Chapter 4: Problem Solutions

Digital Filters

Problems on Non-Ideal Filters

Problem 4.1

We want to design a Discrete Time Low Pass Filter for a voice signal. The specifications are:

- Passband $P_0 = 4 \text{ kHz}$, with 0.8 dB ripple;
- Stopband $P_S = 4.5 \text{ kHz}$, with 50dB attenuation;
- Sampling Frequency $F_s = 22 \text{ kHz}$.

Determine a) the discrete time Passband and Stopband frequencies, b) the maximum and minimum values of $|H(\omega)|$ in the Passband and the Stopband, where $H(\omega)$ is the filter frequency response.

Solution

a) Recall the mapping from analog to digital frequency $\omega = 2\pi f / F_s$, with $F_s$ the sampling frequency. Then the passband and stopband frequencies become $\omega_p = 2 \pi 4 / 22 \text{ rad} = 0.36 \pi \text{ rad}$, $\omega_s = 2 \pi 4.5 / 22 \text{ rad} = 0.41 \pi \text{ rad}$.

b) A 0.8 dB ripple means that the frequency response is the passband is within the interval $1 \pm \delta$ where $\delta$ is such that $10 \log_{10} (1 + \delta) = 0.8$. This yields $\delta = 10^{0.04} - 1 = 0.096$. Therefore the frequency response within the passband is within the interval $0.9025 < |H(\omega)| < 1.096$. Similarly the stopband the maximum value is $|H(\omega)| < 10^{-0.5 \delta} = 0.0025$.

Problem 4.2

A Digital Filter has frequency response $H(\omega)$ such that

\[0.95 < |H(\omega)| \leq 1.05 \quad \text{for} \quad 0 \leq \omega \leq 0.3 \pi\]
\[0.0 \leq |H(\omega)| \leq 0.065 \quad \text{for} \quad 0.4 \pi \leq \omega \leq \pi\]
Also let the sampling frequency be \( f_s = 8 \text{ kHz} \). Determine the Passband and Stopband frequencies in kHz, the Passband ripple and the Stopband attenuation in dB.

**Solution**

The passband ripple is given by \( 20 \log_{10} (1.05) = 0.42 \text{ dB} \) and the attenuation in the stopband \( -20 \log_{10} 0.005 = 46 \text{ dB} \). The analog passband frequency is \( 0.3 \pi f_s / 2 \pi = 1.2 \text{ kHz} \) and the stopband \( 0.4 \pi f_s / 2 \pi = 1.6 \text{ kHz} \)

**Problem 4.3**

A continuous time filter has frequency response

\[
H (F) = \frac{1}{1 + 10^F}
\]

Determine the passband and stopband frequencies in Hz, assuming a passband ripple of 1dB and attenuation of 40dB in the stopband. Also determine the half power frequency \( f_c \).

**Solution.**

A passband ripple of 1dB means that the frequency response is within the interval \( 1 - \delta \leq |H(F)| \leq 1 + \delta \) with \( 20 \log_{10} (1 + \delta) = 1 \), which yields \( \delta = 0.12 \). Since

\[
|H(F)| = \frac{1}{\sqrt{1 + (10^F)^2}}
\]

then we determine the passband from the equation

\[
\frac{1}{\sqrt{1 + (10^F)^2}} = 1 - 0.12 = 0.88
\]

which yields \( F = 85.9 \text{Hz} \). Similarly for the stopband, we need to determine the frequency where

\[
|H(F)| = 1 - 40/20 = 0.01 \text{ which yields } F = 15.914 \text{ Hz}
\]

Notice that this filter has a very long transition region, as we can see from the plot of its magnitude.
b) Since the transfer function is of the form
\[ H(z) = \frac{1}{\beta} \frac{1 - e^{-2\pi i}}{z^N (1 - e^{-2\pi i})} \]
the zeros are of the form \( z = e^{2\pi i k}, k = 1, \ldots, N = 1 \) and the poles are all at \( z = 0 \).

c) Since \( y[n] = \frac{1}{\beta} \left( x[n-1] + \ldots + x[n-N] \right) \) and
\( y[n-1] = \frac{1}{\beta} \left( x[n-2] + \ldots + x[n-N-1] \right) \) by comparing \( y[n] \) and \( y[n-1] \), we see that
\[ y[n] = y[n-1] + \frac{1}{\beta} x[n-1] - \frac{1}{\beta} x[n-N-1] \]
This yields the transfer function:
\[ H(z) = \frac{1}{\beta} \frac{1 - e^{-2\pi i}}{1 - e^{-2\pi i}} = \frac{1}{\beta} \frac{1 - e^{2\pi i}}{1 - e^{2\pi i}} \]
as we saw before. This is an example of a recursive filter with finite impulse response (FIR).

**Problems on FIR Filters**

- **Problem 4.6**

  We want to design a Low Pass FIR Filter with the following characteristics:
Passband 10kHz.
Stopband 1kHz, with attenuation of 50dB.
Sampling frequency 44kHz

Determine the causal impulse response $h[n]$, and an expression for the phase within the passband. Use one of the standard windows listed in section 4.3.

**Solution**

First we have to determine the specifications in the digital freq. domain.

**Passband:** \( \omega_p = \frac{2 \pi 10}{44} = 0.4545 \pi \text{ rad} \)

**Stopband:** \( \omega_s = \frac{2 \pi 11}{44} = 0.5 \pi \text{ rad} \)

Therefore we choose the passband of the ideal filter as \( \omega_c = \frac{1}{2} (\omega_p + \omega_s) = \frac{45}{44} \pi \approx 0.477 \pi \). We need a Blackman window to satisfy the 50dB attenuation in the stopband. With this window the transition region has a width of \( 12 \pi / N \). Since we want a transition region \( \omega_s - \omega_p = 2 \pi / 44 \) we determine the filter length \( N \) as

\[
\frac{2 \pi}{44} \approx 12 \frac{\pi}{N}
\]

which yields \( N \approx 12 \times 22 = 264 \). Therefore we choose \( N = 265 \) and a shift \( L = 132 \). Finally the impulse response is

\[
h[n] = h_d[n - 132] \cdot \text{Blackman}[n] = \frac{\sin(0.4545 \pi (n-132))}{(n-132)} \cdot \text{Blackman}[n]
\]

which is shown below.
Within the passband the phase is linear and it is given by the expression

$$\angle H(\omega) = -\omega L = -132 \omega$$

**Problem 4.7**

Repeat Problem 2.1 with an equiripple filter using the "remez" function in Matlab. Plot the two frequency responses and compare the two filters in terms of performance and complexity.

**Solution**

With Matlab we need first to determine the order of the filter. Use the function "remezord" as follows:

```matlab
[N, fo, mo, k] = remezord([10000, 11000], [1, 0], [delta, delta], 44000);
```

with `delta = 10^-(-50,-20)` the maximum deviation corresponding to 50dB's. This yields an order $N = 114$, in the sense that the transfer function is of the form

$$H(z) = h[0] + h[1] z^{-1} + \ldots + h[114] z^{-114}$$

The impulse response $h[n]$ is obtained as

```matlab
h = remez(N, fo, mo, k);
```

where $fo$, $mo$ and $k$ are from remezord.
Notice that the order of the equiripple filter $N = 112$ is considerably smaller than the order of the filter designed with the Blackman window in Problem 4.6.

**Problem 4.8**

Repeat Problem 4.6 using the Kaiser window.

**Solution**

The Kaiser window we have to determine the parameters $N$ and $\beta$ from the specifications. In particular, we want an attenuation $A = 50$ dB which yields a factor $\beta$ from the expression

\[
\beta = 0.5842 (A - 21)^{0.4} + 0.07886 (A - 21) = 4.53
\]

The filter length is determined from the expression

\[
N \geq \frac{L}{2.285 \Delta f} = \frac{L}{2.285 \frac{1}{T_{90\%}}} = 128.717
\]

We can choose $N = 129$ and $L = 64$. The frequency response of the filter therefore becomes

\[
\frac{\sin (0.4545\pi \cdot n/64)}{n/n - 64} \cdot \text{Kaiser}[n]
\]

Amplitude is shown below.
Problems on IIR Filters

Problem

Using the Bilinear Transformation, determine the order $N$ and the cut off frequency $\Omega_c$ of the analog prototype filter for the following discrete time design:

a) passband 8 kHz;
b) stopband 9 kHz;
c) passband ripple 0.5 dB;
d) stopband attenuation 40 dB;
e) sampling frequency $f_s = 44$ kHz.

Solution

First we define the problem in the digital frequency domain:

$$\omega_p = \frac{\omega_c}{f_s} = 1.1424 \text{ rad}$$
$$\omega_s = \frac{2\pi}{f_s} = 1.2852 \text{ rad}$$

Then we determine the specifications of the Analog Prototype:

$$\Omega_p = f_s \tan \left( \frac{\frac{\pi}{4}}{2} \right) = 56554.2 \text{ rad/ sec}$$
$$\Omega_s = f_s \tan \left( \frac{\frac{3\pi}{4}}{2} \right) = 65876.0 \text{ rad/ sec}$$

Now from the Passband Ripples we determine $\epsilon$ as

$$\Delta_p = 10^{-0.5/20} = 0.944$$
$$\epsilon = \sqrt{(\frac{1}{\Delta_p})^2 - 1} = 0.349$$

Then we determine the order of the filter
which yields \( N = 38 \). Finally, we determine the cut-off frequency of the filter as

\[ \Omega_c = \frac{\xi D}{\pi^{1/2}} = 581.41 \text{ rad/sec} \]

and therefore its frequency response becomes

\[ H(\Omega) = \frac{\Omega}{1 + \left(\frac{\Omega}{\Omega_c}\right)^2} \]

The following plot shows the poles of the filter in the s-plane.

The frequency response (magnitude only) is shown next:
Problem 4.15

A 4-th order Butterworth filter has cut off frequency \( \omega_c = 200 \pi \text{ rad/sec} \).

a) Determine the zeros and poles of the transfer function;

b) What would be its passband and stopband frequencies if we want 1dB ripple in the passband and 40dB attenuation in the stopband?

c) If we apply a Bilinear Transformation with sampling frequency \( F_s = 1 \text{ kHz} \), determine the zeros and poles in the z-plane.

Solution

a) Recall that an \( N \)-th order Butterworth Filter has poles on a circle with radius \( \Omega_p = 200 \pi \text{ rad/sec} \) spaced by an angle of \( 360 / 2N = 360 / 8 = 45 \text{ degrees} \). The poles are shown in the figure below.
and they are given by $\rho_1, s = 200 \pi e^{j57.78}$ and $\rho_2, s = 200 \pi e^{j214.8}$, All zeros are at $s = \infty$.

b) From the frequency response $H(s) = \frac{1}{\sqrt{1 + \frac{1}{s^2} \rho^2}}$ we solve for $\Omega_p$ and $\Omega_s$ as

$$H(\Omega_p) = 10^{-120} = 530.673 \text{ rad/sec}$$
$$H(\Omega_s) = 10^{-45/20} = 1986.83 \text{ rad/sec}$$

c) Applying the formula for the Bilinear Transformation each pole is mapped as

$$z = \frac{s + \frac{1}{\rho}}{s - \frac{1}{\rho}}$$

This yields poles in the $z$-plane at $0.673045 \pm j0.633479$ and $0.53679 \pm j0.143193$ and four zeros at $z = 1$.

### Problem 4.14

Repeat Problem 4.14 using a Chebyshev filter. Which one would you choose if complexity is an issue?

**Solution**

The specifications of the prototype filter are the same, since they depend on the original specifications of the filter and on the bilinear transformation. Recall them here for convenience:

$$\omega_p = \frac{\pi \omega}{\omega_c} = 1.1424 \text{ rad}$$
$$\omega_s = \frac{2 \pi \omega}{\omega_c} = 1.2853 \text{ rad}$$

$$\Omega_p = \omega_p 2 \tan \left( \frac{\pi}{2} \right) = 56554.2 \text{ rad/sec}$$
$$\Omega_s = \omega_s 2 \tan \left( \frac{\pi}{2} \right) = 55876.0 \text{ rad/sec}$$
\[ \delta_p = 10^{-1/20} = 0.944 \]
\[ \delta_s = 10^{-40/20} = 0.01 \]
\[ e = \sqrt{\left( \frac{1}{\delta_p} \right)^2 - 1} = 0.349 \]

The formulas for the Chebyshev Filter, from section 4.4, to obtain \( N = 12 \) (the complexity of the frequency response as shown below).

The order is determined from the formula

\[ N = \frac{\log \left( \frac{\sqrt{1 - \delta_p^2} + \sqrt{\delta_s^2 - \delta_p^2}}{\delta_s + \sqrt{\left( \frac{\delta_s}{\delta_p} \right)^2 - 1}} \right)}{\log e} \]

As \( N = 12 \). Then the poles of the filter in the s-plane are computed as

\[ \beta = \left( \frac{\sqrt{\delta_p^2 + 1} + 1}{\delta_p} \right)^{1/2N} \]
\[ r_1 = \frac{\Omega_p (\beta^2 + 1)}{2 \beta} \]
\[ r_2 = \frac{\Omega_p (\beta^2 - 1)}{2 \beta} \]

The \( \left( \frac{\pi (2k + 1)}{2N} \right) + j r_1 \sin \left( \frac{\pi (2k + 1)}{2N} \right) \), \( k = 0, \ldots, N-1 \)

The poles in the s-plane are shown below. Notice the two different scales for the Real and Imaginary
The plot of the frequency response is shown below. If you compare it with the Butterworth filter in Problem 4.14, notice that you obtain the same attenuation with a lower complexity \(2^7 = 128\) for Chebyshev and \(N = 38\) for Butterworth.

\[
\begin{array}{cccccc}
20000 & 45000 & 80000 & 120000 & 180000 \\
-20 & -40 & -60 & -80 & -100
\end{array}
\]

### Problem 4.17

We want to implement the analog filter with transfer function

\[ H(s) = \frac{-2s+1}{s^2 + s + 1} \]

by a discrete time approximation, using the Bilinear Transform method. Let \( F_s = 1.0 \text{ Hz} \) be the sampling frequency.

a) Determine zeros and poles of both the analog filter \( H(s) \) and the discrete time implementation \( H(z) \);

b) Determine the Linear Difference Equation of the discrete time implementation;

c) Plot the frequency responses of the digital filter \( H(z) \) and the analog filter \( H(s) \). Verify that you can obtain one from the other by the appropriate frequency transformation.

**Solution**

a) The zeros of the analog system are \( s = -1/2 \), and \( s = \infty \) (yes this is a zero too). The poles are the solution of \( s^2 + s + 1 = 0 \) which yields \( s = e^{\frac{\pm \sqrt{3}i\pi}{3}} \).

Applying the mapping \( s \rightarrow z \)

\[ z = \frac{s + \frac{\sqrt{3}}{2}}{s - \frac{\sqrt{3}}{2}} \]

we can verify that the zeros are mapped as
Problem 4.19

You want to design an analog Band Pass Filter which passes the frequencies in the interval

\( 5 \text{ kHz} \leq f \leq 6 \text{ kHz} \)

with 1dB ripple in the passband. Let the filter be Butterworth with order \( N = 4 \).

a) Determine the frequency transformation Low Pass to Band Pass you would use;

b) Determine the frequency response of the corresponding Low Pass Filter, together with its zeros and poles;

c) Determine the zeros and poles of the Band Pass Filter and its transfer function.

Solution

From the specifications we determine the lower and upper frequencies as

\[
\Omega_L = 2\pi \times 5000, \quad \Omega_H = 2\pi \times 6000
\]

Then we choose a prototype Butterworth filter of order \( N = 4 \) and with arbitrary cut-off frequency, say

\( \Omega_0 = 1 \)

which has poles at

\[
P_k = \Omega_0 e^{\frac{k\pi j}{N}}, \quad k = 0, 1, 2, 3
\]

In order to apply the proper transformation (Low Pass to Band Pass)

\[
q(s) = \frac{2}{\Omega_0} \frac{(s^2 + \Omega_L \Omega_0)}{s(\Omega_H - \Omega_L)}
\]
(b) Butterworth Low pass filter prototype

\[ |H_p(\omega)|^2 = \frac{1}{1+\omega^2} \]

\[ |H_p(s)| = |H_p(-s)| = |H_p(\omega)|^2 \left|_{\omega = \frac{\omega_0}{\sqrt{2}}} \right. \]

\[ = \frac{1}{s^2 + 1} \]

The poles \( P_k = e^{j \frac{2\pi k + 1}{8}}, \quad k = 0, 1, \ldots \)

\[ P_k = e^{j \frac{5\pi}{8}}, e^{j \frac{9\pi}{8}}, e^{j \frac{13\pi}{8}}, e^{j \frac{17\pi}{8}} \]

The corresponding poles in the \( H_p(s) \)

are the roots of

\[ P_k' = \frac{\sqrt{2}}{s} \left( s^2 + \omega_0 s \right) = \frac{s^2 + \omega_0 s}{s \left( s^2 - \omega_0 s \right)} \]

\[ S = \frac{P_k' \left( s^2 - \omega_0 s \right) + \sqrt{P_k'^2 \left( s^2 - \omega_0 s \right)^2 - 4P_k' \omega_0 s}}{4} \]

\[ = \text{poles} \]
we compute the poles of the bandpass filter from the equations

\[ q(z) = p(z), \quad k = 0, 1, 2, 3 \]

Each equation is quadratic and it yields two solutions. As a total we have \(2 \times 4 = 8\) poles for the bandpass filter which are given by

\[
\text{poles} = \{-1303.33 + 37418.3i, -1101.24 - 31613.4i,
-3608.15 + 35615.3i, -2800.76 - 33110.8i, -3004.15 - 35515.3i,
-2800.76 + 33110.8i, -1303.33 - 37418.3i, -1101.24 + 31613.4i\}
\]

The filter has also four zeros at \(\pi = 0\), due to the fact that \(q(0) = 0\). From the zeros and the poles we determine the transfer function. The magnitude of the frequency response is shown below.
Chapter 5: Problem Solutions
Digital Filter Implementation

State Space Realizations

Problem 5.1.

Problem

Given the system with transfer function

\[ H(z) = \frac{z^2 - 3}{z^2 - 2z + 1} \]

determine:

a) the difference equation relating the input \( x[n] \) to the output \( y[n] \);

b) a block diagram realization, together with its state space equations;

c) its diagonal state space realization.

A realization made of blocks of first and second order only.

Solution

a) Call \( x[n] \) and \( y[n] \) the input and output sequences respectively. Then the difference equation is


b) Block Diagram representation for a Type I realization:
The diagram can be determined the state space equations as

\[ \begin{align*}
    s_1[n+1] &= s_2[n] \\
    s_2[n+1] &= s_3[n] \\
    s_3[n+1] &= s_1[n] - s_2[n] - 2 s_3[n] + x[n] \\
    y[n] &= s_1[n] + 2 s_2[n]
\end{align*} \]

From these become

\[ \begin{bmatrix}
    s[n+1] \\
    x[n+1] \\
    y[n]
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 1 \\
    -1 & -1 & -2 \\
    1 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
    s[n] \\
    x[n] \\
    x[n]
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]

where \( s = (s_1, s_2, s_3) \) is the state vector.

The eigenvalues and eigenvectors of the matrix \( A \) are the following:

\[ \begin{align*}
    \lambda_1 &= -0.1226 + j0.7449, \\    \lambda_2 &= -0.1226 - j0.7449, \\    \lambda_3 &= -1.7549
\end{align*} \]

Calling matrices

\[ Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \]

The \( 3 \times 3 \) matrix of eigenvectors, the diagonal realization is given by the

\[ X = Q^{-1} A Q =
\begin{bmatrix}
    -0.1226 + j0.7449 & 0 & 0 \\
    0 & -0.1226 - j0.7449 & 0 \\
    0 & 0 & -1.7549
\end{bmatrix} \]

\[ B = Q^{-1} B = (0.3972 + j0.3395, 0.3972 - j0.3395, -1.1440) \]

\[ C = (-0.056) + j(0.2120, -0.0561 + j(0.1220, -0.5815) \]
As you notice the eigenvalues are complex and in general we want to avoid using complex operations if we do not need to. A better realization would be to define the transformation matrix $Q$ using the real and imaginary parts of the eigenvectors. In this way define

$$Q = \begin{pmatrix}
\text{Re}(q_1), & \text{Im}(q_1), & q_3
\end{pmatrix} = \begin{pmatrix}
-0.6526 & 0.2981 & -0.2715 \\
0.5052 & -0.4570 & 0.4765 \\
0.3032 & 0.2819 & -0.8362
\end{pmatrix}$$

and therefore we obtain

$$A = Q^{-1} A Q = \begin{pmatrix}
-0.1226 & 0.7449 & 0 \\
0.7449 & -0.1226 & 0 \\
0 & 0 & -1.7569
\end{pmatrix}$$

$$B = Q^{-1} B = (-0.1226, -0.5750, -1.1440)^T$$

$$C = C Q = (-0.9561, -1.2120, 0.6815)^T$$

d) Factor the transfer function in terms of zeros and poles

$$H(z) = \frac{2(z+1)}{(z-0.4522)(z-0.7400)(z-1.7549)} = \left(\frac{2(z+1)}{(z-0.4522)(z-0.7400)(z-1.7549)}\right)$$

Problem 5.2

Repeat problem 5.1 for the transfer function

$$H(z) = \frac{2(z+1)}{z^2+0.2452z+0.5659}$$

Solution

a) Difference equation


b) The matrices for the state space realization
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]
\[ B = (0, 1)^T \]
\[ C = (-2, 5) \]
\[ D = 0 \]

c) Eigenvalues and eigenvectors
\[ \lambda_1 = -0.5 + j0.866, \quad q_1 = [0.6124 - j0.3536, \quad j0.7071]^T \]
\[ \lambda_2 = -0.5 - j0.866, \quad q_2 = [0.6124 + j0.3536, \quad -j0.7071]^T \]

Then the diagonal realization
\[ \bar{A} = Q^{-1} A Q = \begin{bmatrix} -0.5 + j0.866 & 0 \\ 0 & -0.5 - j0.866 \end{bmatrix} \]
\[ \bar{B} = Q^{-1} B = (0.4082 - j0.7071, \quad 0.4082 + j0.7071)^T \]
\[ \bar{C} = C \bar{Q} = (-1.2247 + j4.2426, \quad -1.2247 - j4.2426) \]

d) The filter is already a second order system with complex poles, and it cannot be reduced any further.

Problem 5.3

Repeat problem 5.1 for the transfer function
\[ H(z) = \frac{2.16}{z^2 - 1.6 z^1 + 0.64} \]

Solution

a) The difference equation
\[ y[n] = 1.6 y[n-1] - 0.64 y[n-2] - 1.24 y[n-3] + 0.8192 y[n-4] + 2 x[n-2] + 2 x[n-3] + 2 x[n-4] \]

b) Matrices for the State Space equations:
\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8192 & -1.024 & -0.64 & 1.6 \end{bmatrix} \]
\[ B = (0, 0, 0, 1)^T \]
\[ C = (2, 2, 2, 0) \]

c) Eigenvalues and eigenvectors:
\( \lambda_1 = 0.8 + j0.8 \),
\( q_1 = [0.3487 - j0.2113, 0.4480 + j0.1099, 0.2704 + j0.4463, -0.1407 + j0.5734] \)
\( \lambda_2 = 0.8 - j0.8 \),
\( q_2 = [0.3487 + j0.2113, 0.4480 - j0.1099, 0.2704 - j0.4463, -0.1407 - j0.5734] \)
\( \lambda_3 = 0.8 \),
\( q_3 = [0.6577, 0.5262, 0.4209, 0.3367] \)
\( \lambda_4 = -0.8 \),
\( q_4 = [0.6577, -0.5262, 0.4209, -0.3367] \)

A real diagonal realization (with all real entries) is obtained from the transformation matrix

\[ Q = [\text{Re}(q_1), \text{Im}(q_1), q_3, q_4] \]

which yields

\[ \tilde{X} = Q^{-1} A Q = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \beta = Q^{-1} \beta = (-2.1353, 0.1735, 1.4848, -0.2970)^T \]

\[ \tilde{C} = C Q = (2.1343, 0.6901, 3.2096, 1.1049) \]

d) Factorize numerator and denominator:

\[ H(z) = \frac{jz^{2}+1}{(z^{2}-1.5z+1.29)(z^{2}-0.8)(z^{0.8})} = \frac{jz^{2}+1}{(z-0.5)(z-2)(z^{0.8})} \]

\[ H(\omega) = \frac{-j(\omega)^{2}+1}{(\omega^{2}-1.5\omega+1.29)(\omega^{2}-0.8)(\omega^{0.8})} \]

\[ H(\omega) \mid_{\omega=2} > 2 \]

Problem 5.4

You want design a Low Pass Filter by appropriately placing zeros and poles. You come up with the transfer function

\[ H(z) = K \frac{(z-0.5)^{2}}{(z-2)(z^{0.8})} \]

a) Determine the value of the gain \( K \) so that \[ |H(\omega)| \mid_{\omega=2} = 1 \];

b) Determine the difference equation associated to this system;

c) Determine a Type I state space realization;

d) In the transfer function, perturb the denominator coefficient of \( z^5 \) by any value of your choice smaller in magnitude than \( |0.5|^2 \). Is the system still stable? Would you trust this filter implementation on fixed point arithmetic?

e) Implement the filter as the cascade of low order sections (first or second order). How can you guarantee stability in the presence of numerical errors?