05 Review of Linear Algebra and Transformations

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Part of the slides taken and modified based on materials from Prof. M. Vasilescu’s Computer Vision Course taught in Stony Brook University (SUNY)
Overview

• n-dimensional vectors
• Dot (Scalar) product
• Bases and Frames
• Homogeneous Coordinates
• 2D and 3D Geometric Transformations
Vectors

Notation:

\[ x \in \mathbb{R}^n, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x = (x_1, x_2, \ldots, x_n)^T \]

Length:

\[ \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \]

Geometric Meanings:

- An origin + \( n \) pairwise perpendicular vectors \( \rightarrow \) defines a frame or a coordinate system
- For a fixed frame, a point \( \rightarrow \) a \( n \)-dimensional vector
Dot Product

Dot Product of two vectors:
\[
\langle x, y \rangle = x \cdot y = x^T y = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}
\]

Geometric Meanings:
\[
x \cdot y = \|x\| \|y\| \cos \theta
\]

Properties
- Commutative: \( x \cdot y = y \cdot x \)
- Distributive: \( (x + y) \cdot z = x \cdot z + y \cdot z \)
- Linearity:
  \[
  (cx) \cdot y = c(x \cdot y) \\
  x \cdot (cy) = c(x \cdot y)
  \]
  \[
  (c_1 x) \cdot (c_2 y) = (c_1 c_2)(x \cdot y)
  \]
- Non-negativity:
  \[
  \forall x \neq 0: \langle x, x \rangle > 0 \\
  \langle x, x \rangle = 0 \iff x = 0
  \]
- Orthogonality:
  \[
  \forall x \neq 0, y \neq 0 \quad x \cdot y = 0 \iff x \perp y
  \]
Norms

- Euclidean norm (sometimes called 2-norm):

\[ \|\mathbf{x}\| = \|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \]

- The length of a vector is defined to be its (Euclidean) norm.
- A unit vector is of length 1.
- Non-negativity properties also hold for the norm:

\[ \forall \mathbf{x} \neq 0 : \|\mathbf{x}\|^2 > 0 \quad \|\mathbf{x}\|^2 = 0 \iff \mathbf{x} = 0 \]
Linear Dependence

- Linear combination of vectors $x_1, x_2, \ldots x_n$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

- A set of vectors $X = \{x_1, x_2, \ldots x_n\}$ are linearly dependent if there exists a vector $x_i \in X$ that is a linear combination of the rest of the vectors.

- In $\mathbb{R}^n$
  - sets of $n+1$ vectors are always dependent
  - there can be at most $n$ linearly independent vectors
Bases

- A basis is a linearly independent set of vectors that spans the “whole space”. i.e., we can write every vector in our space as linear combination of vectors in that set.

- Every set of $n$ linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$

- A basis is called
  - **orthogonal**, if every basis vector is orthogonal to all other basis vectors
  - **orthonormal**, if additionally all basis vectors have length 1.
Bases

- Standard basis in $\mathbb{R}^n$ is made up of a set of unit vectors:

  $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \hat{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$

- We can write a vector in terms of its standard basis:

  $\begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix} = 4 \hat{e}_1 + 7 \hat{e}_2 - 3 \hat{e}_3.$

- Observation: To find the coefficient for a particular basis vector, we project our vector onto it.

  $x_i = \hat{e}_i \cdot x$
Change of Bases

- Suppose we have a new basis $B = [b_1 \cdots b_n]$, $b_i \in \mathbb{R}^m$ and a vector $x \in \mathbb{R}^m$ that we would like to represent in terms of $B$.

- Compute the new components.

- When $B$ is orthonormal:
  - $\tilde{x}$ is a projection of $x$ onto $b_i$.
  - Note the use of a dot product.

Note: $B$ is an orthonormal matrix, whose inverse is its transpose. Therefore, we have $\tilde{x} = B^{-1}x$.
Rank of a Matrix

The rank of a matrix is the number of linearly independent rows or columns.

Examples: \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\] has rank 2, but \[
\begin{pmatrix}
2 & 1 \\
4 & 2
\end{pmatrix}
\] only has rank 1.

Equivalent to the dimension of the range of the linear transformation.

A matrix with full rank is called non-singular, otherwise it is singular.
Cross Product

Consider two vectors \( \mathbf{a} \) and \( \mathbf{b} \) in the 3D space

- the cross product \( \mathbf{a} \times \mathbf{b} \) is a vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \), and therefore normal to the plane containing them

\[
\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \mathbf{n}
\]

- If the two given vectors are parallel or one vector has zero length, then the cross product is zero

- Anti-commutative: \( \mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) \)

- Geometric meaning:
  - Magnitude: the area of a parallelogram
  - Direction: right-hand rule

- Example 1: Prove if \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \) then \( \mathbf{a} \cdot \mathbf{c} = 0 \)
Homogeneous Coordinates (2D)

Consider a point \( \mathbf{x} \) in the 2D plane:

- Conventional Notation (inhomogeneous coordinates)
  - A 2D point can be represented by a pair of coordinates \((x, y) \in \mathbb{R}^2\)
  - A 2D point \(\iff\) a 2D vector \( \mathbf{x} = (x, y)^T \); the 2D plane \(\iff\) \(\mathbb{R}^2\) vector space

- Homogeneous Notation (homogeneous coordinates)
  - \( \tilde{\mathbf{x}} = (x, y, 1)^T \), referred to as an augmented vector
  - 3D vectors \((x, y, w)^T\) can be converted to its corresponding augmented vectors \(w(x/w, y/w, 1)\) if \(w \neq 0\);
  - In this case, we consider \((x, y, w)^T\) and \(\left(\frac{x}{w}, \frac{y}{w}, 1\right)^T\) to be equivalent
  - The set of equivalence classes of vectors from 3D space \(\mathbb{R}^3 - (0,0,0)^T\) forms the projective space \(P^2\)
  - Homogeneous vector representation of a point \(\tilde{\mathbf{x}} = (x_1, x_2, x_3)^T\) corresponds to the point with the 2D inhomogeneous coordinates \(\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)^T\)

- Example 2:
  - A 2D point \((3,4)\), what is its homogeneous coordinates?
    - \(\rightarrow (3,4,1)\)
  - What about \((6,8,2)\)?
Homogeneous Coordinates (cont.)

- Motivation of using Homogeneous Coordinates:
  - To consider and represent linear transformations using matrices
  - Simplify a lot of computations

- A line in the plane: \( ax + by + c = 0 \), can be put as \((a, b, c)^T\) in Homogeneous Notation
  - The correspondence is **NOT** one-to-one: \(k(a, b, c)^T, k \neq 0\) represents the same line
  - The set of equivalence classes of vectors in \(R^3 - (0,0,0)^T\) forms the projective space \(P^2\)

- A planar point \(x = (x, y)^T\) lies on the line \(l = (a, b, c)^T\) iff \(ax + by + c = 0\)
  - If we write the point as an augmented vector: \(\tilde{x} = (x, y, 1)^T\)
  - \(\rightarrow\) inner product of vectors: \((x, y, 1)(a, b, c)^T = 0\), or simply \(\tilde{x} \cdot l = 0\)
Homogeneous Coordinates for 3D Points

- Inhomogeneous Coordinates \((x, y, z)^T\) → homogeneous coordinates \((x, y, z, 1)^T\)

- A 3D plane can be represented as \(m = (a, b, c, d)^T\), because any point \(x = (x, y, z)^T\) on this plane satisfies:
  \[ax + by + cz + d = \tilde{x} \cdot m = 0\]
2D Transformations

- Basic Transformations: Translations, Rotations, Scaling, Affine, Projective
- Euclidean Transformation = Rigid Transformation = Translations + Rotations
- Similarity Transformation = Uniform scaling + Translations + Rotations + Reflection
- Affine Transformation = Similarity + non-uniform scaling + shear mapping
- Projective Transformation

**Figure 2.4** Basic set of 2D planar transformations.
Linear Transformations

- A transformation $T(x) = y, x \in \mathbb{R}^n, y \in \mathbb{R}^m$, is a linear transformation if it can be represented by a matrix: $y = Mx, M \in \mathbb{R}^m \times \mathbb{R}^N$, where $M$ is called the transformation matrix.

- With homogeneous coordinates, all basic transformations listed in the previous slide can be treated as linear transformations.
Example: 2D Rotation

To rotate the left figure to the right figure (the left bottom corners of both objects remain at the origin)

\[ T(e_1) = f_1, \quad T(e_2) = f_2 \]

\[ \text{To find a transformation } T \text{ such that } T(e_1) = f_1, \quad T(e_2) = f_2 \]

\[ \Rightarrow \text{ We can solve a linear system using elementary trigonometry:} \]

\[ f_1 = Me_1; \quad f_2 = Me_2 \]

under the coordinate system of \((e_1, e_2)\),

\[ f_1 = (\cos \theta, \sin \theta)^T; \quad f_2 = (-\sin \theta, \cos \theta) \]

\[ M = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]
Example: Scaling

2D scaling transformation matrix:

$$
M = \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
$$

as

$$
M [x] = \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix} [x] = [ax]
$$

where $a$ and $b$ are called the scaling factors along $x$ and $y$ axes.

- If $a = b$: uniform (isotropic) scaling, shape preserved, only size changed
- If $a = b = 1$: identity transform

• Directly extendible to 3D
Example: Shear

Starting from the 2D scaling transformation matrix:

\[
M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}
\]

Let one of the off-diagonal element be non-zero:

\[
\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}
\]

\[
M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + cy \\ by \end{bmatrix}
\]

- A general definition: shearing = to shift the figure along a direction by an amount proportional to its signed distance from the line parallel to that direction
- Shearing changes the shape (angles changes under shearing) but preserves the area

\[ a = b = 1, c = 1 \]
Examples: Reflection & Orthography

- **Reflection** = scaling with negative factors
  
  \[ M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]

- **Orthographic Projection**
  
  \[ M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

- Does not map a basis onto another basis
- A singular transformation (can’t be inverted, lost depth information along a direction)
Example: Translation

\[
\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} x + d_1 \\ y + d_2 \end{bmatrix}
\]

Not a linear transformation (can’t be represented as a matrix-vector multiplication)

But if the homogeneous coordinates are used:

\[
\begin{bmatrix} 1 & d_1 \\ 1 & d_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_1 \\ y + d_2 \\ 1 \end{bmatrix}
\]

Rigid Transformation = translation + rotation

- If a transformation preserves distance \( \rightarrow \) called an isometry
- Rigid transformations are definitely isometries, but isometries may not be rigid
  - e.g. bending a paper to a cylinder
Example: General Projection

A general projection, or perspective transformation, or also called homography, is defined on homogeneous coordinates,

\[ \tilde{x}' = H \tilde{x} \]

where \( H \) is an arbitrary \( 3 \times 3 \) matrix, defined up to a scale.

- Converting back to inhomogeneous coordinates:

\[
x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \quad \text{and} \quad y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}}.\]

- Perspective transformations preserve straight lines such that they remain straight after the transformation.
### Summary of 2D Transformations*

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Matrix</th>
<th># DoF</th>
<th>Preserves</th>
<th>Icon</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation</td>
<td>$\begin{bmatrix} I &amp; t \end{bmatrix}_{2\times3}$</td>
<td>2</td>
<td>orientation</td>
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<tr>
<td>rigid (Euclidean)</td>
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<td>lengths</td>
<td>◻️</td>
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<tr>
<td>similarity</td>
<td>$\begin{bmatrix} sR &amp; t \end{bmatrix}_{2\times3}$</td>
<td>4</td>
<td>angles</td>
<td>◻️ ◻️</td>
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<tr>
<td>affine</td>
<td>$\begin{bmatrix} A \end{bmatrix}_{2\times3}$</td>
<td>6</td>
<td>parallelism</td>
<td>◻️</td>
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<tr>
<td>projective</td>
<td>$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times3}$</td>
<td>8</td>
<td>straight lines</td>
<td>◻️ ◻️ ◻️</td>
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#DOF : the Number of Degrees of Freedom = the number of independent variables (unknowns) that can change freely
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<td>straight lines</td>
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