Lecture 6 – Geometry (1)
Curves and Surfaces

Xin (Shane) Li

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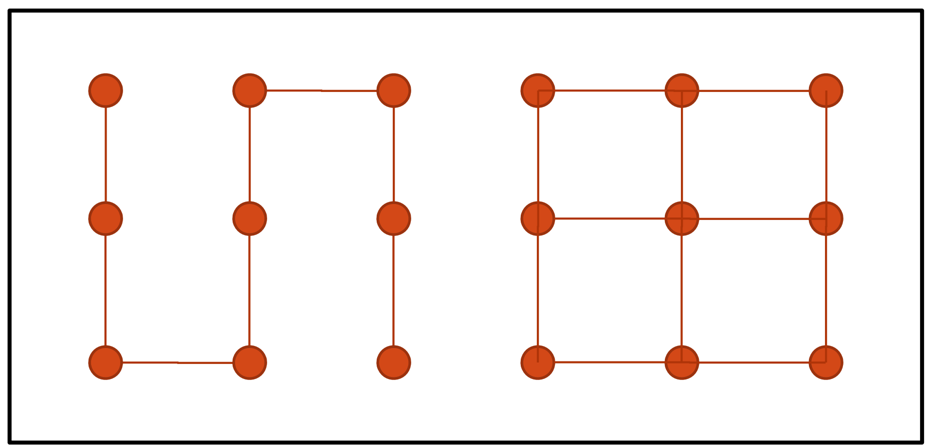
Outline

- What are curves, surfaces, and solids?
- Continuous Case
  - Topology, Topological Space, Open Set
  - Topological Equivalence
  - Curves, Surfaces, Solids…
- Discrete Case
  - Simplicial Complex
  - Neighborhoods in the Discrete Case
- Topological Classification of Surfaces

What is it about?

- Key to understand and manipulate modeling, processing, and rendering shapes is to understand geometric shapes.
- Curves, surfaces, and solid shapes are three common concepts in our surrounding world.
- Curves can be in R^1, R^2, and R^3, surfaces can be in R^2 and R^3, what is the fundamental difference between curves and surfaces? Why do we keep saying a curve is "1D" and a surface is "2D"…

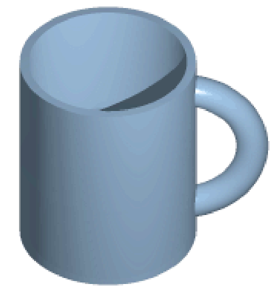
➔ Topology



Topological Space

- Topological space = a set X + some structural relationship T defined on X
- Open Set: a subset U of X, if it belongs to T
- (X, T) is a topological space, we usually say: X is a topological space with topology T
- Neighborhoods of a point p in X
- Given a point set, different topologies can be defined

Topological Space → Curves, Surfaces…



- Homeomorphism
  - Geometric Intuition: topologically equivalent = geometrically stretching and bending an object continuously
  - Example
    - (1,2) and (3,5)
    - Continuous Deforming (Translation, Scaling,…) are homeomorphism.
- Topological space → "Analytically good" topological space
  Hausdorff space: if different points have disjoint neighborhoods
- A topological Hausdorff space is <u>a surface (2D-manifold)</u>
  if each point has a neighborhood homeomorphic to an open set of Euclidean space R^2 (*)
- Similarly, <u>curves (1D-manifolds), solids (3D-manifolds), …</u> can be defined

From Discrete Aspects

- A surface → represented by a triangular mesh [Simplicial Complex]
- Simplexes
  - 0-simplex (vertex): $v \in \mathbb{R}^n$.
  - 1-simplex (edge): $(v_0, v_1) = \{\sum_{i=0}^{1} \lambda_i v_i : \sum_{i=0}^{1} \lambda_i = 1, \lambda_i \geq 0\}$
  - 2-simplex (face): $(v_0, v_1, v_2) = \{\sum_{i=0}^{2} \lambda_i v_i : \sum_{i=0}^{2} \lambda_i = 1, \lambda_i \geq 0\}$

λs (Barycentric Coordinates):

$$\begin{array}{rcl} \text{interior of a simplex} & \rightarrow & \text{all } \lambda_i > 0; \\ \text{boundary of a simplex} & \rightarrow & \exists \lambda_i = 0. \end{array}$$

A simplex is the smallest convex set containing its vertices.

Barycentric Coordinates

- Why Barycentric Coordinates are so defined uniquely?
  - From $v = \sum_{i=1}^{3} \lambda_i v_i$ and $\sum_{i=1}^{3} \lambda_i = 1, \quad \lambda_1, \lambda_2, \lambda_3 > 0$

  We have: $\lambda_3 = 1 - \lambda_1 - \lambda_2$

  Without loosing generality,
  we can defined a local 2D coordinate system on (v_1, v_2, v_3) :

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + (1 - \lambda_1 - \lambda_2) x_3 \\ y = \lambda_1 y_1 + \lambda_2 y_2 + (1 - \lambda_1 - \lambda_2) y_3 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 (x_1 - x_3) + \lambda_2 (x_2 - x_3) + (x_3 - x) = 0 \\ \lambda_1 (y_1 - y_3) + \lambda_2 (y_2 - y_3) + (y_3 - y) = 0 \end{cases} \quad \text{i.e.} \quad \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}$$

Barycentric Coordinates (cont.)

$$\ldots \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}$$

now $\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$

Since (r_1-r_3) and (r_2-r_3) are linear independent

has inversed matrix:

$$\frac{1}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)} \begin{bmatrix} y_2 - y_3 & -(y_1 - y_3) \\ -(x_2 - x_3) & x_1 - x_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)} \begin{bmatrix} y_2 - y_3 & -(x_2 - x_3) \\ -(y_1 - y_3) & x_1 - x_3 \end{bmatrix} \begin{bmatrix} x - x_3 \\ y - y_3 \end{bmatrix}$$

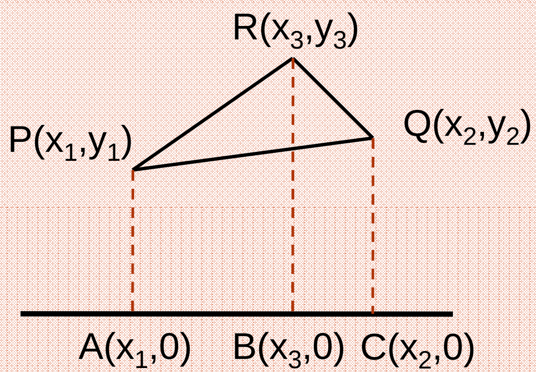
$$\lambda_1 = \frac{(x - x_3)(y_2 - y_3) - (x_2 - x_3)(y - y_3)}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)}$$

Barycentric Coordinates (cont.)

The denominator:

$$(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) = \boxed{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}$$

Twice the signed area of triangle $((x_1,y_1), (x_2, y_2), (x_3,y_3))$



Area(PQR)=Area(PABR)+Area(RBCQ)-Area(PACQ)

$\text{Area(PABR)}=(x_3-x_1)(y_1+y_3)/2$

$\text{Area(RBCQ)}=(x_2-x_3)(y_2+y_3)/2$

$\text{Area(PACQ)}=(x_2-x_1)(y_1+y_2)/2$

➔ $\text{Area(PQR)}=(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))/2$

Similarly, the numerator $(x - x_3)(y_2 - y_3) - (x_2 - x_3)(y - y_3)$

is twice the signed area of triangle $((x,y), (x_2, y_2), (x_3,y_3))$

➔ $\lambda_1 = \text{Area}(v,v_2,v_3)/\text{Area}(v_1,v_2,v_3)$

Simplicial Complex

- A simplicial complex is a finite set K of simplexes, satisfying the intersection condition:

$$\text{if } s, t \in K \quad \text{then } s \cap t = \begin{cases} \emptyset \\ \text{a simplex both of } s \text{ and of } t \end{cases}$$

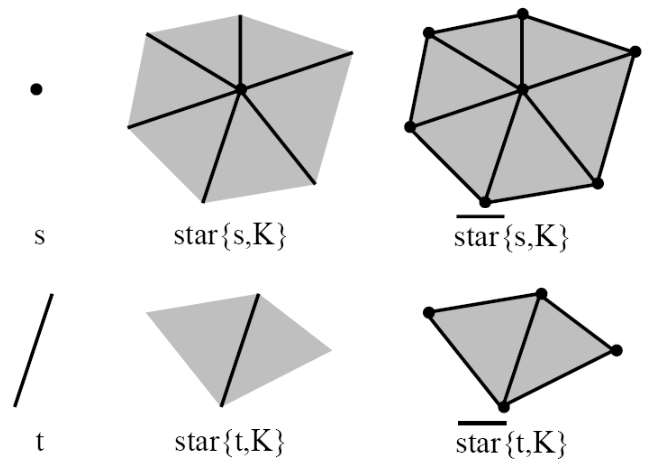
- Neighborhoods in a simplicial complex:

For simplex $s \in K$

$$\mathrm{star}(s;K) = \{s' \in K \ : \ s \le s'\}$$

$$\overline{\mathrm{star}}(s;K) = \{s' \in K \ : \ \exists t \in \mathrm{star}(s;K) \ : \ s' \le t\}$$

$$\mathrm{link}(s;K) = \overline{\mathrm{star}}(s;K) \setminus \mathrm{star}(s;K).$$



Continuous VS Discrete

	Continuous Def.	Discrete Def.
Closed Surface:	Each point's neighborhood is homeomorphic to 2D plane	Each vertex's "star-bar" simplicial complex is a simple closed polygon
Open Surface:	Boundary point's neighborhood is homeomorphic to 2D half-plane	Boundary vertex's "star-bar" simplicial complex is a simple closed polygon, with this vertex on the boundary

Similarly, for 1D-, 3D-, … nD- manifolds…

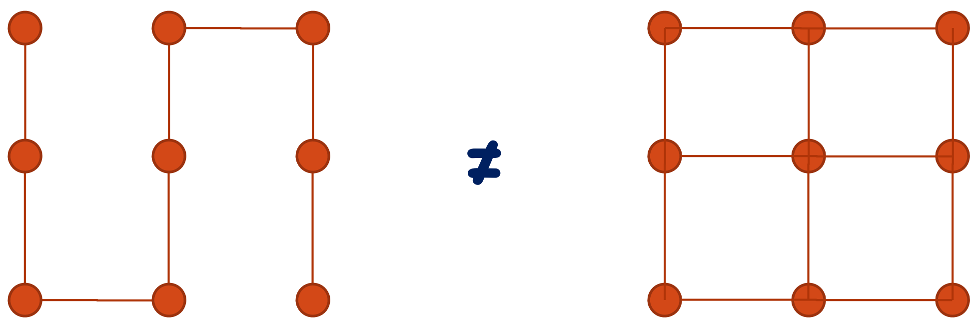
Curves, Surfaces, Solids…

Surfaces
= the space in which every point has a neighborhood ~ an open disk

People living at a point on a surface, (if their perception is limited to a small neighborhood of their point), are unable to distinguish their situation from that of people actually living on a plane.
← Our living on the earth fits the description

Topology → Curves ≠ Surfaces

The neighboring information ➔ Curves ≠ Surfaces



- Topologically, curves and surfaces are inequivalent
- Are all surface equivalent?
  - A surface can continuously deform to another arbitrary surface?

Topological Classification of Surfaces

Sphere ≠ Disk ≠ Torus ≠ … ≠ 2 spheres …
≠ Disk with an inner hole …

What characterizes topological equivalence-relationship?

- # of Connected Components → c
- # of Boundaries → b
- # of Genus → g
- Orientability → o (y/n)

Theorem: c, b, g, and o ➔ topological equivalence

How to compute c, b, and g of a given surface?
→c and b are intuitively straightforward
→g : Euler Characteristic Number

Euler Characteristic

Euler Characteristic #:

$$\chi = 2 - 2g - b$$

(for each component)

Given a discretization ,
i.e. a tessellation with F faces, E edges, and V vertices:

$$\chi = V - E + F \quad \text{(Euler Formula)}$$

Euler Characteristic # (A topological invariance):
→ a most important topological property of the shape
(defined over a specific discretization, but characterizes the surface topological properties regardless of how the division is made)

Euler Characteristic # (cont.)

- For higher dimensional manifolds.

  e.g. a solid object (3D manifold):

$$\chi = V - E + F - S$$

- Many applications/examples:
  - Map coloring

    (4 on the plane, 6 on the sphere… Heawood's conjecture)
  - Regular Polyhedra
  - …

Euler Characteristic # (cont.)

An example: Platonic Solids (Regular polyhedra)

Regular Polyhedron = one in which all faces have the same # of edges, and all vertex have same valence

Ancient Greeks discovered there are only five regular polyhedra…

Proof: (by Euler Formula)

Suppose each face has x vertices (edges), and each vertex's valence is y, then

1) $x * F = 2 * E$, $x > 2$, and

2) $y * V = 2 * E$, $y > 2$

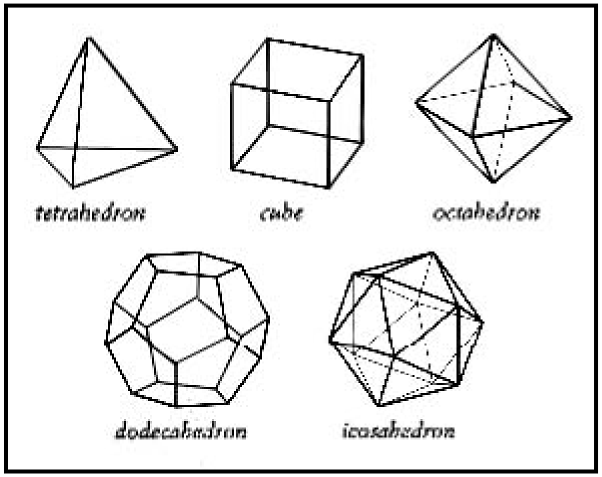
Euler Formula ➔ $V-E+F=2$

→ $2E/x - E + 2E/y = 2$, i.e. $1/x + 1/y - \frac{1}{2} = 1/E$

From $E>0$, and x, y both $>= 3$,

→ Both x and $y < 6$

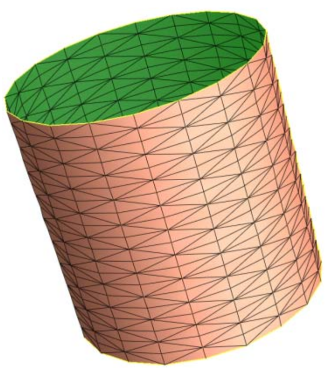
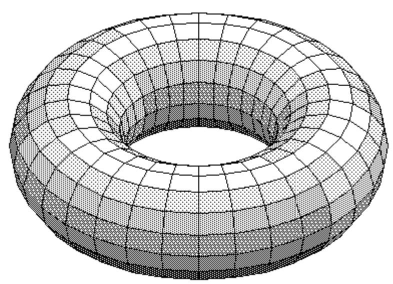
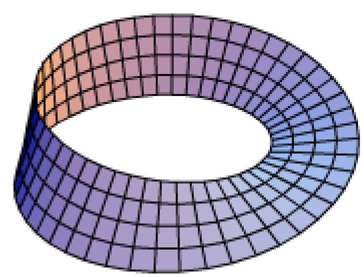
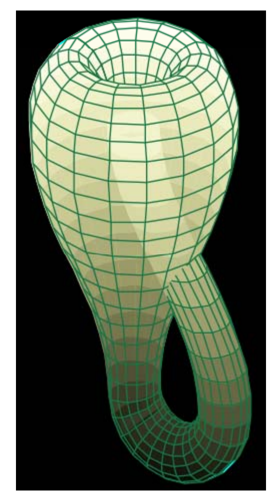
The result can be enumerated (only 5 types)



Orientability

(Intuitively) Given a surface, whether we can clearly define "outward"-side and "inward"-side

A non-orientable manifold has a path which brings a traveler back to his starting point mirror-reversed.

Orientable Surface Examples	Non-orientable Surface Examples
 	 

Topological Surface Classification

1. Connected-Components
2. Boundary
3. Genus
4. Orientability

Theorem:

Any closed surface uniquely falls into one of the following categories:

1) Sphere (genus-0, orientable)
2) k-Klein bottle (genus-k, non-orientable)
3) g-Torus (genus-g, orientable)

What happen if we use half-edge data structure to represent a non-orientable surface?

Puzzle

1. How do we find out the topology of our planet? (e.g. is it a sphere or a donut?)
   a) Intuitively, if we can jump out of it, and look from outside, maybe we can somehow figure it out.
   b) But if we are ants with local perceptions, do we have anyway to get to know the space we stay in?

Tessellate the planet! Then get the Euler Characteristic #.

But from such a number, do you get enough intuitions?
→ what geometrically differentiates a torus and a sphere?

Poincaré studied this problem, and founded modern Algebraic Topology…
An intuitive explanation → loops on surfaces

Homework 1

1. A more-complicated "program1.cpp"
2. A mesh library with better efficiency

3. See the homework description pdf file for more details

   http://www.ece.lsu.edu/xinli/teaching/EE4700Fall2009/Homework1.pdf

Next Class

Level of Details (LOD) Paradigm: Progressive Meshes…