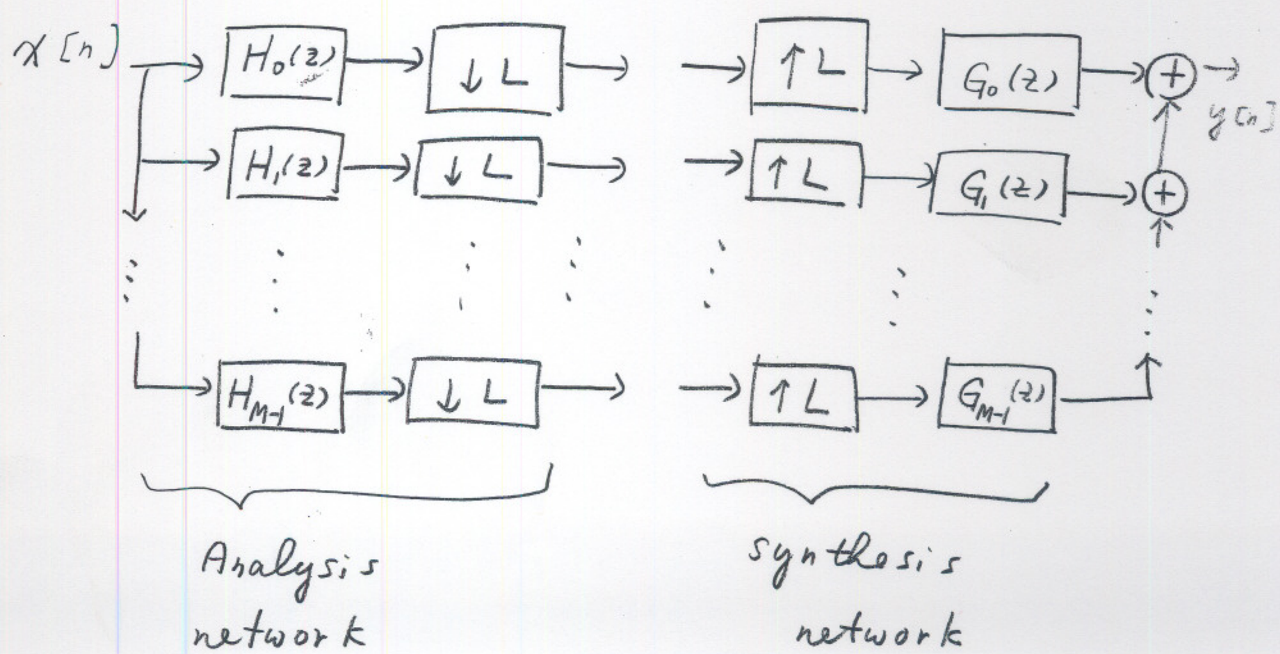
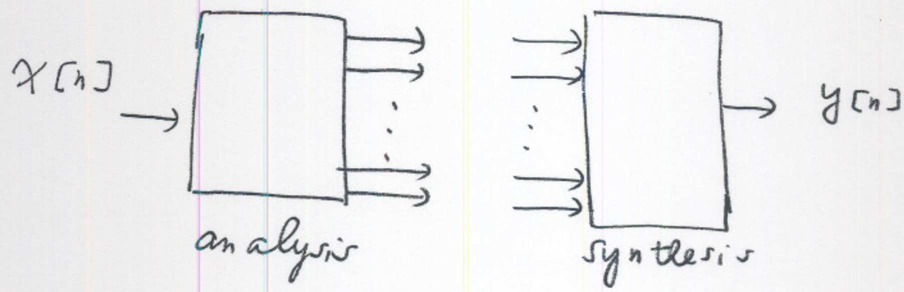
Chapter 7. DFT Filter Banks and Transmultiplexers

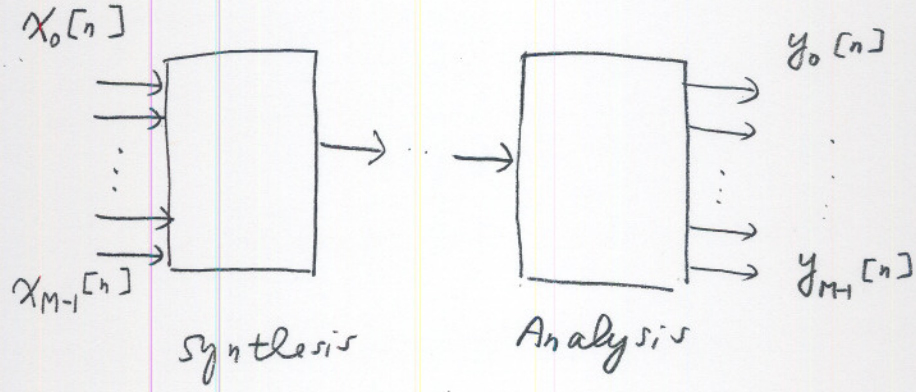
7.1 In several applications, we need to either decompose a signal into a number of components or to assemble a number of signals into one signal. These two operations are called analysis (one signal to many signals) and synthesis (many signals to one signal), respectively as shown in the following figure



If the network is in the order of analysis-then-synthesis, it is called the filter banks, otherwise, it is called the transmultiplexers (synthesis-then-analysis).



Filter Bank



Transmultiplexer

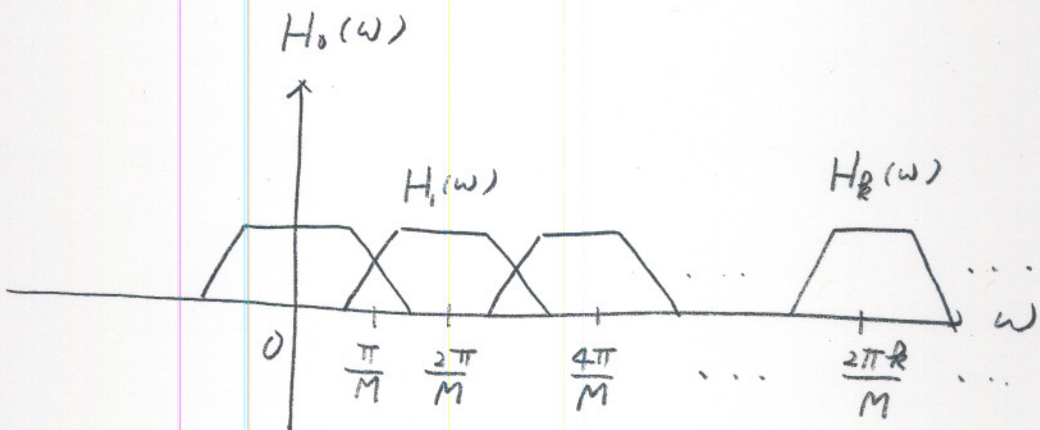
Application of filter banks are in instantaneous spectral decomposition and in digital communications

In a typical spectral decomposition problem, we subdivide the frequency spectrum into a number of bands, with the intent of determining the signal component within each band. The goal can be either to estimate a time-varying frequency spectrum or to encode/compress a signal based on the energy distribution among subbands.

On the other hand, the concept of transmultiplexing is the basis of digital modulations. The goal is to combine a number of digital signals (in general coming from different users) into one discrete-time signal to be transmitted over a single medium. A the receiver end, the signals are then separated and routed to the corresponding users.

7.2 DFT Filter Banks

The problem is the design of a network that decomposes the signal into M components within different, equally spaced frequency bands.



frequency responses of filter banks

Each filter is nonideal and assumed to be derived from a common prototype filter $H(\omega)$ such that

$$H_k(\omega) = H\left(\omega - \frac{2\pi k}{M}\right), \quad \text{for } k=0,1,2,\ldots,M-$$

that is, each filter is obtained by shifting the prototype in the frequency domain

The transfer function of each of these filters can be easily determined by

$$H_k(z) = H\left(e^{-j\frac{2\pi k}{M}} z\right)$$

$$= H\left(W_M^k z\right)$$

where $W_M = e^{-j\frac{2\pi}{M}}$, $\quad k = 0, 1, \ldots, M-1$

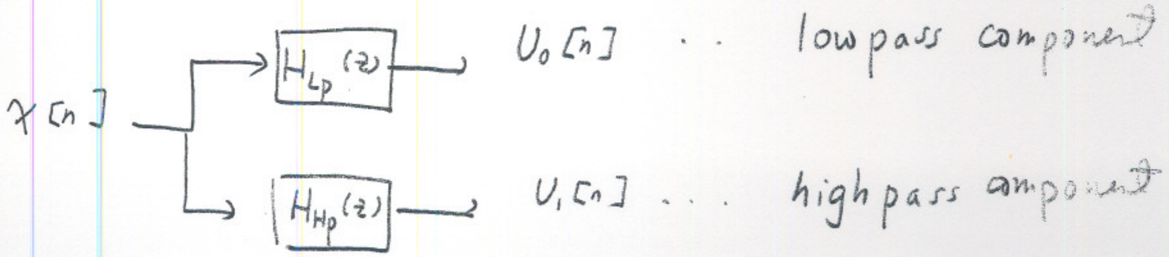
The design involves a lowpass filter or the prototype and then the rest of filters can be implemented very efficiently by an M-point FFT.

Example :

$$H_{Lp}(\omega) = H(\omega), \quad H_{Hp}(\omega) = H(W_2^1 z)\Big|_{z=e^{j\omega}}$$

$$= H(\omega - \pi)$$

form a 2-band filter bank.



Thus, $H_{LP}(z) = H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

$H_{HP}(z) = H(-z) = \sum_{n=-\infty}^{\infty} (-1)^n h[n] z^{-n}$

Using the polyphase decomposition, we have

$$H_{LP}(z) = \underbrace{\sum_{m=-\infty}^{\infty} h[2m] z^{-2m}}_{E_\phi(z^2)} + \underbrace{\sum_{m=-\infty}^{\infty} h[2m-1] z^{-2m+1}}_{z E_{-1}(z^2)}$$

$$H_{HP}(z) = \underbrace{\sum_{m=-\infty}^{\infty} h[2m] z^{-2m}}_{E_\phi(z^2)} - \underbrace{\sum_{m=-\infty}^{\infty} h[2m-1] z^{-2m+1}}_{z E_{-1}(z^2)}$$

This means that we simply have to implement the two subfilters $E_\phi(z^2)$ & $E_{-1}(z^2)$ separately and then sum the outputs to establish a lowpass filter and subtract the outputs to

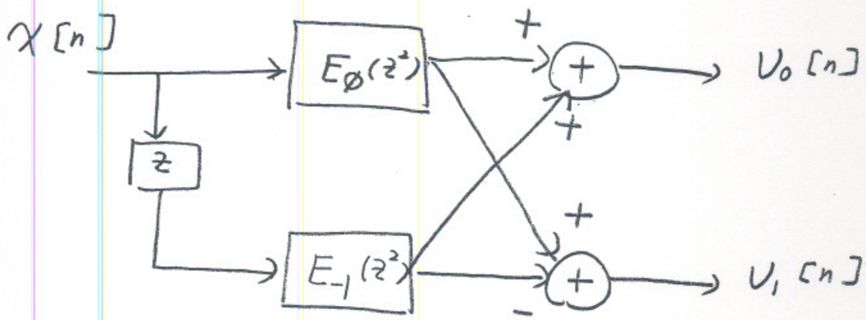
$$\begin{bmatrix} H(z) \\ H(-z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_\emptyset(z^2) \\ zE_{-1}(z^2) \end{bmatrix}$$

where $E_\emptyset(z^2) = \sum_{m=-\infty}^{\infty} h[2m]\, z^{-2m}$,

$$E_{-1}(z^2) = \sum_{m=-\infty}^{\infty} h[2m-1]\, z^{-2m}$$

and $H(z) = E_\emptyset(z^2) + z\, E_{-1}(z^2)$

This can be depicted as follows:



Implementation of polyphase filters. $(M=2)$

Example: If $E_\emptyset'(z^2) = \sum_{m=-\infty}^{\infty} h[2m]\, z^{-2m}$,

$$E_1'(z^2) = \sum_{m=-\infty}^{\infty} h[2m+1]\, z^{-2m},$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

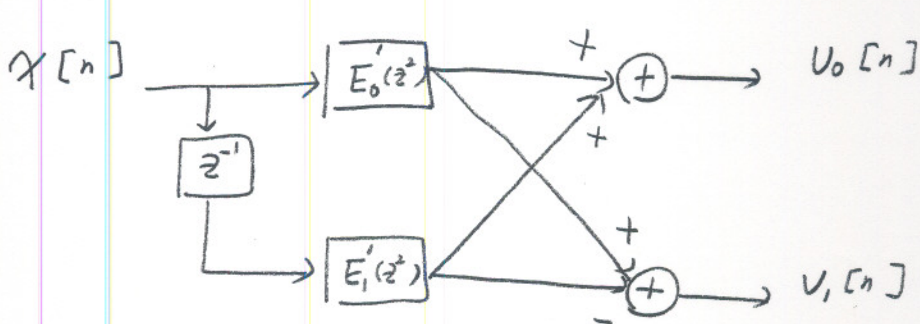
$$= \sum_{m=-\infty}^{\infty} h[2m]\, z^{-2m} + \sum_{m=-\infty}^{\infty} h[2m+1]\, z^{-2m-1}$$

$$= E_{\phi}'(z^2) + z^{-1} E_1'(z^2)$$

$$H_{LP}(z) = H(z) = E_{\phi}'(z^2) + z^{-1} E_1'(z^2)$$

$$H_{HP}(z) = H(-z) = E_{\phi}'(z^2) - z^{-1} E_1'(z^2)$$

Thus,



Hence, there are two choices of polyphase filter implementation.

The result can be generalized to a bank of M equally spaced bandpass filters with transfer functions $H_k(z) = H(W_M^k z)$,

for $k = 0, 1, \ldots, M-1$

First, we can decompose $H(z)$ into its polyphase components:

$$H(z) = \sum_{l=0}^{M-1} z^{l} E_{-l}(z^M)$$

where $$E_{-l}(z^M) = \sum_{m=-\infty}^{\infty} h[Mm - l]\, z^{-Mm}$$

Therefore, $$H_k(z) = H(W_M^k z)$$

$$= \sum_{l=0}^{M-1} z^{l} W_M^{kl} E_{-l}(z^M \underbrace{W_M^{kM}}_{=1})$$

$$= \sum_{l=0}^{M-1} z^{l} W_M^{kl} E_{-l}(z^M)$$

Since $W_M^{kM} = e^{-j\frac{2\pi kM}{M}} = 1$.

Thus, each filter $H_k(z)$ can be written in a matrix form:

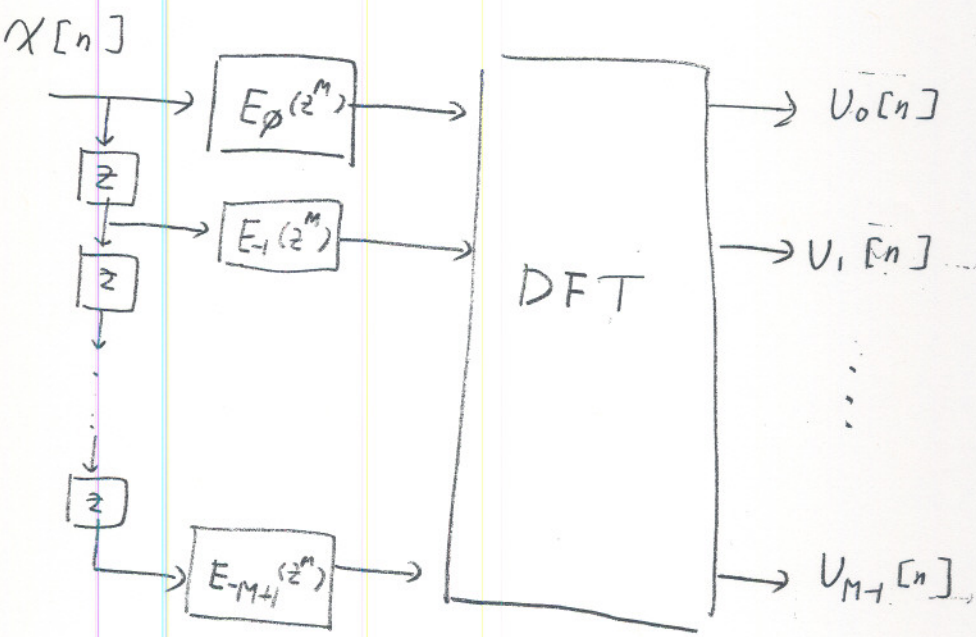
$$H_k(z) = [1 \;\; W_M^k \;\; \cdots \;\; W_M^{(M-1)k}] \begin{bmatrix} E_0(z^M) \\ z E_{-1}(z^M) \\ \vdots \\ z^{M-1} E\;\;(z^M) \end{bmatrix}$$

If we stack all filters $H_k(z)$, $k = 0, 1, \cdot\ M-1$,

we can form

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_M & \cdots & W_M^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{M-1} & \cdots & W_M^{(M-1)^2} \end{bmatrix}}_{\text{Fourier Matrix}} \begin{bmatrix} E_0(z^M) \\ z E_{-1}(z^M) \\ \vdots \\ z^{M-1} E_{-M+1}(z^M) \end{bmatrix}$$

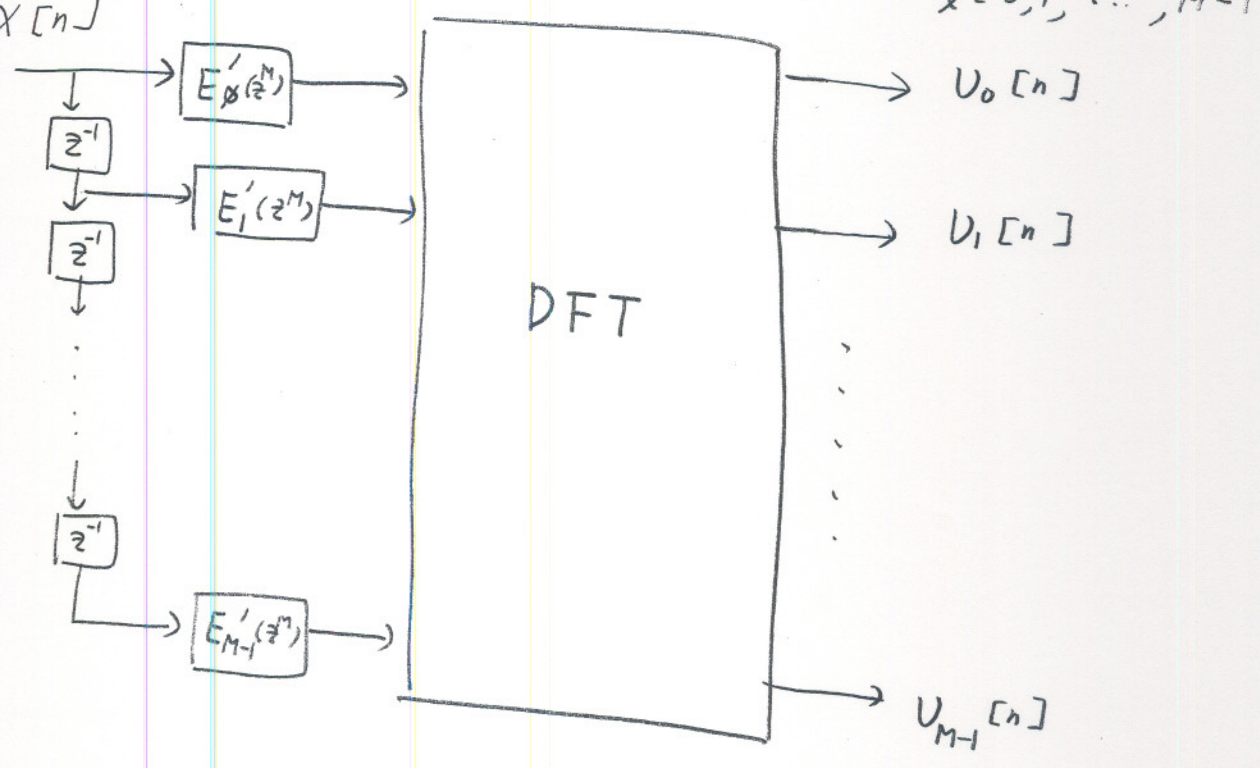
It can be implemented as follows:



$$
\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & W_M & \cdots & W_M^{M-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & W_M^{M-1} & \cdots & W_M^{(M-1)^2}
\end{bmatrix}
\begin{bmatrix} E'_0(z^M) \\ z^{-1}E'_1(z^M) \\ \vdots \\ z^{-M+1}E'_{M-1}(z^M) \end{bmatrix}
$$

where

$$
E'_l(z^M) = \sum_{m=-\infty}^{\infty} h[Mm+l]\, z^{-Mm}, \qquad l = 0, 1, \ldots, M-1
$$



mple: We want to decompose a signal into $M=4$ frequency components.

This can be done by choosing four filters with frequency response

$$H_k(\omega) = H(\omega - k\tfrac{\pi}{2}), \quad \text{for}$$

$k = 0, 1, 2, 3$.

The lowpass prototype filter can have the impulse response (noncausal) as

$$h[n] = \frac{\sin[\frac{\pi}{4} n]}{\pi n} w[n],$$

where $w[n]$ is the appropriate window function.

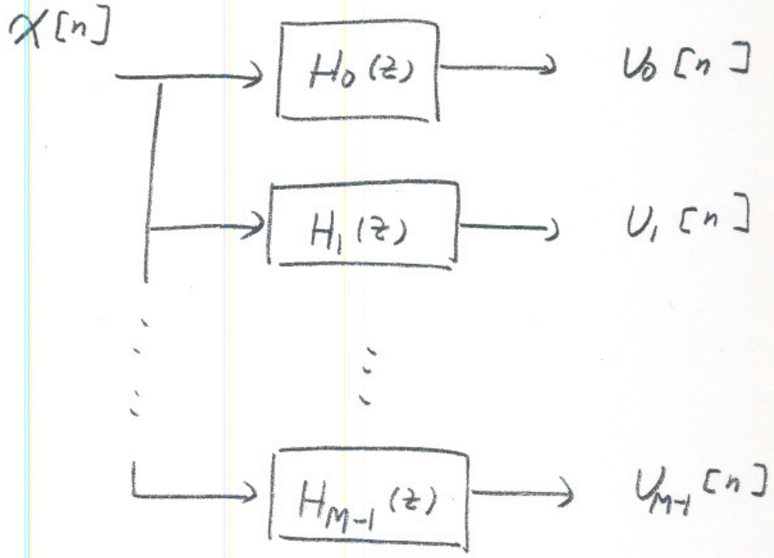
The four polyphase components are given by the transfer functions

$$E_{-k}(z^4) = \sum_{l} h[4l-k]\, z^{-4l}$$

$$= \sum_{l} \frac{\sin[\frac{\pi}{4}(4l-k)]}{\pi(4l-k)} w[4l-k]\, z^{-4l}$$

Signal Reconstruction and M-Band Filters

We would like to fouus on the perfect reconstruction filter banks here. A suffiient condition for perfect reconstruction is that the sum of all outputs of the filter bank yields the original signal $x[n]$.



No information is lost if the condition

$$x[n] = \sum_{k=0}^{M-1} v_k[n]$$

exists.

In the frequency domain,

$$\bar{X}(\omega) = \sum_{k=0}^{M-1} H\left(\omega - \frac{2\pi k}{M}\right) X(\omega)$$

where $V_k(\omega) = \text{DTFT}\{v_k[n]\}$

$$= H_k(\omega)\, X(\omega), \qquad k = 0, 1, \cdots, M-1$$

The condition for perfect reconstruction can be expressed as

$$\sum_{k=0}^{M-1} H\left(\omega - \frac{2\pi k}{M}\right) = 1.$$

Any filter satisfying the perfect reconstruction above can be called an M-band filter. Such a filter has the following properties.

1. In the frequency domain:

$$\sum_{k=0}^{M-1} H\left(\omega - \frac{2\pi k}{M}\right) = 1$$

2. In the time domain:

$$h[nM] = \frac{1}{M}\delta[n].$$

Proof:

$$h'[n] = IDTFT\left\{\sum_{k=0}^{M-1} H(\omega - \frac{2\pi k}{M})\right\}$$

$$= \sum_{k=0}^{M-1} e^{j\frac{2\pi k}{M}n} h[n]$$

$$= \sum_{k=0}^{M-1} h[Mn] = M h[Mn]$$

where $h_k[n] = IDTFT\left\{H(\omega - \frac{2\pi k}{M})\right\}$

$$= e^{j\frac{2\pi k}{M}n} h[n]$$

Since $h'[n] = IDTFT\left\{\sum_{k=0}^{M-1} H(\omega - \frac{2\pi k}{M})\right\}$

$$= IDTFT\{1\}$$

$$= \delta[n]$$

$$\Rightarrow \quad h[Mn] = \frac{1}{M} h'[n] = \frac{1}{M}\delta[n]$$

3. In the z-domain:

$$E_\phi(z) = \mathcal{Z}\left\{ \frac{1}{M}\delta[n] \right\} = \frac{1}{M}$$

$$H(z) = E_\phi(z^M) + z^{-1}E_1(z^M) + \cdots + z^{-(M-1)} E_{M-1}(z^M)$$

Proof:

$$E_\ell(z) \equiv \sum_{n=-\infty}^{\infty} h[nM+\ell]\, z^{-n}, \quad \ell = 0, 1, \cdots, M-1$$

Thus, $$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$

Example:

An ideal lowpass filter with bandwidth $w_c = \frac{\pi}{M}$ is an M-band filter, because clearly

$$\sum_{k=0}^{M-1} H\left[w - k\left(\frac{2\pi}{M}\right)\right] = 1$$

Example:

Let $M = 4$. Choose an FIR filter with transfer function

$$H(z) = 0.25 + z^{-1} + 3.2z^{-2} - 2.8z^{-3} + 1.5z^{-5} + z^{-6} - z^{-7} + z^{-10}$$

Check :

1. $$H(\omega) = H(z)\Big|_{z=e^{j\omega}}$$

$$= 0.25 + e^{-j\omega} + 3.2\, e^{-j2\omega} - 2.8\, e^{-j3\omega} + 1.5\, e^{-j5\omega} + e^{-j6\omega} - e^{-j7\omega} + e^{-j10\omega}$$

$$H(\omega) + H(\omega - \tfrac{\pi}{2}) + H(\omega - \pi) + H(\omega - \tfrac{3\pi}{2})$$

$$= \sum_{k=0}^{10} h[k]\, e^{-j\omega k} \left[1 + e^{j\frac{\pi}{2}k} + e^{j\pi k} + e^{j\frac{3\pi}{2}k} \right]$$

$$= \sum_{k=0}^{10} h[k]\, e^{-j\omega k} \frac{1 - e^{2\pi k}}{1 - e^{j\frac{\pi}{2}k}}$$

$$= \sum_{k=0}^{10} h[k]\, e^{-j\omega k}\, 4\delta[k-4m], \quad m = 0, 1, 2, 3, \ldots$$

$$= h[0] \times 4 = 4 \times 0.25 = 1$$

2. $$h[4n] = \frac{1}{4}\delta[n]$$

3. $$E_0(z) = \mathcal{Z}\{h[4n]\} = \frac{1}{4}$$

$$E_1(z) = \mathcal{Z}\{h[4n+1]\} = 1 + 1.5z^{-1}$$

$$E_2(z) = \mathcal{Z}\{h[4n+2]\} = 3.2 + z^{-1} + z^{-2}$$

$$E_3(z) = \mathcal{Z}\{h[4n+3]\} = -2.8 - z^{-1}$$

$$E_0(z^4) = \frac{1}{4}$$

$$E_1(z^4) = 1 + 1.5\, z^{-4}$$

$$E_2(z^4) = 3.2 + z^{-4} + z^{-8}$$

$$E_3(z^4) = -2.8 - z^{-4}$$

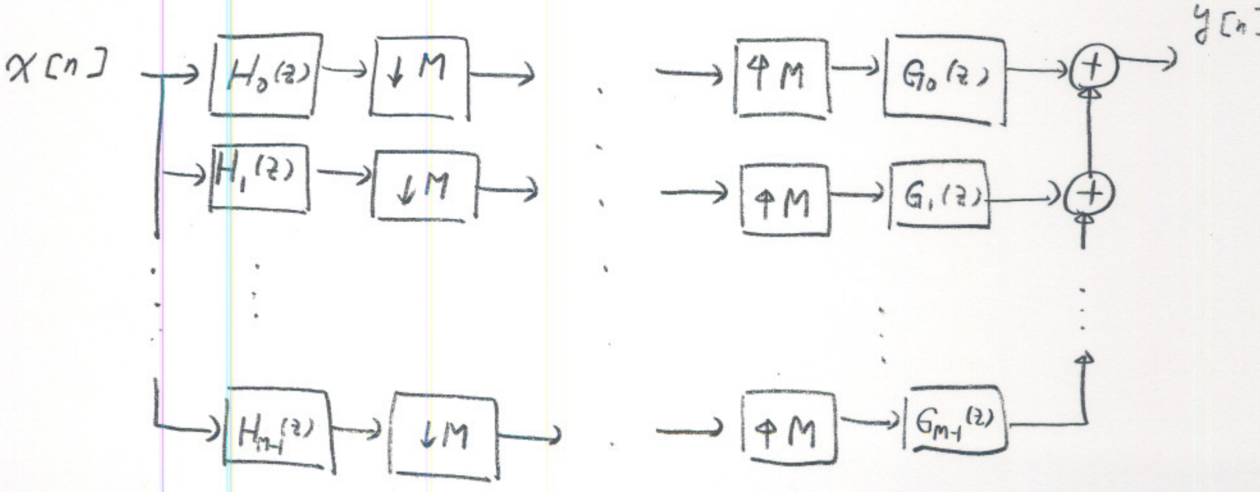
$$H(z) = 0.25 + z^{-1} + 3.2 z^{-2} - 2.8 z^{-3} + 1.5 z^{-5} + z^{-6} - z^{-7} + z$$

$$= \sum_{\ell=0}^{3} z^{-\ell} E_\ell (z^M)$$

Hence $H(z)$ can form a 4-band filter bank.

7.3 Maximally Decimated DFT Filter Banks and Transmultiplexers

In the maximally decimated case, the analysis and synthesis networks do not chang the overall data rate as depicted below:



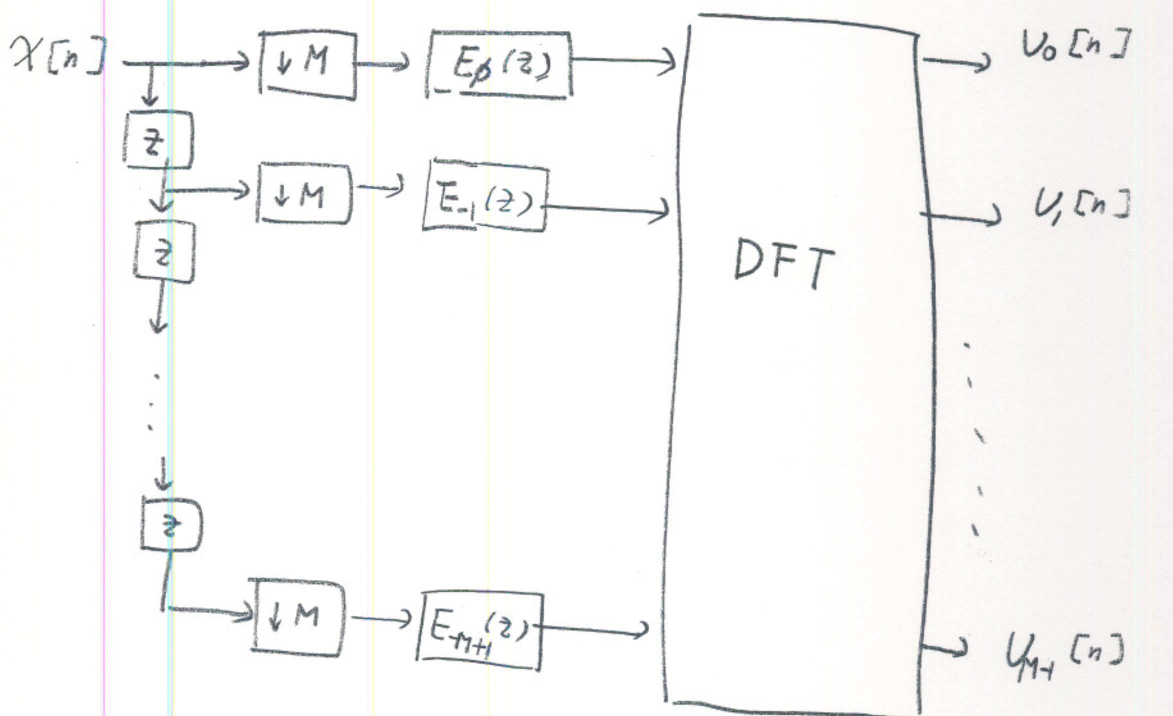
We will show the facts on the perfect reconstruction maximally decimated filter banks here.

Fact 1. The DFT analysis network containing filters such that

$$H_k(\omega) = H\left(\omega - \frac{2\pi k}{M}\right), \quad k = 0, 1, \cdots, M-1$$

is equivalent to the network shown below, where the transfer function is written as

$$H(z) = E_\phi(z^M) + z E_{-1}(z^M) + \cdots + z^{M-1} E_{-M+1}(z^M)$$

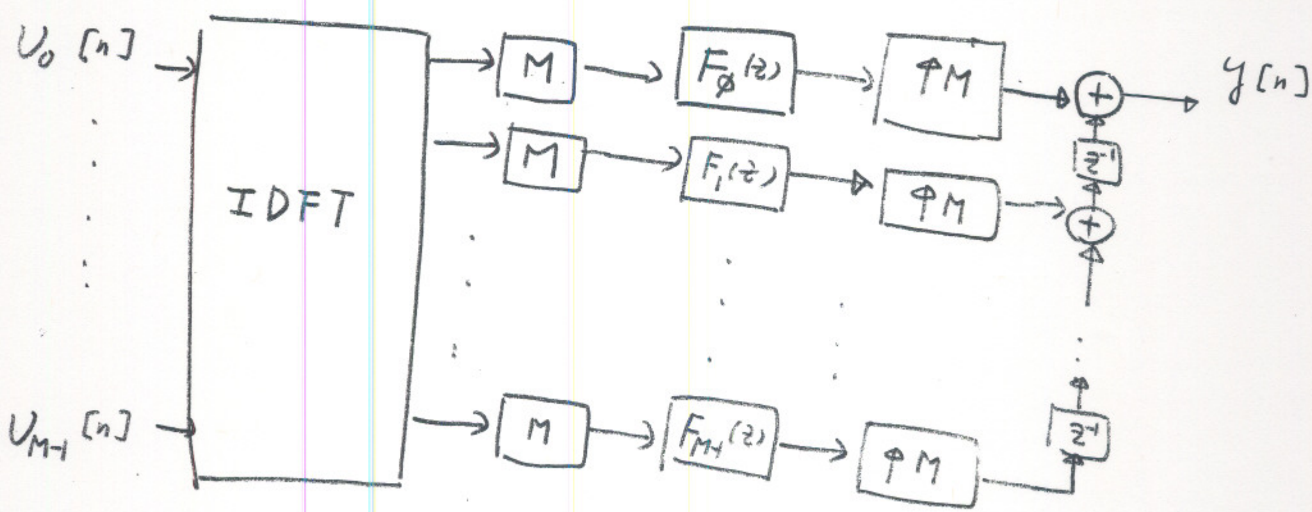


Fact 2. The DFT synthesis network with the filters

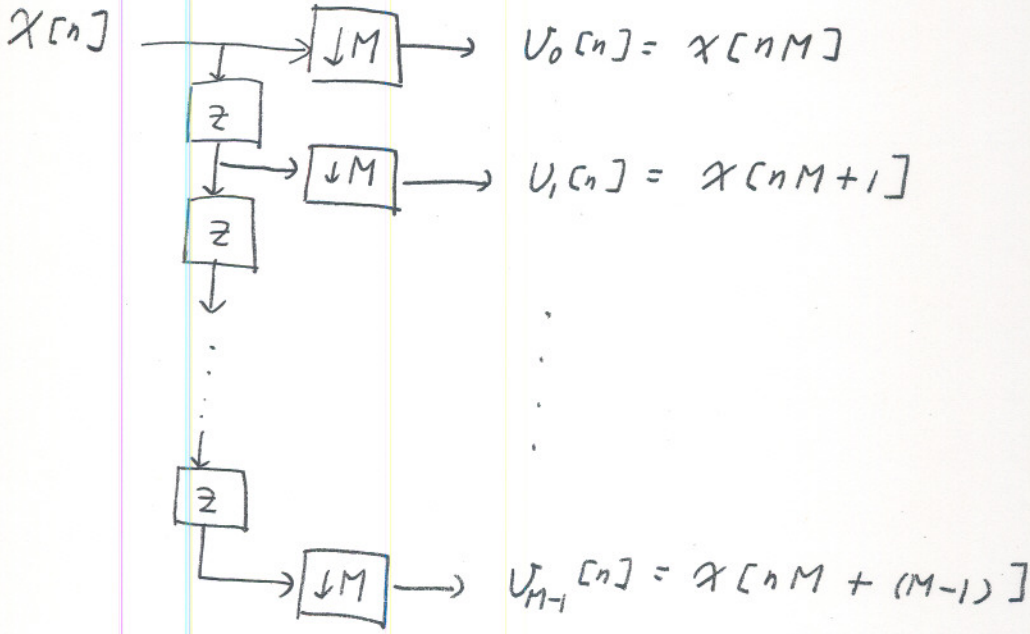
$$G_k(\omega) = G(\omega - \frac{2\pi k}{M}), \quad k = 0, \ldots, M-1$$

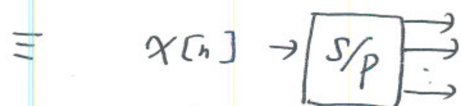
is equivalent to the network shown below where the transfer function is written as

$$G(z) = F_\phi(z^M) + z^{-1} F_1(z^M) + \cdots + z^{-(M-1)} F_{M-1}(z^M).$$



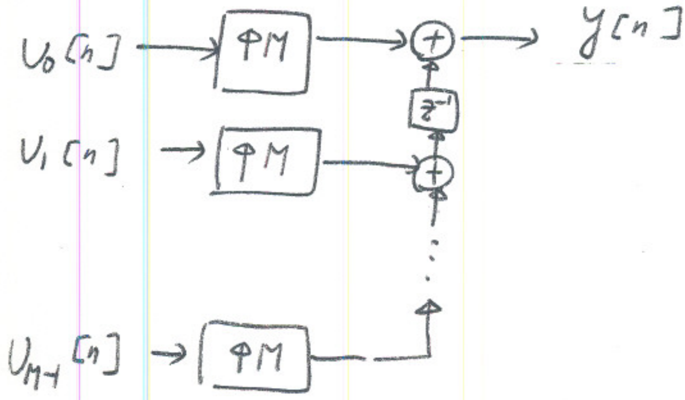
Fact 3. The deinterlacer, as shown below,

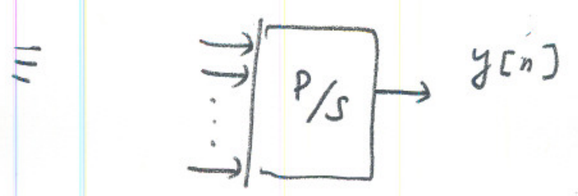




Fact 4. The interlacer takes a number of signals, $v_0[n], v_1[n], \ldots, v_{M-1}[n]$ and interlaces them sequentially into one signal such that

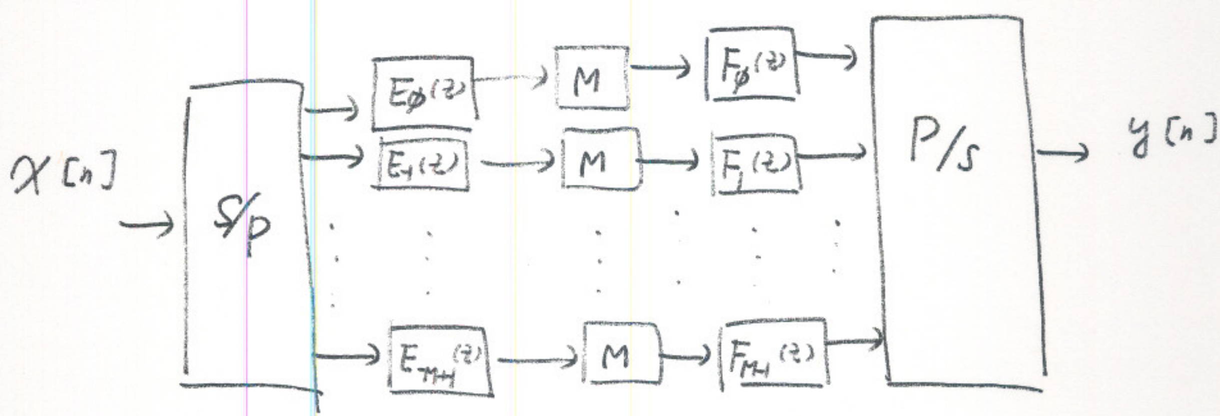
$$y[nM-k] = v_k[n], \quad \text{for } k = 0, 1, \ldots, M-1$$





Maximally Decimated DFT Filter Banks

If we decompose the signal into M frequency components, we decimate each channel by M, and we want to determine conditions on the prototype filter $H(z)$ for perfect reconstruction. If we use a DFT synthesis network, with proto filter $G(z)$, the analysis and synthesis networks can be combined as follows:

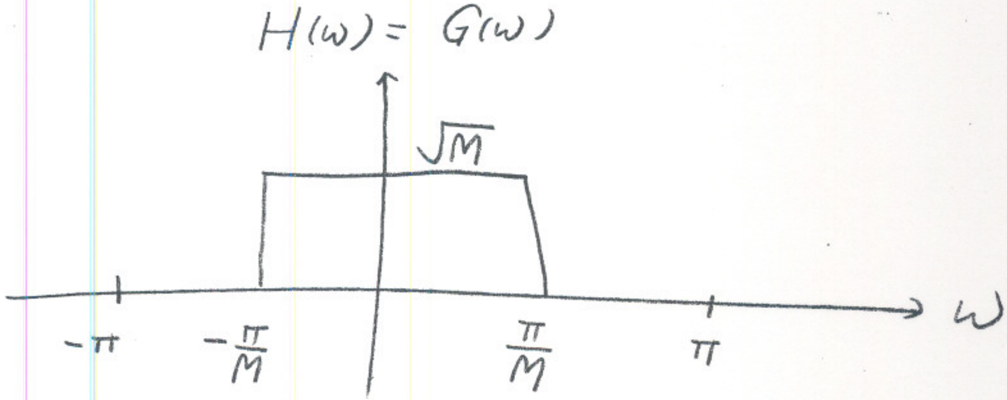


Obviously, the perfect reconstruction can be achieved when

We can obtain a perfect reconstruction DFT filter bank in one of these two cases:

1. If the filters are IIR, both prototype filters $H(z)$ and $G(z)$ (for analysis and synthesis, respectively) have to be ideal lowpass filters with bandwidth $\frac{\pi}{M}$,

where $H(\omega) = G(\omega)$ is bandlimited within $|\omega| < \frac{\pi}{M}$ with amplitude $\sqrt{M}$.



Thus,
$$E_{-k}(\omega) = \text{DTFT}\{h[Mn-k]\}$$
$$= \frac{1}{M} H\left(\frac{\omega}{M}\right) e^{-j\frac{\omega k}{M}} = \frac{1}{\sqrt{M}} e^{-j\frac{\omega k}{M}}$$
$$F_k(\omega) = \text{DTFT}\{g[Mn+k]\}$$
$$= \frac{1}{M} G\left(\frac{\omega}{M}\right) e^{j\frac{\omega k}{M}} = \frac{1}{\sqrt{M}} e^{j\frac{\omega k}{M}}$$

Clearly, in this case, $M E_{-k}(z) F_k(z) = 1$, for all $k = 0, 1, \ldots, M-1$

2. In the case of FIR filters, the perfect reconstruction conditions are satisfied by the prototype filters:

$$H(z) = h[0] + h[-1] z + \cdots + h[-M+1] z^{M-1}$$
$$G(z) = g[0] + g[1] z^{-1} + \cdots + g[M-1] z^{-M+1}$$

Thus, $E_{-k}(z) = h[-k]$, $F_k(z) = g[k]$.

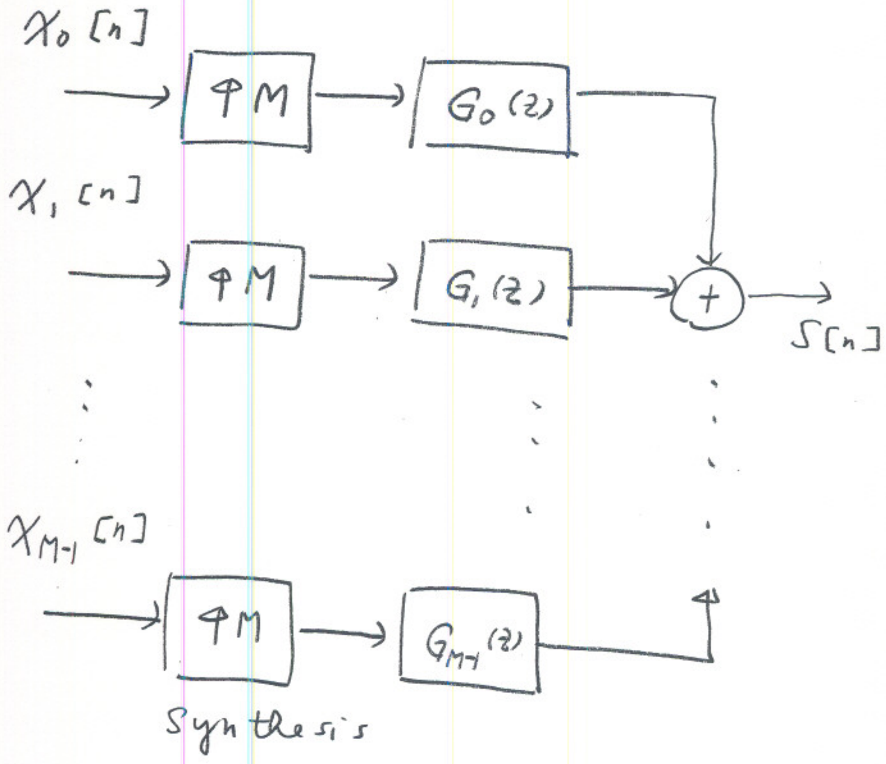
Therefore, the impulse responses of the filters have to be such that

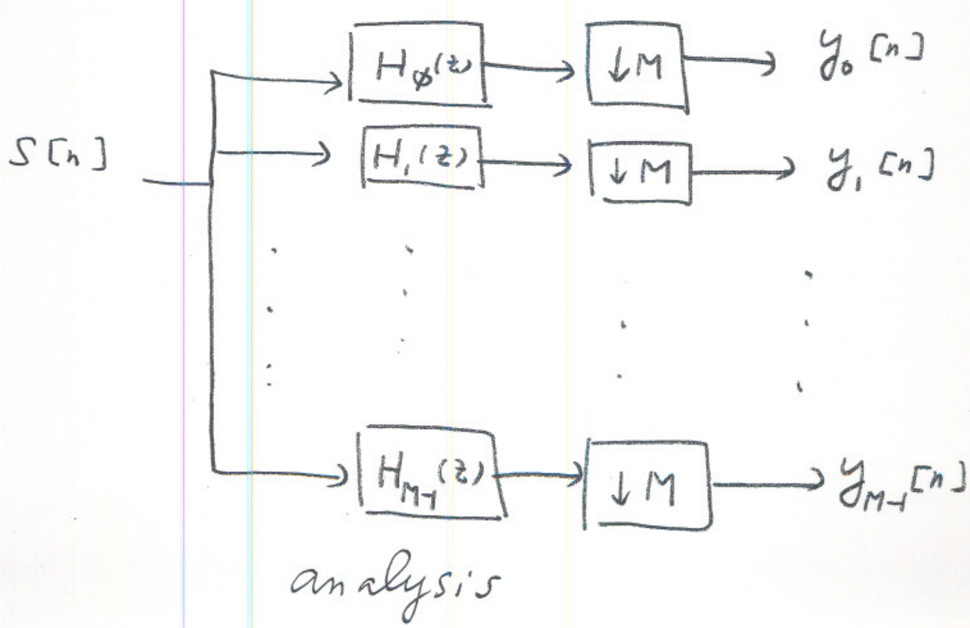
$$h[-n]\, g[n] = \frac{1}{M}, \quad n = 0, 1, \ldots, M-1$$

Example: Given an impulse response of $h[n]$, then design $g[n] = \dfrac{1}{M h[-n]}$ for $n = 0, 1, \cdots, M-1$, which will satisfy the perfect reconstruction condition

7.4 Transmultiplexers

A transmultiplex consists of a synthesis followed by an analysis network depicted as below.





Under ideal conditions, the goal is to have perfect reconstruction of each channel in the absence of crosstalk. In other words, the desired goal is

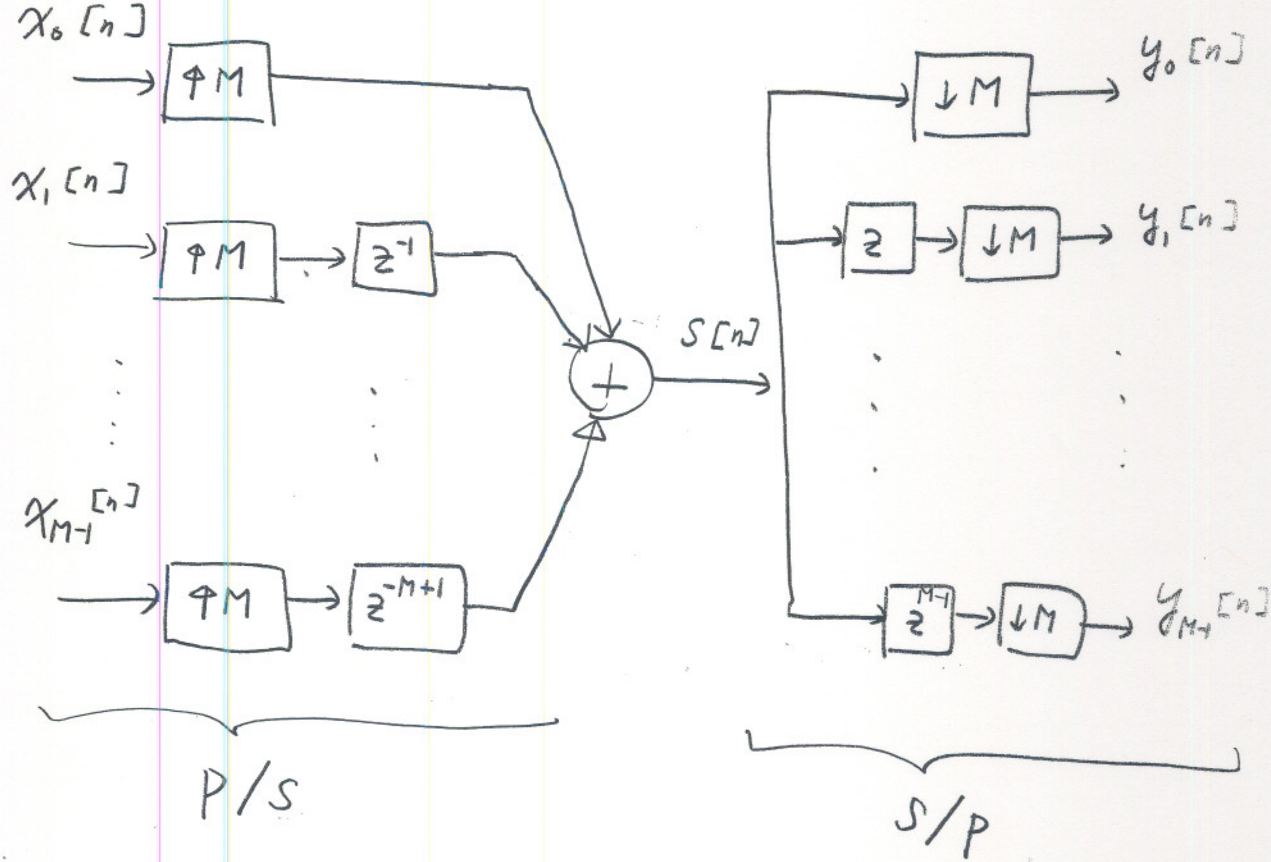
$$y_k[n] = x_k[n], \quad \text{for } k = 0, 1, \ldots, M-1$$

Possible scaling factors and time delays can be easily included.

The transmultiplexer has been applied for the current communication modulation techniques. We can design different $G_k(z)$ and $H_k(z)$ to implement different multi-access communication systems.

Time Division Multiple Access (TDMA)

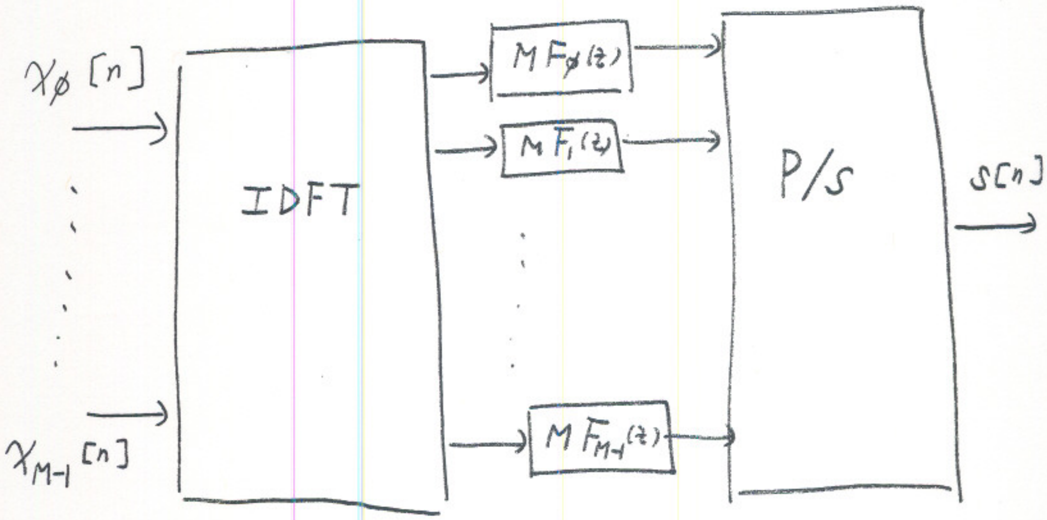
The simplest way of accommodating M different users in one common channel is by the "time multiplexing". It can be depicted

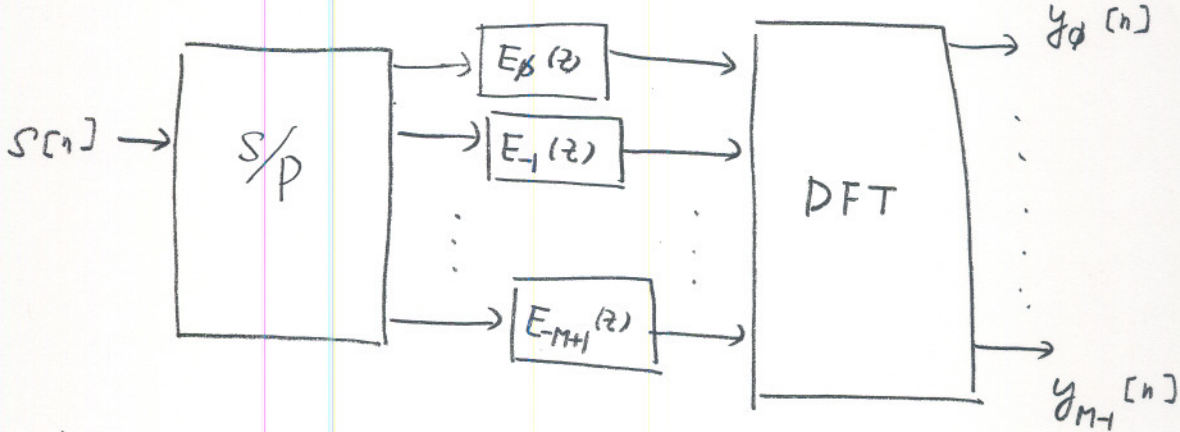


TDMA system

Frequency Division Multiplexing (FDMA)

By a different choice of filters we can allocate a different frequency band, $X_k[n]$, to each user. The DFT transmultiplexer is





The modulation filters $G_k(z)$ and the demodulation filters $H_k(z)$ can be formed according to

$$G_k(\omega) = G(\omega - \frac{2\pi k}{M})$$
$$H_k(\omega) = H(\omega - \frac{2\pi k}{M}),$$

for $k = 0, 1, 2, \ldots, M-1$ The polyphase compon[ents] have to abide by the perfect reconstruction condition such that $E_{-k}(z)\, F_k(z) = \frac{1}{M}$

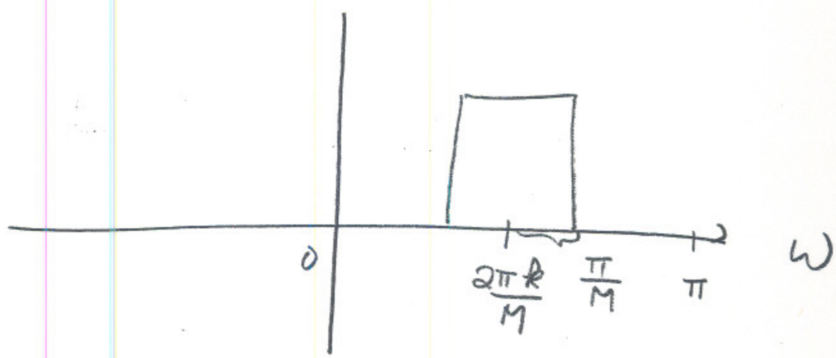
where $H(z) = \sum_{k=0}^{M-1} z^{k} E_{-k}(z^{M})$

and $G(z) = \sum_{k=0}^{M-1} z^{-k} F_{k}(z^{M})$.

As with the DFT filter banks, these conditions are satisfied in two cases:

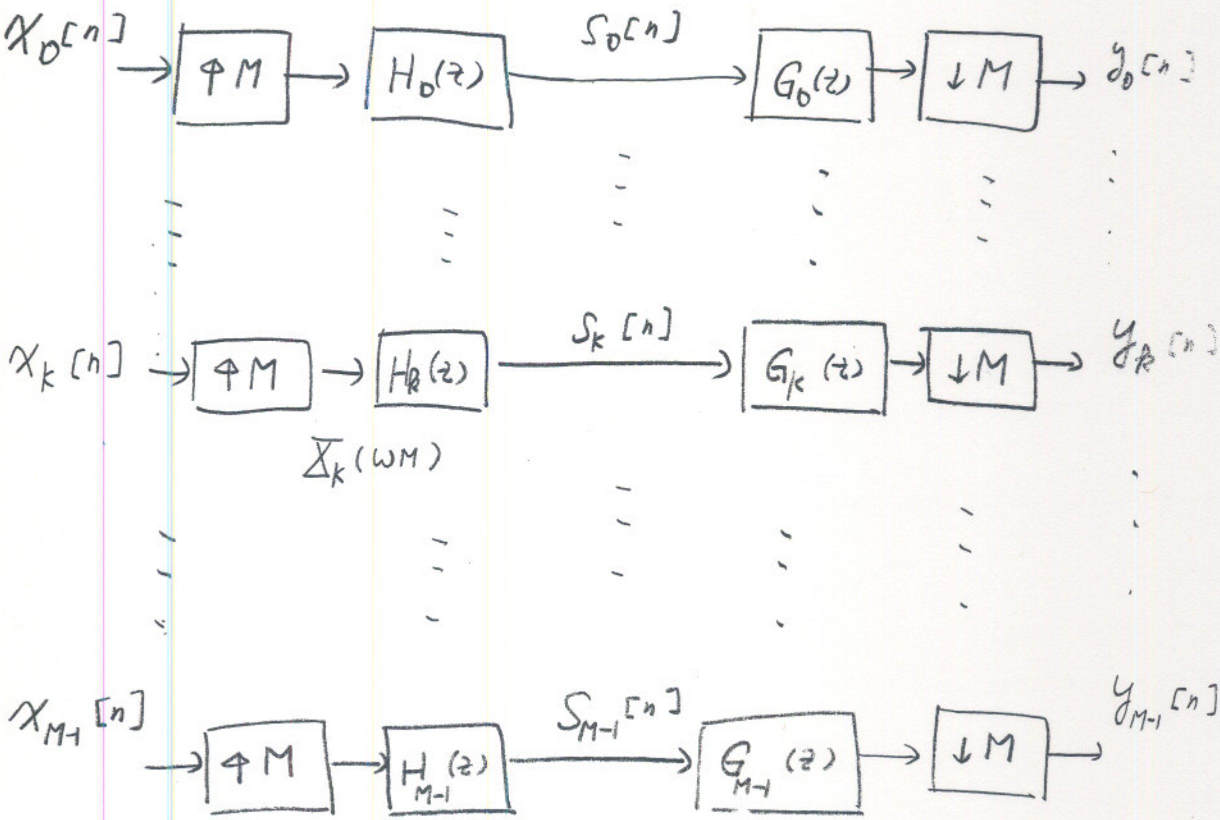
1. Ideal IIR lowpass filters with bandwidth $\frac{\pi}{M}$. In this case, $x_k[n]$ is allocated in its own, nonoverlapping frequency band.

$$H_k(\omega) = G_k(\omega)$$



Since $x_k[n]$ is not overlapped with $x_\ell[n]$, $\forall k \neq \ell$, then there is no crosstalk between $x_k[n]$ and $y_\ell[n]$, $\forall k \neq \ell$

ence, the system can be separated as



$$\mathrm{TFT}\left\{ y_k[n] \right\} = Y_k(\omega) = \frac{1}{M} \sum_{m=0}^{M-1} Q_k\left(\frac{\omega}{M} - \frac{2\pi m}{M}\right) X_k(\omega)$$

where $Q_k(\omega) = G_k(\omega)\, H_k(\omega)$,

$$G_k(\omega) \equiv G_k(z) \Big|_{z=e^{j\omega}}$$

$$H_k(\omega) \equiv H_k(z) \Big|_{z=e^{j\omega}}$$

Since $G_k(\omega) = G(\omega - \frac{2\pi k}{M})$

and $H_k(\omega) = H(\omega - \frac{2\pi k}{M})$,

then $$Y_k(\omega) = \frac{1}{M} \sum_{m=0}^{M-1} Q(\frac{\omega}{M} - \frac{2\pi(m+k)}{M}) X_k(\omega)$$
$$= \frac{1}{M} \sum_{m=0}^{M-1} Q(\frac{\omega}{M} - \frac{2\pi m}{M}) X_k(\omega)$$

where $Q(\omega) = G(\omega) H(\omega)$

Since $G(\omega)$, $H(\omega)$ are both ideal lowpass filters, the product filter $Q(\omega)$ is an M-band filter such that

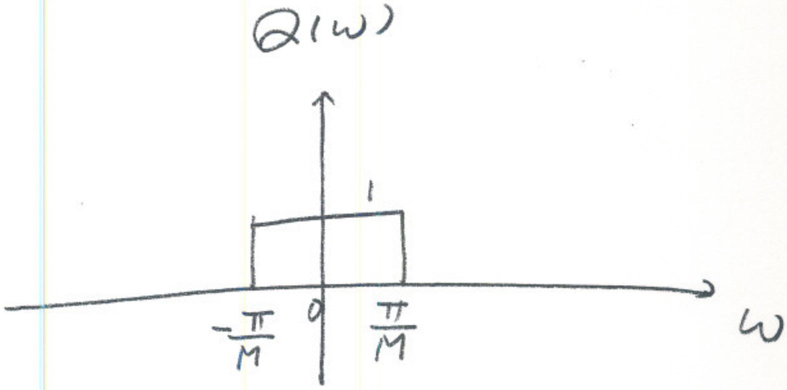
$$\frac{1}{M} \sum_{m=0}^{M-1} Q(\omega - \frac{2\pi m}{M}) = 1, \quad \forall \omega$$

Consequently, $y_k[n] = x_k[n]$.

How to design $G(z)$ & $H(z)$? We introduce two ways here:

Approach 1. perfect bandlimited lowpass filter.

If the product filter $Q(\omega)$ has the perfectly band limited spectrum as depicted below:



then the impulse response of $q[n]$ can be derived as

$$q[n] = \text{IDTFT}\{Q(\omega)\}$$

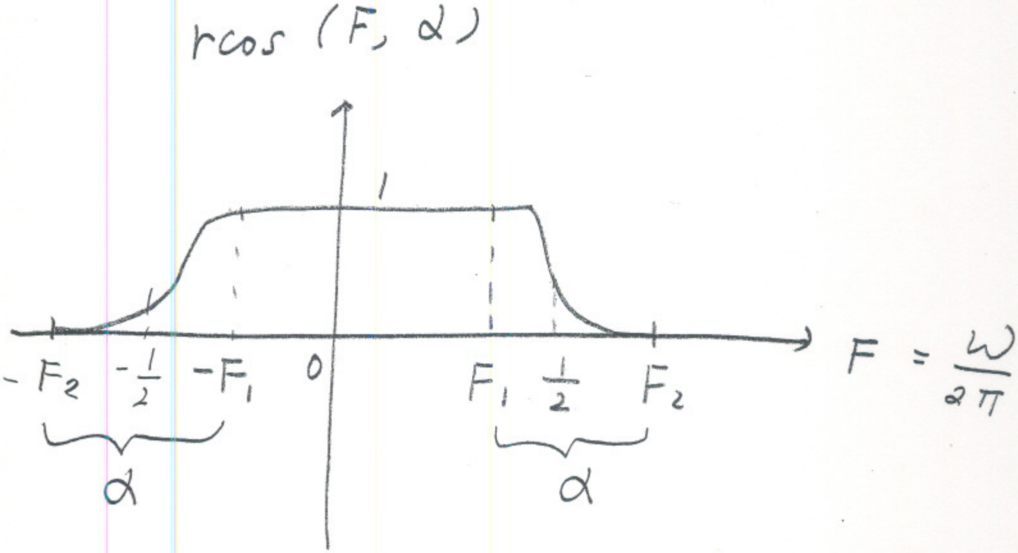
$$= \frac{\sin\left(\frac{\pi n}{M}\right)}{\pi n}$$

Approach 2. Raised-Cosine lowpass filter

The perfectly bandlimited lowpass filter has a long tail in the impulse response, which is not practical. Hence, we need to remove the sharp transition from the passband to stopband and use a raised-cosine function based spectrum to approximate it. The raised-cosine function $rcos(F, \alpha)$ can be formulated as

$$rcos(F, \alpha) \equiv \begin{cases} 1, & \text{if } |F| < F_1 \\ \cos^2\left[\frac{\pi}{2} \frac{|F| - F_1}{\alpha}\right], & \text{if } F_1 < |F| \\ 0, & \text{if } |F| > F_2 \end{cases}$$

where $F_1 = \frac{1-\alpha}{2}$, $F_2 = \frac{1+\alpha}{2}$, $\alpha = F_2 - F_1$



The DTFT $Q(\omega)$ can be defined as

$$Q(\omega) = rcos\left[\frac{\omega M}{2\pi}, \alpha\right]$$

2. Nonideal FIR filter of the form

$$H(z) = h[0] + h[-1]z + \cdots + h[-M+1]z^{M-1}$$

$$G(z) = g[0] + g[1]z^{-1} + \cdots + g[M-1]z^{-M+1}$$

in which channels overlap in the frequency domain. Choose $g[n] = h[-n] = 1$ for $n = 0, 1, \ldots M-1$ as depicted as below: