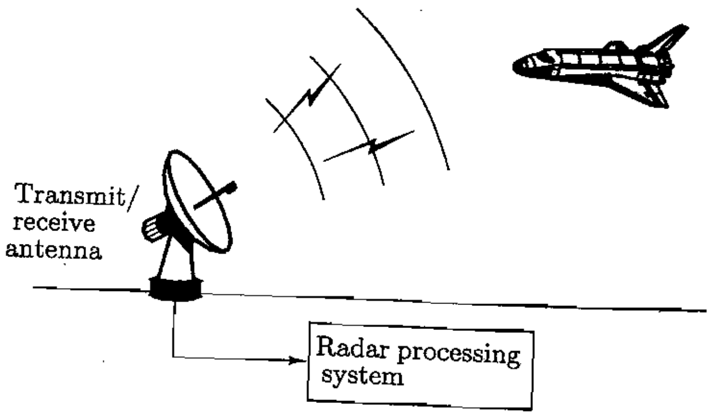
# Detection Theory in Signal Processing

Modern detection theory is fundamental to the design of electronic signal processing systems for decision making the information extraction These systems include

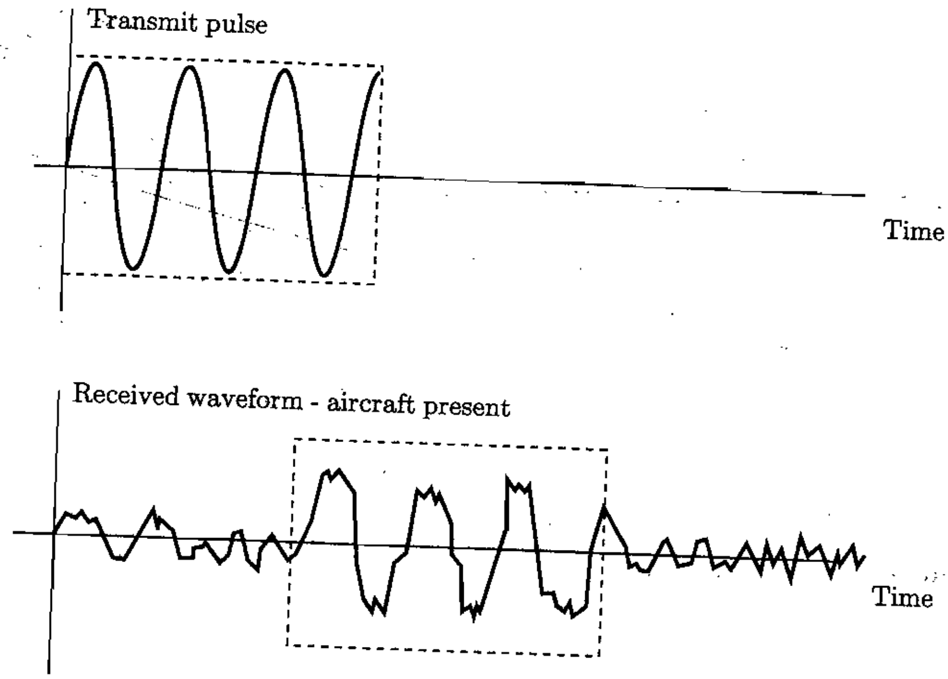
1. radar, 2. Communications, 3 speech, 4. sonar
5. image processing, 6. Biomedicine 7. Control
8. Seismology.

All share the common goal of being able to decide when an event of interest occurs and then to determine more information about that event. The statistical hypothesis test is usually taken for signal detection, or sometimes, we call it Bayesian Test.

Some examples can be found in S. M. Kay, Fundamentals of Statistical Signal Processing, vol II, Detection Theory.


Transmit/
receive
antenna
Radar processing
system

(a)


Transmit pulse
Time
Received waveform - aircraft present
Time

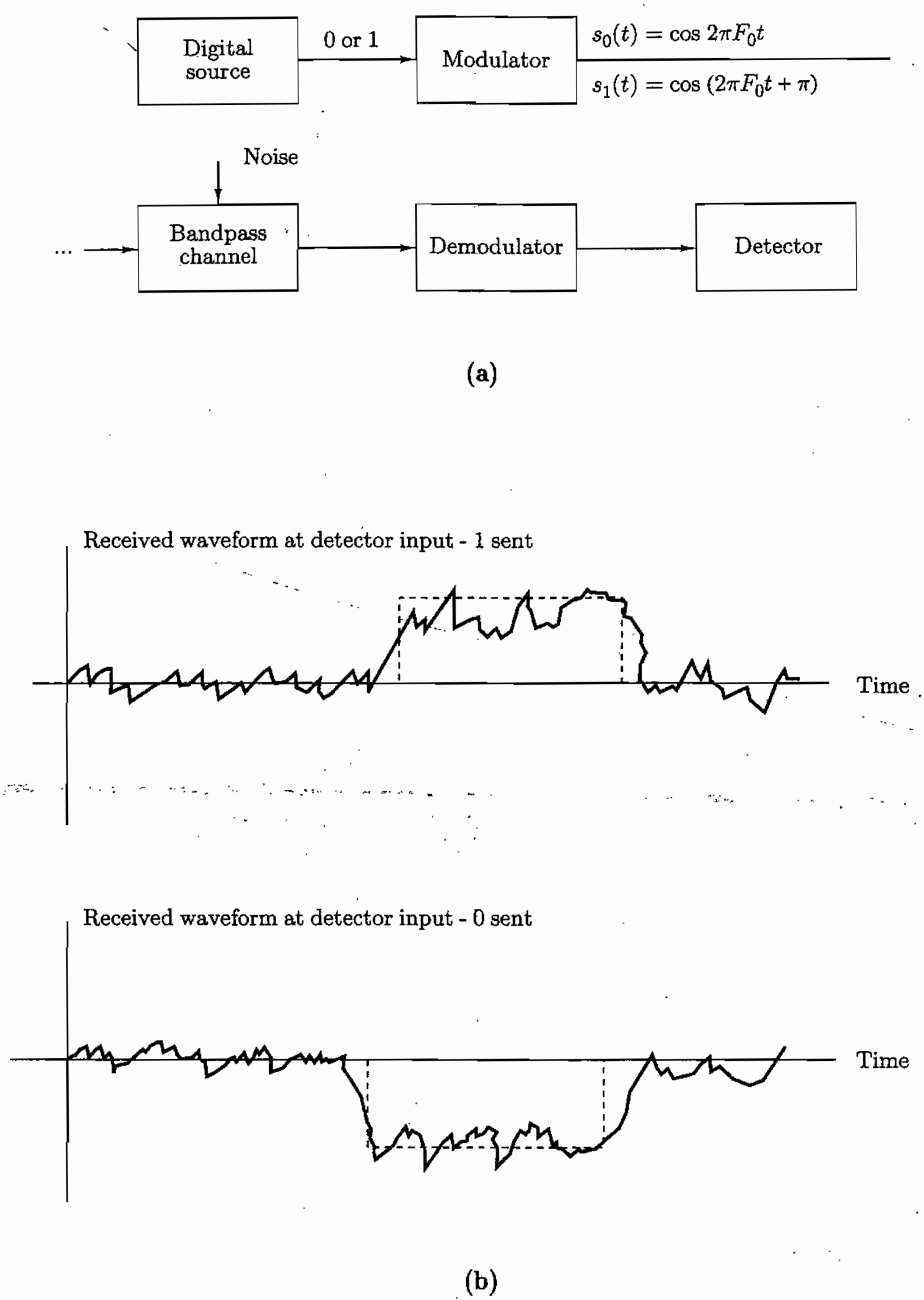
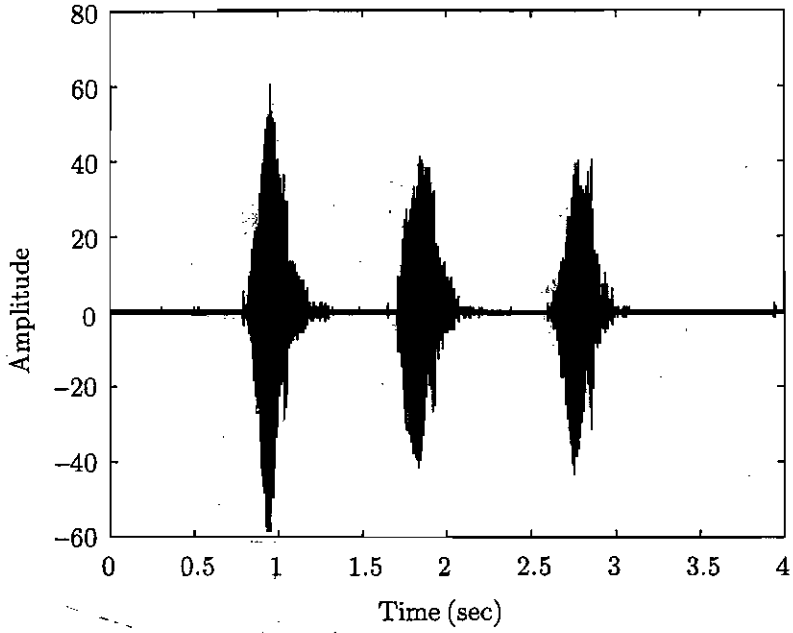
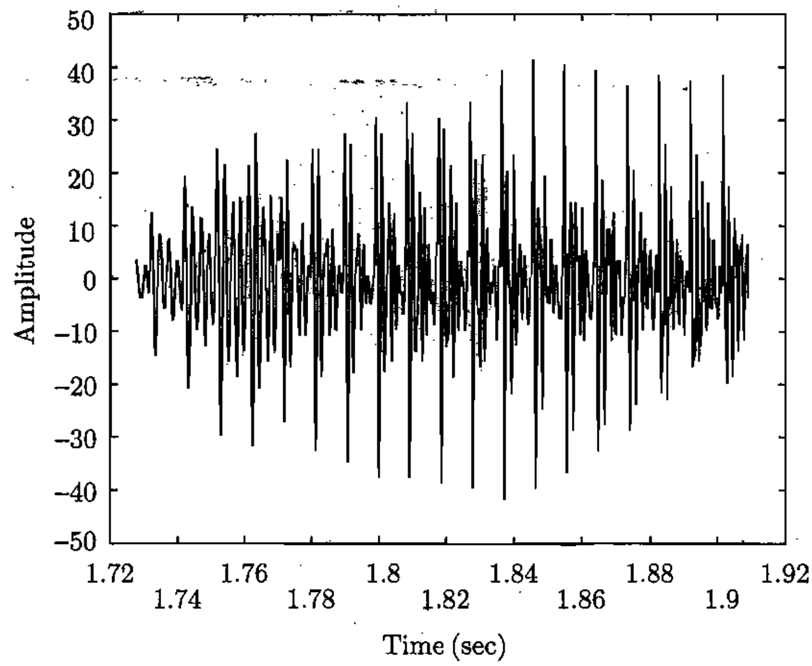



**Figure 1.2.** Binary phase shift keyed digital communication system (a) Basic system (b) BPSK baseband waveforms.

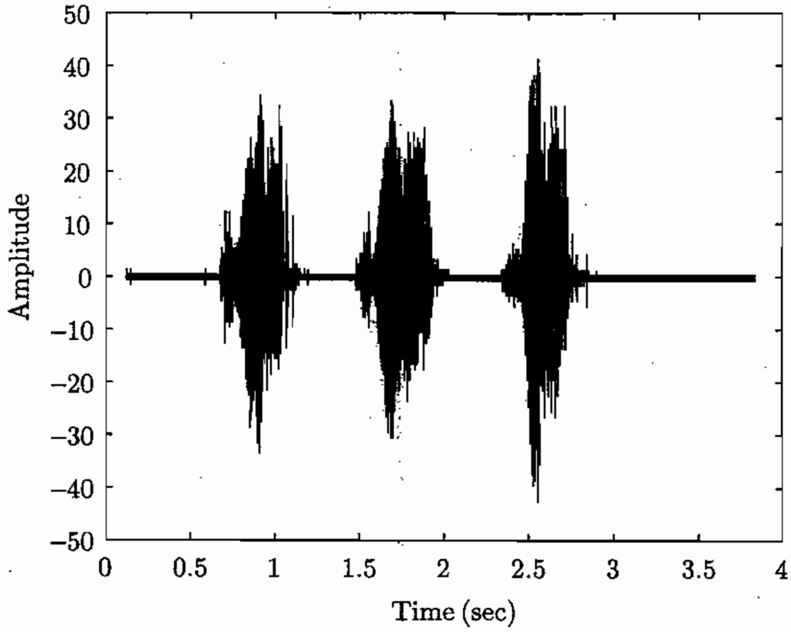



(c)

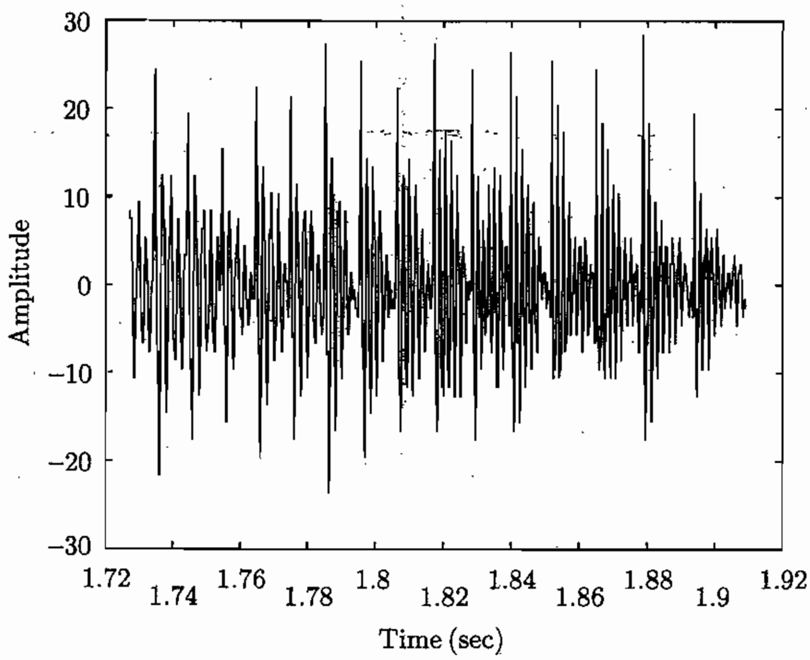



(d)

**Figure 1.3. Continued** (c) "One" spoken three times (d) "One"–portion of utterance.




(a)

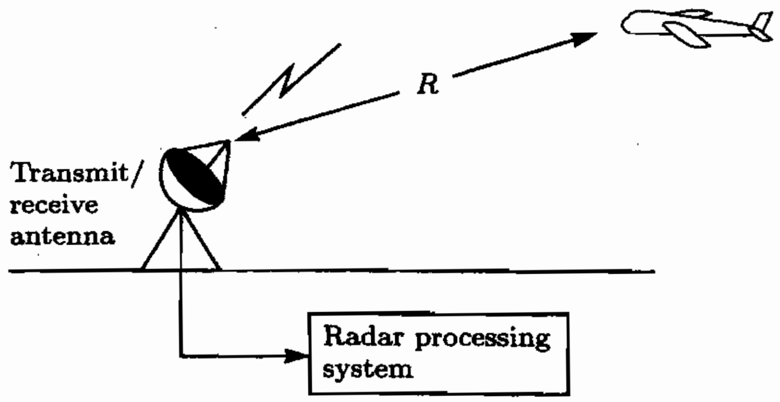



(b)

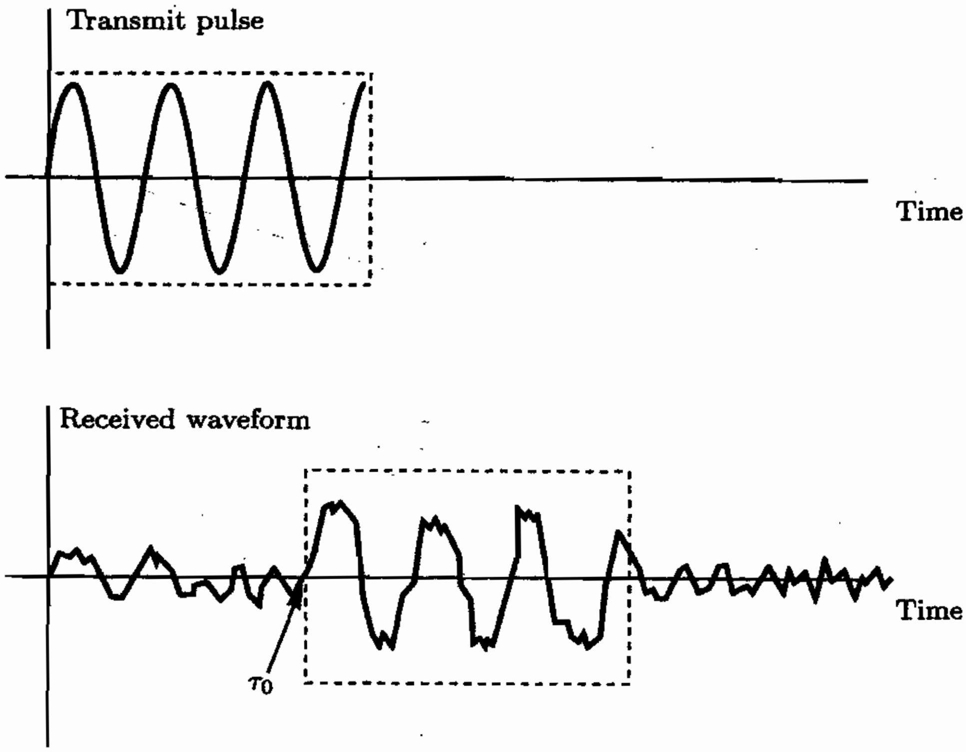
**Figure 1.3.** Speech waveforms for digits "zero" and "one" (a) "Zero" spoken three times (b) "Zero"–portion of utterance.

Modern Estimation Theory can be found at the heart of many electronic signal Processing systems designed to extract information. The aforementioned systems do also have the estimation schemes.

All share the common problem of needing to estimate the values of a group of parameters, such as system coefficients, underlying statistical characteristics, etc. These parameters very often (not always) correspond to some crucial physical phenomena or physical measures, such as traveling time, delay, channel impulse responses, etc.

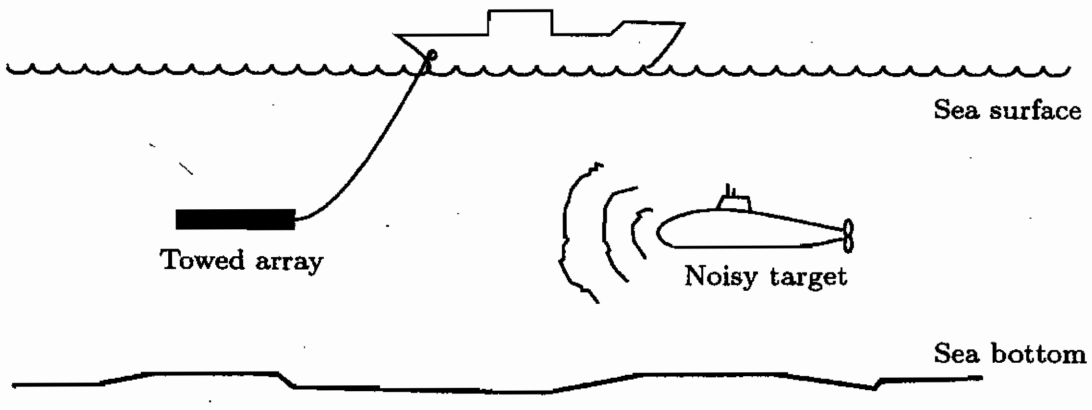



**(a)** Radar

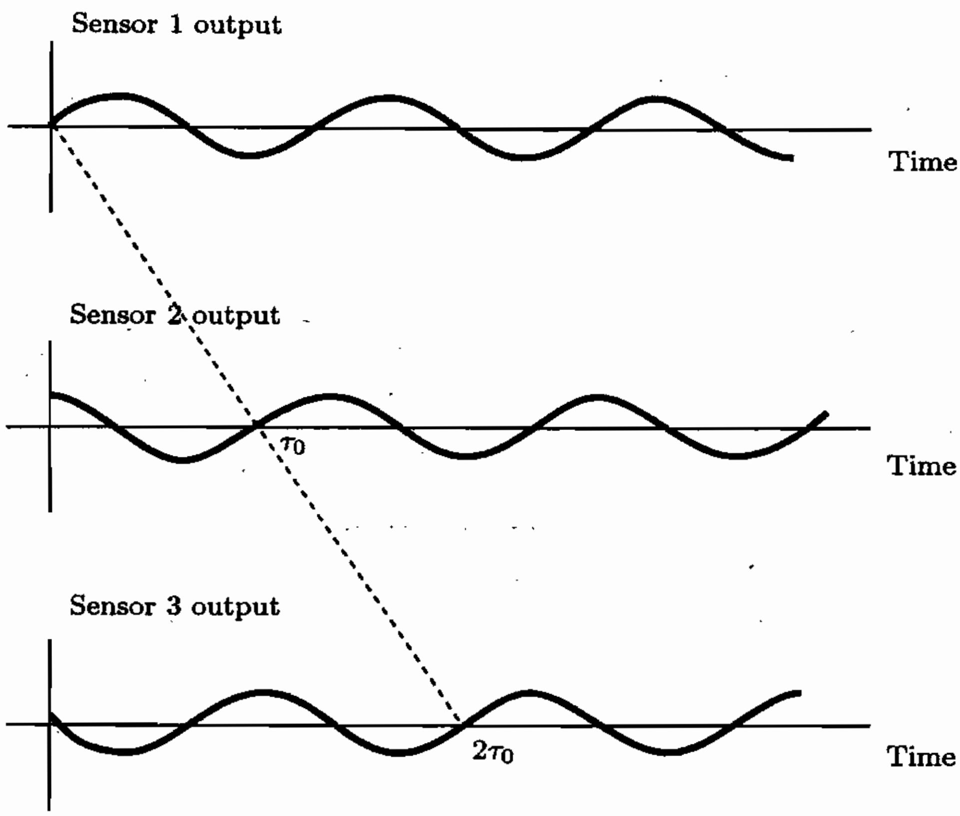



**(b)** Transmit and received waveforms

**Figure 1.1** Radar system




**(a)** Passive sonar




**(b)** Received signals at array sensors

**Figure 1.2** Passive sonar system

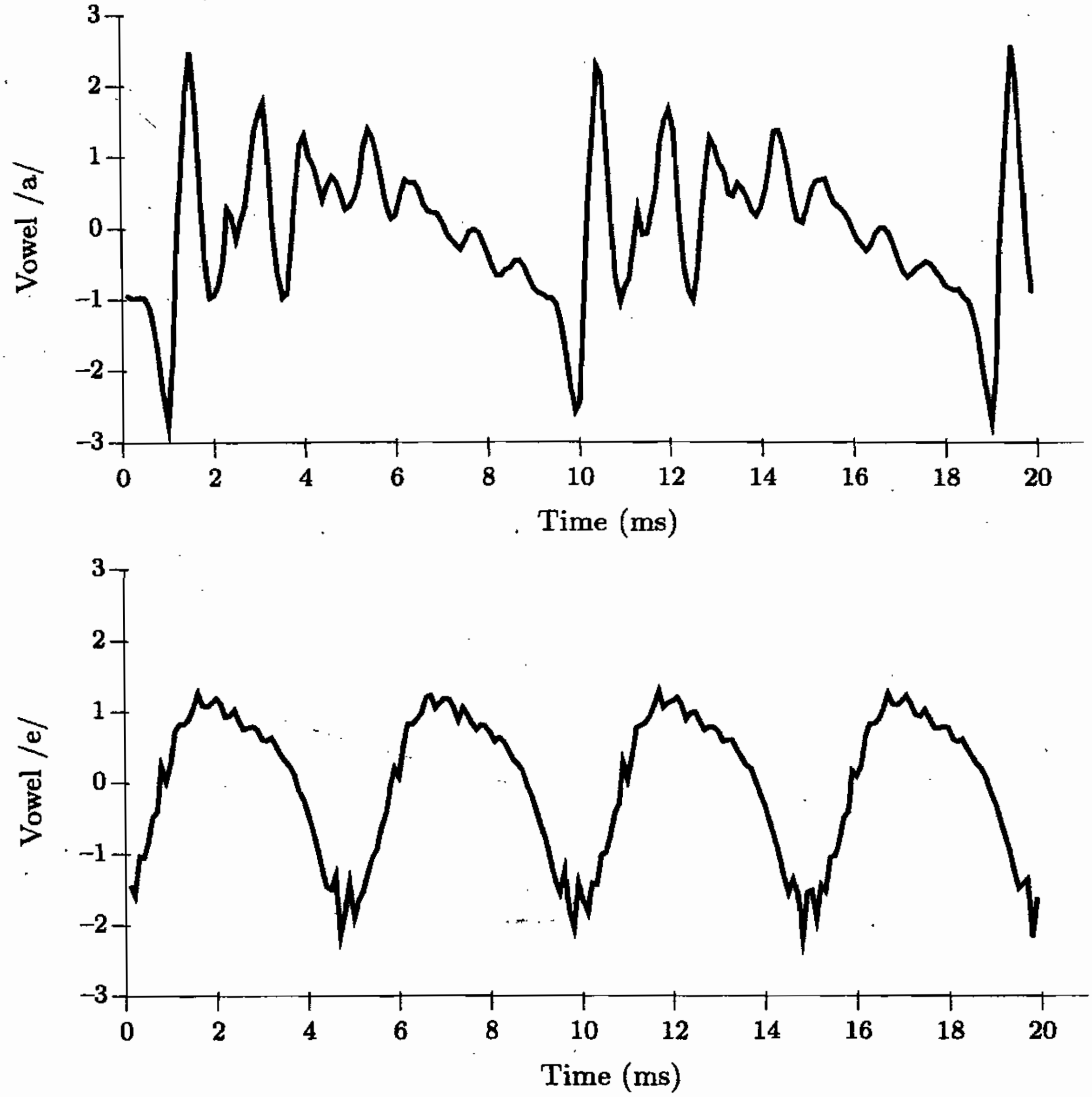



**Figure 1.3** Examples of speech sounds

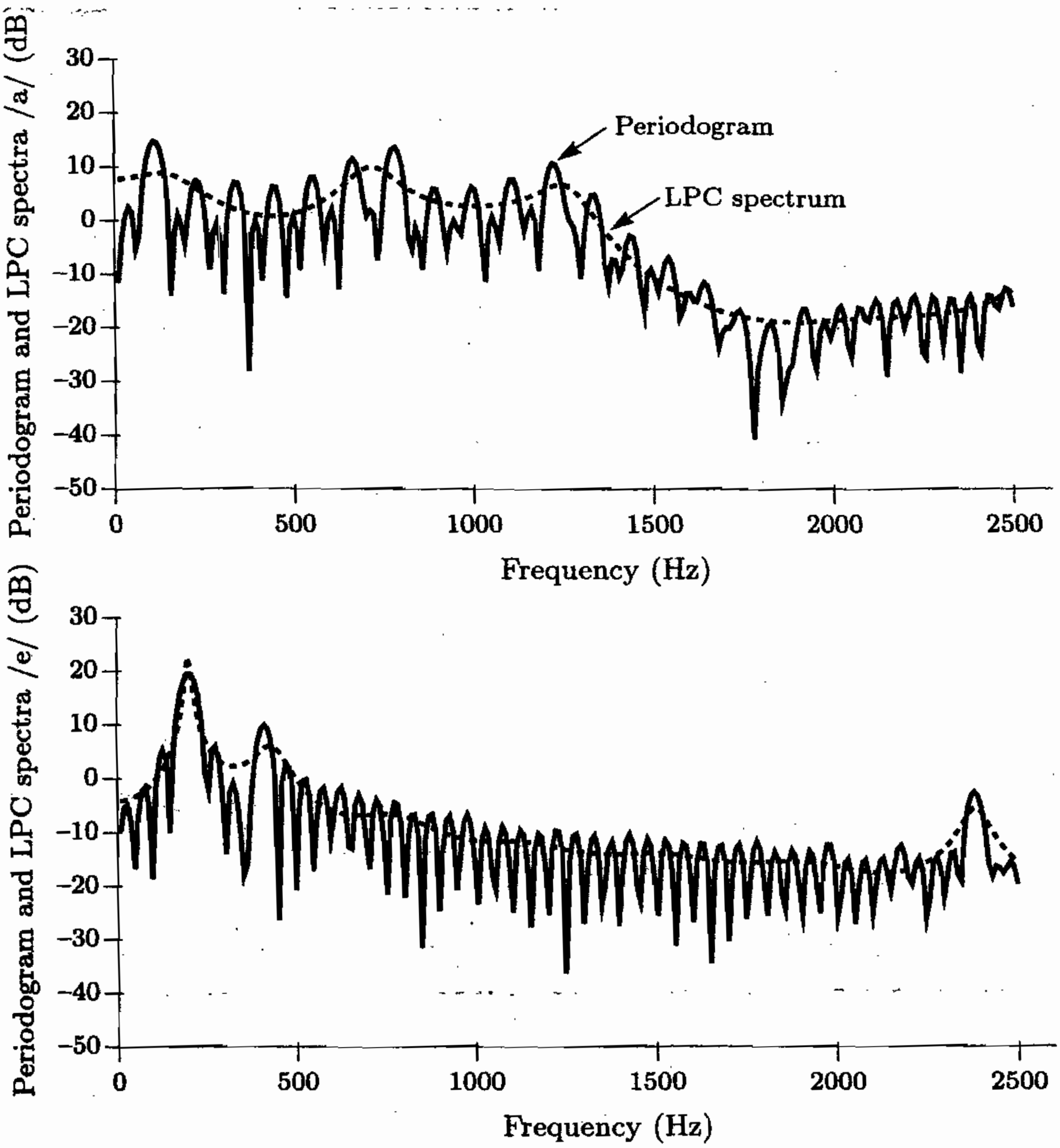



**Figure 1.4** LPC spectral modeling

# Chapter 0

## Revisit of Probabilities and Random Processes

Reference Book: A. Papoulis and S.U. Pillai, Probability, Random Variables and Stochastic Processes.

※ Borel fields:

Suppose that $A_1, A_2, \cdots, A_n, \cdots$ is an infinite sequence of sets in $F$ (the field of nonempty sets). If the union and intersection of these sets also belongs to $F$, then $F$ is called a Borel field,

where If $A \in F$ then $\overline{A} \in F$;

if $A \in F$ and $B \in F$ then $A \cup B \in F$ and $A \cap B \in F$.

※ Axiomatic Definition of an Experiment

In the theory of probability, an experiment is specified in terms of the following concepts:

1. The set of all experimental outcomes.
2. The Borel field of all events of $S$.
3. The probabilitees of these events.

Usually the set $S$ is called the sample space, which identifies not only the certain event but also the entire experiment such that

$P(S) = 1$. (probability of the sample space)

※ Probability Axioms

$S$ is the sample space For any set $A \subset S$ specifying the event, we call the probability of event $A$ as $P(A)$ being a number, such that the following conditions are satisfied:

I. $p(A) \geq 0$

II. $p(S) = 1$

III. If $AB \triangleq A \cap B = \{\phi\}$,

empty set

then $p(A \cup B) = p(A) + p(B)$,

where $B \subset S$.

※ Conditional Probability

The conditional probability of an event A assuming another event M, denoted by $p(A|M)$, is by definition as the ratio:

$$p(A|M) = \frac{p(AM)}{p(M)},$$

where $p(M) \neq 0$.

※ Random Processes

The random process speifies a repeated experiment which corresponds to one or multiple values at each trial.

The set of such values for an indefinite trial is called the random variable vector or random vector $\vec{\underline{X}}$. If there is only one value, it is a scalar or simply the random variable $\underline{X}$. Usually, we speify the real valued random variable $\underline{X}$ in regions such that a particular event

$$A \triangleq \{\underline{X} \leq x\}.$$

$\dot{X}$ Distribution and Density Functions

The elements of the set $S$ that are contained in the even $\{\underline{X} \leq x\}$ change as the number $x$ takes various values. The probability $P\{\underline{X} \leq x\}$ of the event $\{\underline{X} \leq x\}$ is, therefore, a number that depends on $x$. This number is denoted by $F_{\underline{X}}(x)$ and is called the (cumulative) distribution function of the random variable $X$.

Definition: The distribution of the random variable $X$ is the function

$$F_X(x) \triangleq P\{X \le x\}, \quad -\infty < x < \infty.$$

Since there is no ambiguity, we omit the subscripts $X, Y, Z$, such that

$$F_X(x) \triangleq F(x)$$

$$F_Y(y) \triangleq F(y)$$

$$F_Z(z) \triangleq F(z).$$

Properties of Distribution Functions

1. $F(\infty) = 1, \quad F(-\infty) = 0$
2. if $x_1 < x_2$, then $F(x_1) < F(x_2)$
3. if $F(x_0) = 0$ then $F(x) = 0$ for $x \le x_0$
4. $F(X > x) = 1 - F(x)$
5. $F(x^+) = F(x)$

$$\triangleq \lim_{\varepsilon \to 0} F(x+\varepsilon)$$

6. $P\{x_1 < X \le x_2\} = F(x_2) - F(x_1)$

7. $P(X = x) = F(x) - F(x^-)$

8. $P(x_1 \le X \le x_2) = F(x_2) - F(x_1^-)$

※ Probability Density Function (PDF)

The derivative of the probability distribution function $F_X(x)$ is called the probability density function $f_X(x)$ of the random variable $X$. Thus

$$f_X(x) \triangleq \frac{dF_X(x)}{dx}$$

Since

$$\frac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} \ge 0,$$

it follows that

$$f_X(x) \ge 0, \quad \text{for all } x.$$

We can obtain

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

and $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Also, we get

$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$$
$$= \int_{x_1}^{x_2} f_X(x)\,dx.$$

X. Speific Random Variables

Continuous-type Random Variables

1. Normal (Gaussian) Distribution

We say that $X$ is a normal or Gaussian random variable with parameters $\mu$ (mean), and $\sigma^2$ (variance), whose PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where the corresponding CDF is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$
$$\triangleq Q\left(\frac{x-\mu}{\sigma}\right)$$

and the Gaussian-tailed function $Q(x)$ is defined as

$$Q(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

2. Exponential Distribution

We say $X$ is exponential with parameter $\lambda$ if the PDF is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda} \quad (\text{mean free path}).$$

where the corresponding CDF is

$$F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$$

Memoryless Property of exponential distributions:

Let $s, t \geq 0$. Consider two events $\{X > t+s\}$ and $\{X > s\}$. Thus,

$$P\{X > t+s \mid X > s\} = \frac{P\{X > t+s\}}{P\{X > s\}} = \frac{e^{-(t+s)}}{e^{-s}}$$

$$= e^{-t} = P\{X > t\}.$$

Hence, $P\{X > t+s \mid X > s\} = P\{X > t\}$

3. Gamma Distribution

We say $X$ to be a Gamma random variable with parameter $\alpha, \beta$, for $\alpha, \beta > 0$, if the PDF is

$$f_X(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}, & x \geq 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$.

If $\alpha$ is an integer, we get

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!.$$

4. Chi-Square Distribution

We say $X$ to be the $\underline{\chi^2(n)}$ random variable

chi-square with degree of freedom $n$

if the PDF is

$$f_X(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

※ $n=2$, $\chi^2(2)$ is an exponential distribution.

5: Rayleigh Distribution

We say $X$ to be a Rayleigh distribution with parameter $\sigma^2$ if the PDF is

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

6. Nakagami-m Distribution

A generalization to the Rayleigh distribution with an addition parameter $m$ is given by the Nakagami distribution whose PDF is

$$f_X(x) = \begin{cases} \dfrac{2}{\Gamma(m)} \left(\dfrac{m}{\Omega}\right)^m x^{2m-1} e^{-\frac{mx^2}{\Omega}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Notice that $m=1$ corresponds to the Rayleigh distribution. For $m<1$, the tail distribution decays slowly compared to the Rayleigh distribution while it decays faster for $m>1$.

7. Uniform distribution

We say $X$ to be uniformly distributed in the interval $(a,b)$, $-\infty<a<b<\infty$, if the PDF is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

※ Disnete - Type Random Variables

1. Bernouli Distribution

We say $X$ to be Bernoulli distributed if $X$ takes the values 1 or 0 and

$$P\{X=1\} = p \quad \text{and} \quad P\{X=0\} = q = 1-p$$

2. Binomial Distribution

We say $Y$ to be a Binomial distribution if $Y$ takes the values from $0, 1, 2, \ldots, n$

with

$$P\{Y=k\} = \binom{n}{k} p^k q^{n-k}, \quad p+q=1, \quad k=0,1,2,\ldots n$$

3. Poisson Distribution

$X$ is said to be a Poisson random variable with parameter $\lambda$ if $X$ takes the values $0, 1, 2, \cdots, \infty$, with

$$P\{X=k\} = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\ldots,\infty$$

$P_k \triangleq P\{X=k\}$ ; then

$$\frac{P_{k-1}}{P_k} = \frac{e^{-\lambda}\lambda^{k-1}/(k-1)!}{e^{-\lambda}\lambda^k/k!} = \frac{k}{\lambda}$$

If $\lambda < 1$, $P_k$ is maximum for $k=0$;

If $\lambda > 1$, $\lambda \notin Z^+$, $P_k$ increases as $k$ increases, and reach its maximum for $k = \lfloor\lambda\rfloor \leftarrow$ integer rounding down;

If $\lambda \in Z^+$, $P_k$ is maximum for $k=\lambda-1$ and $k=\lambda$ ($P_\lambda = P_{\lambda-1}$).

※ Joint Distribution

Let $\overrightarrow{X}_N$ denote a random vector such that $\overrightarrow{X}_N = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$ ; assume that the PDF for each random element $X_i$, $i=1,2,\dots,N$ is

$$f_i(x_i), \quad i=1,2,\dots,N.$$

The joint PDF can be denoted as

$$f_{\overrightarrow{X}}(x_1, x_2, \dots, x_N)$$

If $X_i$ are statistically independent of each other, then

$$f_X(x_1, x_2, \dots, x_N) \triangleq f_{\overrightarrow{X}}(\overrightarrow{x}) = \prod_{i=1}^{N} f_i(x_i)$$

The marginal PDF can be calculated as

$$f_{X_i}(x_i) = \underbrace{\iint \cdots \int}_{\substack{(N-1)\text{-tuple} \\ \text{integration}}} f_{\overrightarrow{X}'}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_N)\, dx_1\, dx_2 \cdots dx_N$$

## X. Statistical Expectation

Given a random variable $X$ together with its PDF $f_X(x)$, we can establish the statistical expectation $E\{g(x)\}$ as

$$E\{g(x)\} \triangleq \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Example: Given a Gaussian random variable, Calculate the expectation $E\{x\}$ (mean).

Solution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$g(x) = x$$

$$E\{x\} = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu$$

Example: Given a Gaussian random variable, Calulate the central variance $E\{(x-\mu)^2\}$

Solution:

$$g(x) = (x-\mu)^2$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E\{(x-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \sigma^2$$

Example: Given a Gaussian random variable, calculate the odd-central moments $E\{(x-\mu)^p\}$, where $p = 1, 3, 5, 7, \ldots$

Solution:

$$g(x) = (x-\mu)^p, \quad p = 1, 3, 5, \ldots$$

$$E\{(x-\mu)^p\} = \int_{-\infty}^{\infty} \underbrace{(x-\mu)^p}_{\substack{\text{odd function} \\ \text{with respect} \\ \text{to } x=\mu}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{\substack{\text{even function} \\ \text{with respect} \\ \text{to } x=\mu}} dx$$

$$= 0, \quad p = 1, 3, 5, 7, \ldots$$

Given a random vector $\vec{X}$, together with the joint PDF $f_{\vec{X}}(\vec{x})$, we can estimate the expectation

$$\underset{\substack{\downarrow \\ \text{scalar, vector, matrix}}}{g(\vec{x})} = \underbrace{\iint \cdots \int}_{N\text{-tupe integration}} g(\vec{x}) \, f_{\vec{X}}(\vec{x}) \, d\vec{x}$$

Example: Given the multi-variate Gaussian random vector $\vec{X}$, calculate the mean vector $E\{\vec{X}\}$ and covariance matrix $E\{(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^H\}$

Solution: The multivariate Gaussian PDF is

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{2\pi}^N \sqrt{|\tilde{\Sigma}|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^H \tilde{\Sigma}^{-1} (\vec{x}-\vec{\mu})\right\}$$

where $\vec{\mu} \triangleq E\{\vec{x}\}$, $\tilde{\Sigma} \triangleq E\{(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^H\}$

$$E\{\vec{x}\} = \underbrace{\iint \cdots \int}_{N\text{-tuple}} \vec{x} \, \frac{1}{\sqrt{2\pi}^N \sqrt{|\tilde{\Sigma}|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^H \tilde{\Sigma}^{-1} (\vec{x}-\vec{\mu})\right\} d\vec{x}$$

$$= \vec{\mu}$$

$$E\{(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^H\} = \underbrace{\iint \cdots \int}_{N\text{-tuple}} (\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^H f_{\vec{X}}(\vec{x}) \, d\vec{x}$$

$$= \tilde{\Sigma}$$