C}}(\sigma_1, \sigma_2) = \tfrac{1}{2}(\sigma_1 - \sigma_2)^2 \qquad E_{\mathrm{D}}(\sigma_1, \sigma_2) = \tfrac{1}{2}({\sigma_1}^2 + {\sigma_2}^2)$$

Another one: MIPS energy [Hormann 02]

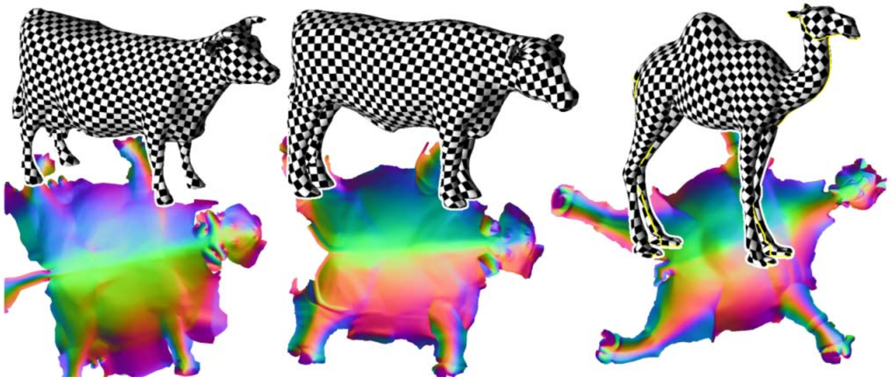
$$E_{\mathrm{M}}(\sigma_1, \sigma_2) = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \frac{{\sigma_1}^2 + {\sigma_2}^2}{\sigma_1 \sigma_2}$$

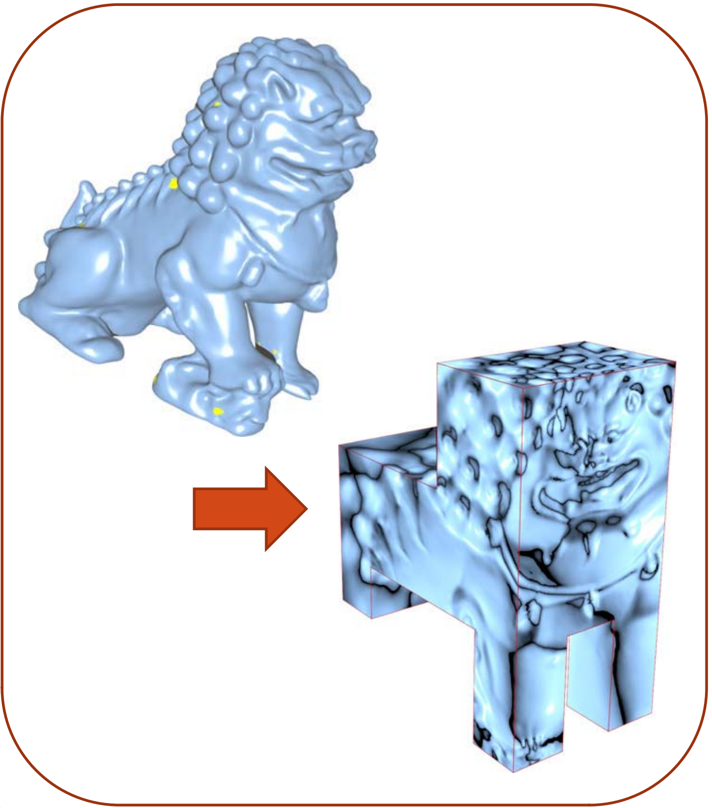
- Advantage: (1) symmetry: $E_{\mathrm{M}}(\sigma_1^T, \sigma_2^T) = E_{\mathrm{M}}(\sigma_1^t, \sigma_2^t)$

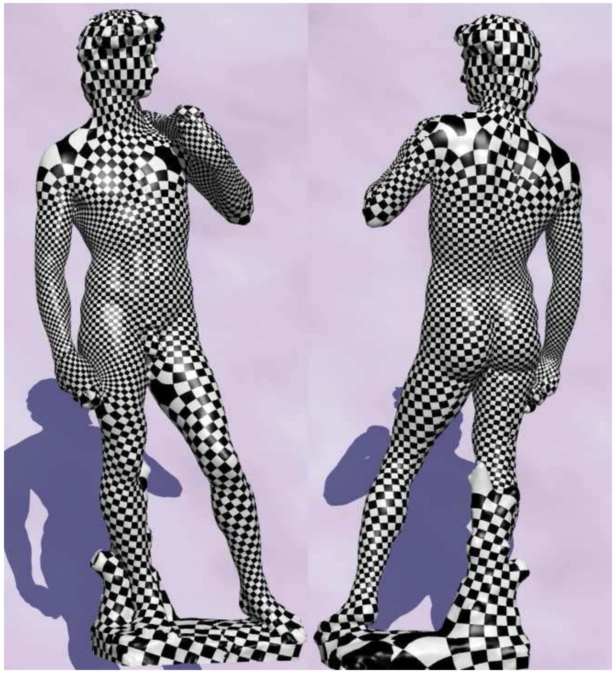
  (2) bijectivity
- Disadvantage: non-linear

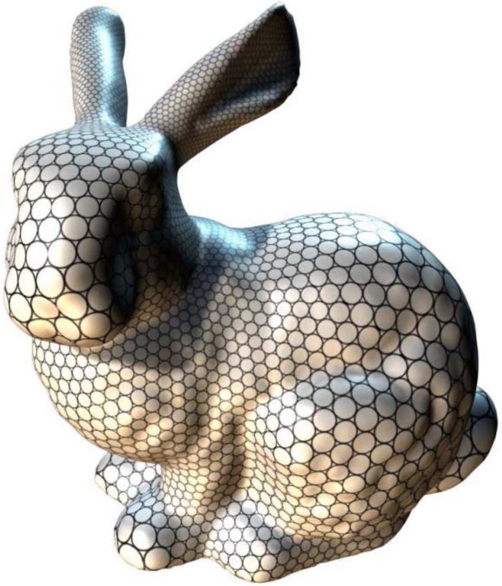
Many more about mapping…

- Free-boundary mapping
- Deforming the metric
- Global parameterization
- Inter-shape mapping









Application on meshing

With the parameterization, we can do

Remeshing – to generate high quality mesh