1. The probability density of \( r_i, i = 1, 2, \ldots, N \), given \( a_1 \) and \( a_2 \) is
   \[
   f_{R_i|a_1,a_2}(r_i|a_1,a_2) = \frac{1}{\sqrt{2\pi a_2}} \exp \left[ -\frac{(r_i - a_1)^2}{2a_2} \right],
   \]
   where \( a_1 \) is the mean and \( a_2 \) is the variance.
   
   (a) Find the joint maximum-likelihood estimates \( \hat{a}_1, \hat{a}_2 \) of \( a_1 \) and \( a_2 \) respectively by using \( N \) independent observations.
   
   (b) Are they biased?
   
   (c) Another variance estimator is described as
       \[
       \hat{a}_2' = \frac{1}{N-1} \sum_{i=1}^{N} \left[ r_i^2 - \frac{1}{N} \sum_{i=1}^{N} r_i \right]^2.
       \]
       Is this biased?
   
   (d) Check if the mean estimator \( \hat{a}_1 \) achieves the Cramer-Rao lower bound.
2. We want to transmit two parameters, $a_1$ and $a_2$. In a simple attempt to achieve a secure communication system we construct two signals to be transmitted over separate channels such that

\[
\begin{align*}
    s_1 &= x_{11}a_1 + x_{12}a_2, \\
    s_2 &= x_{21}a_1 + x_{22}a_2,
\end{align*}
\]

where $x_{ij}, i, j = 1, 2,$ are known. The received variables are

\[
\begin{align*}
    r_1 &= s_1 + n_1, \\
    r_2 &= s_2 + n_2.
\end{align*}
\]

The additive noise processes $n_1, n_2$ are independent, identically distributed and zero-mean Gaussian, $N(0, \sigma_n)$. The parameters $a_1$ and $a_2$ are nonrandom.

(a) what are the maximum-likelihood estimates $\hat{a}_1$ and $\hat{a}_2$?

(b) Are $\hat{a}_1$ and $\hat{a}_2$ biased?

(c) Compute the variances of $E\{(\hat{a}_1 - a_1)^2\}$ and $E\{(\hat{a}_2 - a_2)^2\}$.

(d) Do the maximum-likelihood estimates $\hat{a}_1$ and $\hat{a}_2$ achieve the Cramer-Rao lower bounds?
3. The joint probability density function for the two observations \( r_1 \) and \( r_2 \) is

\[
f_{R_1,R_2|\rho}(r_1,r_2|\rho) = \frac{1}{2\pi \sqrt{(1 - \rho^2)}} \exp \left[ -\frac{(r_1^2 - 2\rho r_1 r_2 + r_2^2)}{2(1 - \rho^2)} \right].
\]

(a) Determine the maximum-likelihood estimate \( \hat{\rho} \) in terms of \( r_1 \) and \( r_2 \).

(b) Is this \( \hat{\rho} \) biased?

(c) Determine the Cramer-Rao lower bound for \( \rho \).