Lecture 6 – Geometry (1)
Curves and Surfaces

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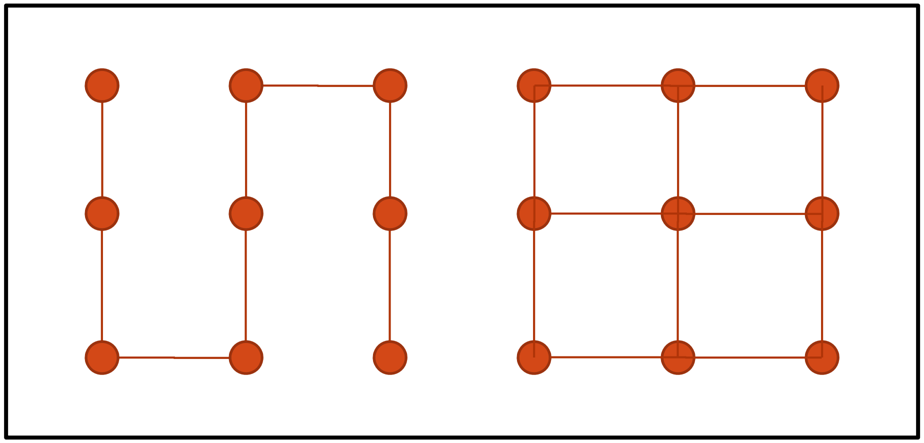
Outline

- What are curves, surfaces, and solids?
- Continuous Case
  - Topology, Topological Space, Open Set
  - Topological Equivalence
  - Curves, Surfaces, Solids…
- Discrete Case
  - Simplicial Complex
  - Neighborhoods in the Discrete Case
- Topological Classification of Surfaces

What is it about?

- Key to understand and manipulate modeling, processing, and rendering shapes is to understand geometric shapes.
- Curves, surfaces, and solid shapes are three common concepts in our surrounding world.
- Curves can be in R^1, R^2, and R^3, surfaces can be in R^2 and R^3, what is the fundamental difference between curves and surfaces? Why do we keep saying a curve is "1D" and a surface is "2D"…

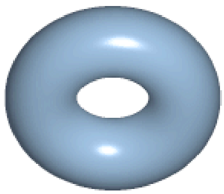
➔ Topology



Topological Space

- Topological space = a set X + some structural relationship T defined on X
- Open Set: a subset U of X, if it belongs to T
- (X, T) is a topological space, we usually say: X is a topological space with topology T
- Neighborhoods of a point p in X
- Given a point set, different topologies can be defined

Topological Space
→ Curves, Surfaces…



- Homeomorphism
  - Geometric Intuition: topologically equivalent = geometrically stretching and bending an object continuously
  - Example
    - (1,2) and (3,5)
    - Continuous Deforming (Translation, Scaling,…) are homeomorphism.
- Topological space → "Analytically good" topological space
  Hausdorff space: if different points have disjoint neighborhoods
- A topological Hausdorff space is a surface (2D-manifold)
  if each point has a neighborhood homeomorphic to an open set of Euclidean space R^2 (*)
- Similarly, curves (1D-manifolds), solids (3D-manifolds), … can be defined

From Discrete Aspects

- A surface → represented by a triangular mesh [Simplicial Complex]
- Simplexes
  - 0-simplex (vertex): $v \in \mathbb{R}^n$.
  - 1-simplex (edge): $(v_0, v_1) = \{\sum_{i=0}^{1} \lambda_i v_i : \sum_{i=0}^{1} \lambda_i = 1, \lambda_i \geq 0\}$
  - 2-simplex (face): $(v_0, v_1, v_2) = \{\sum_{i=0}^{2} \lambda_i v_i : \sum_{i=0}^{2} \lambda_i = 1, \lambda_i \geq 0\}$

λs (Barycentric Coordinates):

$$\begin{array}{rcl} \text{interior of a simplex} & \to & \text{all } \lambda_i > 0; \\ \text{boundary of a simplex} & \to & \exists \lambda_i = 0. \end{array}$$

A simplex is the smallest convex set containing its vertices.

Barycentric Coordinates

- Why Barycentric Coordinates are so defined uniquely?
  - From $v = \sum_{i=1}^{3} \lambda_i v_i$ and $\sum_{i=1}^{3} \lambda_i = 1, \quad \lambda_1, \lambda_2, \lambda_3 > 0$

    We have: $\lambda_3 = 1 - \lambda_1 - \lambda_2$

    Without loosing generality,
    we can defined a local 2D coordinate system on (v_1, v_2, v_3) :

$$\begin{cases} x = \lambda_1 x_1 + \lambda_2 x_2 + (1 - \lambda_1 - \lambda_2) x_3 \\ y = \lambda_1 y_1 + \lambda_2 y_2 + (1 - \lambda_1 - \lambda_2) y_3 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 (x_1 - x_3) + \lambda_2 (x_2 - x_3) + (x_3 - x) = 0 \\ \lambda_1 (y_1 - y_3) + \lambda_2 (y_2 - y_3) + (y_3 - y) = 0 \end{cases}$$

i.e.

$$\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}$$

Barycentric Coordinates (cont.)

$$\ldots \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}$$

now $\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$ → Since $(r_1\text{-}r_3)$ and $(r_2\text{-}r_3)$ are linear independent

has inversed matrix:

$$\frac{1}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)} \begin{bmatrix} y_2 - y_3 & -(y_1 - y_3) \\ -(x_2 - x_3) & x_1 - x_3 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)} \begin{bmatrix} y_2 - y_3 & -(x_2 - x_3) \\ -(y_1 - y_3) & x_1 - x_3 \end{bmatrix} \begin{bmatrix} x - x_3 \\ y - y_3 \end{bmatrix}$$

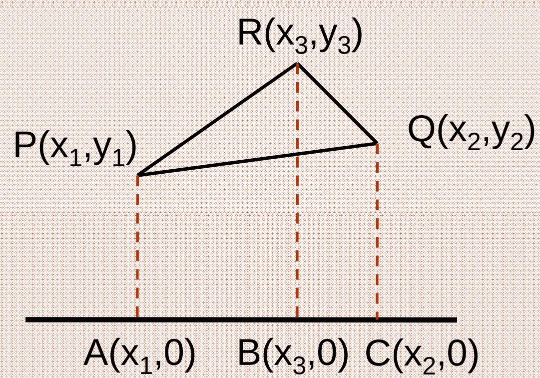
$$\lambda_1 = \frac{(x - x_3)(y_2 - y_3) - (x_2 - x_3)(y - y_3)}{(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)}$$

Barycentric Coordinates (cont.)

The denominator:

$$(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) = \boxed{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}$$

Twice the signed area of triangle $((x_1,y_1), (x_2, y_2), (x_3,y_3))$



$Area(PQR)=Area(PABR)+Area(RBCQ)-Area(PACQ)$

$Area(PABR)=(x_3-x_1)(y_1+y_3)/2$

$Area(RBCQ)=(x_2-x_3)(y_2+y_3)/2$

$Area(PACQ)=(x_2-x_1)(y_1+y_2)/2$

➔ $Area(PQR)=(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))/2$

Similarly, the numerator $(x - x_3)(y_2 - y_3) - (x_2 - x_3)(y - y_3)$

is twice the signed area of triangle $((x,y), (x_2, y_2), (x_3,y_3))$

➔ $\lambda_1 = Area(v,v_2,v_3)/Area(v_1,v_2,v_3)$

Simplicial Complex

- A simplicial complex is a finite set K of simplexes, satisfying the intersection condition:

$$\text{if } s,t \in K \quad \text{then } s \cap t = \begin{cases} \emptyset \\ \text{a simplex both of } s \text{ and of } t \end{cases}$$

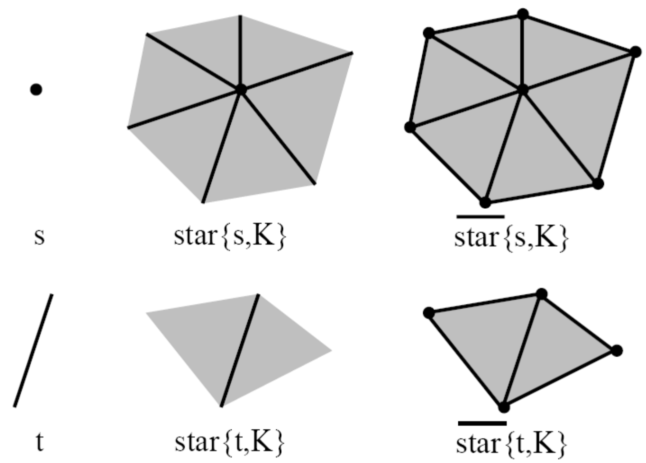
- Neighborhoods in a simplicial complex:

For simplex $s \in K$

$$\text{star}(s;K) = \{s' \in K \ : \ s \leq s'\}$$

$$\overline{\text{star}}(s;K) = \{s' \in K \ : \ \exists t \in \text{star}(s;K) \ : \ s' \leq t\}$$

$$\text{link}(s;K) = \overline{\text{star}}(s;K) \setminus \text{star}(s;K).$$



Continuous VS Discrete

	Continuous Def.	Discrete Def.
Closed Surface:	Each point's neighborhood is homeomorphic to 2D plane	Each vertex's "star-bar" simplicial complex is a simple closed polygon
Open Surface:	Boundary point's neighborhood is homeomorphic to 2D half-plane	Boundary vertex's "star-bar" simplicial complex is a simple closed polygon, with this vertex on the boundary

Similarly, for 1D-, 3D-, … nD- manifolds…