Nondegeneracy of Harmonic Volumetric Parameterization on Star-shaped Domains

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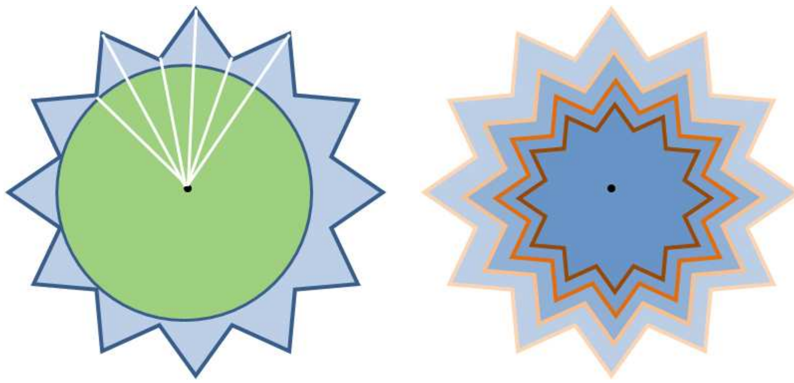


Figure 1: Parameterizing a Star Shape onto a Solid Sphere. (Left) Initial u and v coordinates can be projected from the boundary surface to the inscribed sphere. (Right) The w coordinate (depth) can be computed using methods of fundamental solutions [4].

sectionHarmonic Volumetric Parameterization using MFS After the decomposition of a given object M, we get a set of star shapes $\{M_i\}$, each region being guarded by a point g_i. Then we can parameterize each subregion onto a solid sphere. A key property that we will show shortly is that such a harmonic map is guaranteed to be bijective. The harmonic map can be computed using the **method of fundamental solutions (MFS)** [4], by setting the boundary condition of the volumetric map to be the spherical parameterization of ∂M_i. We use the harmonic spherical parameterization [2] to get the harmonic surface map $f' : \partial M_i \to S^2$. [2] takes the normal map as the initial mapping and conduct the optimization on local tangential plane before projecting the adjusted position back to the sphere. If the initial map (like the normal map) has large flip-over regions, the optimization will be slow and could trap locally. Since each M_i now is a star shape, the following approach efficiently gets a bijective initial spherical mapping. Figure 1 illustrates our idea. In the left picture, full visibility of the local region guarantees a bijective projection from the boundary points onto the sphere. When the boundary mapping is decided, we only need to compute the third dimensional coordinate w of the parameter. Figure 1 (Right) illustrates the concept of the different iso-w level sets. Such a harmonic function w can be computed using the MFS method. We refer to [4] for technical details, and only briefly recap the idea as follows.

MFS for Volumetric Parameterization. Based on the Green's function and the maximum principle of harmonic functions, the harmonic function w defined on a region D never reaches maximal or minimal values in the interior of D, and w is fully determined by the boundary condition and can be computed by fundamental solutions of the 3D Laplace equation. The kernel function for the 3D Laplace equation coincides with the electric potential produced by point charges. Therefore, its intuitive physical explanation is to design a potential field that approximates the boundary condition. The potential field, guaranteed to be harmonic by the fundamental solutions, is the function w that we seek for. The parameterization therefore is converted to a boundary fitting problem for the potential field, and can be solved using a linear system effectively.

1 Bijectiveness of Harmonic Mapping on Star Regions

In the surface case, a harmonic map is a minimizer of the Dirichlet energy and indicates a minimal surface [6]. It can be effectively approximated through FEM analysis of harmonic energy [1]. The theoretic foundation for harmonic surface mapping is built upon the Radó Theorem, which states that, on a simply connected surface M with a Riemannian metric, suppose a harmonic function $f : M \to D \subset \mathbb{R}^2$ maps M to a convex planar domain D, if f on the boundary is a homeomorphism then f in the interior of M is also a diffeomorphism.

FEM analysis of harmonic energy can also be conducted [7] on 3-manifolds using tetrahedral meshes. However, the Radó theorem does not hold for 3-manifolds. Therefore, fundamental theoretic obstacles remain for volumetric harmonic mapping. We tackle this fundamental parameterization problem for volumetric data through star decomposition. It can be proved that for specific domains such as convex shapes, bijective harmonic parameterization exists. Then in order to compute a bijective mapping, we can first decompose volumetric data into a set of solvable sub-domains for local piecewise mapping computation. We show ([3] gives a rigorous mathematic proof through analyzing the induced foliation) the existence of harmonic volumetric parameterization on a star-shaped region and that the bijective map can be constructed using the MFS-based framework effectively. The idea is as follows.

Lemma 1 In a star-shaped domain M guarded by a point g, the harmonic function $w : M \to [0, 1]$ has its only critical point at g.

When M is visible to a guard g and a harmonic function w is defined over M, with $w|_{\partial M} = 1$ and $w|_g = 0$. We can use g as the origin O and create a local coordinate system. Then we analyze the gradient of the harmonic function ∇w within this local coordinate system. At $p(x_1, x_2, x_3)$, we can define another harmonic function $h(p) =< p, \nabla w >$, where $<, >$ denotes the dot product. Since w is harmonic, $\frac{\partial w}{\partial x_i}$ is also harmonic, then we can verify $h(p)$ is harmonic by:

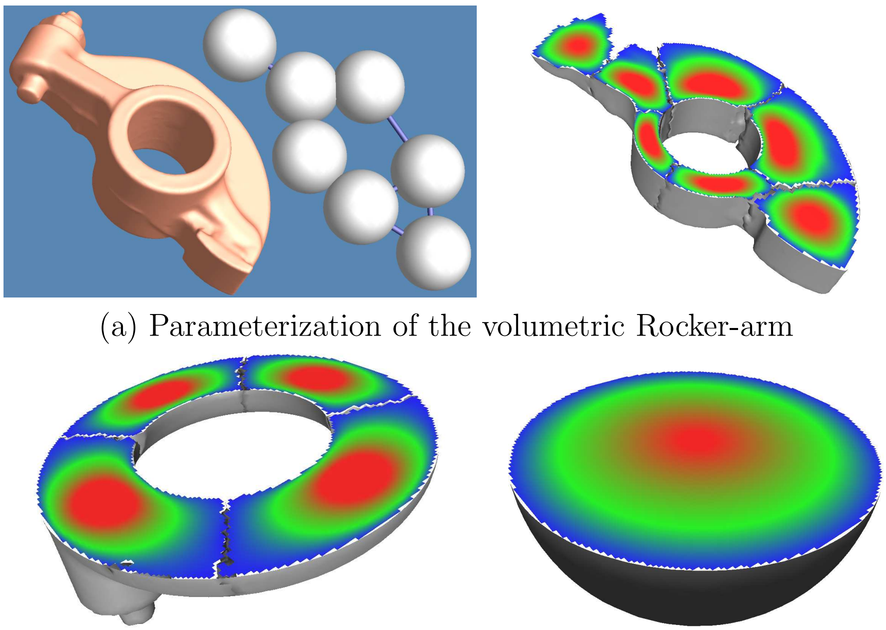
$$\Delta h = (\sum_k \frac{\partial^2}{\partial x_k^2})(\sum_i x_i \frac{\partial w}{\partial x_i}) = 2\Delta w + \sum_i x_i \Delta(\frac{\partial w}{\partial x_i}) = 0.$$

The maximum principle of harmonic map guarantees h reaches its max and min values only on the boundary of the harmonic field, i.e. surface boundary ∂M and the infinitely small ball bounding O, $\partial B(O, \epsilon)$. ∇w has same direction of p so it is easy to see that $f > 0$ in all the region bounded by $\partial B(O, \epsilon)$ and ∂M. Therefore, for arbitrary ϵ, $f \neq 0$, we have $\nabla w \neq 0$ in $M/\{O\}$, the harmonic potential is proved to have no critical point in M except O.

Theorem 1 Given a potential value r, the level set $w^-1(r)$ is a topological sphere.

This is guaranteed by Morse theory [5], which says that two level sets share the same topology if there does not exist critical point between these two layers. Therefore, based on Lemma 1, all interior iso-layer $w^{-1}(r)$ has the same topology with the region boundary. This demonstrates the harmonic function w computed using MFS, together with the other two coordinates u, v defined by surface mapping, induce a bijective spherical map $M \to S^2$, which is also a diffeomorphism.

We conduct our harmonic volumetric parameterization on decomposed sub-regions. And verify the signed volume of each tetrahedron when mapped onto the target object. Under the mapping, all the deformed tetrahedra still preserve the positive volume. Which demonstrates the non-degeneracy of our parameterization, meanwhile, decomposition improves the mapping distortion, namely,the quasi-conformality of the harmonic mapping also effectively increases.



(a) Parameterization of the volumetric Rocker-arm

(b) Torus-cone (4-parts)

(c) Spherical level set

Figure 2: Harmonic Parameterization of Rocker-arm and Torus-cone. (a) The volumetric rocker-arm model, its spherical parametric domain, and the volumetric parameterization visualized in one cross-section. (b) The volumetric torus-cone model is decomposed to 4 star regions, also, the color-encoded w distance field is visualized in one cross-section. (c) Each sub-region is parameterized onto a solid sphere, whose color-encoded level set is visualized.

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