Chapter 6 Multirate Signal Processing

6.2 Upsampling and Downsampling

※ Definition: Given an integer L, we define the upsampling operator S_L, by

$$y[n] = S_L\{x[n]\} = \begin{cases} x[\frac{n}{L}], & \text{if } n/L \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

The operator S_L increases the sampling frequency by a factor L, by adding more samples with zero value.

※ Definition: Given an integer D, we define the downsampling operator $S_{1/D}$, by

$$y[n] = S_{1/D}\{x[n]\} = x[nD]$$

The operator $S_{1/D}$ decreases the sampling frequency by a factor D, by keeping one sample out of D samples

※ Symbolic Notations

Upsampling

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y[n] = S_L\{x[n]\}$$

Downsampling

$$x[n] \longrightarrow \boxed{\downarrow D} \longrightarrow y[n] = S_{1/D}\{x[n]\}$$

Example: Let $x[n] = \{\cdots, \underset{\uparrow}{1}, 2, 3, 4, 5, \cdots\}$

$y[n] = S_2\{x[n]\}$

then $y[n] = \{\cdots, 0, \underset{\uparrow}{1}, 0, 2, 0, 3, 0, 4, \cdots$

Example: Let $x[n] = \{\cdots, \underset{\uparrow}{1}, 2, 3, 4, 5, \cdots\}$

$y[n] = S_{1/2}\{x[n]\}$

then $y[n] = \{\cdots, \underset{\uparrow}{1}, 3, 5, 7, \cdots\}$

6.3 Analysis of Upsampling and Downsampling

A sampling sequence $\delta_D[n]$ is defined as

$$\delta_D[n] \equiv \begin{cases} 1, & \text{if } n/D \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

A subset of sequence $x[n]$ inserted with zeros is defined as

$$v[n] \equiv \delta_D[n]\, x[n].$$

It has the z-transform:

$$V(z) = \mathcal{Z}\{v[n]\}$$

$$= \sum_{n=-\infty}^{\infty} v[n]\, z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[mD]\, z^{-mD}.$$

Since $\delta_D[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{j\frac{2\pi k n}{D}}$,

$$= \begin{cases} 1, & \text{if } n/D \text{ is an integer} \\ 0, & \text{otherwise} \end{cases},$$

$$v[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n]\, e^{-j\frac{2\pi k n}{D}}$$

$$V(z) = \mathcal{Z}\{v[n]\}$$

$$= \frac{1}{D} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{D-1} x[n] \left(z e^{j\frac{2\pi k}{D}}\right)^{-n}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{j\frac{2\pi k}{D}} z\right)$$

where $X(z) = \mathcal{Z}\{x[n]\}$.

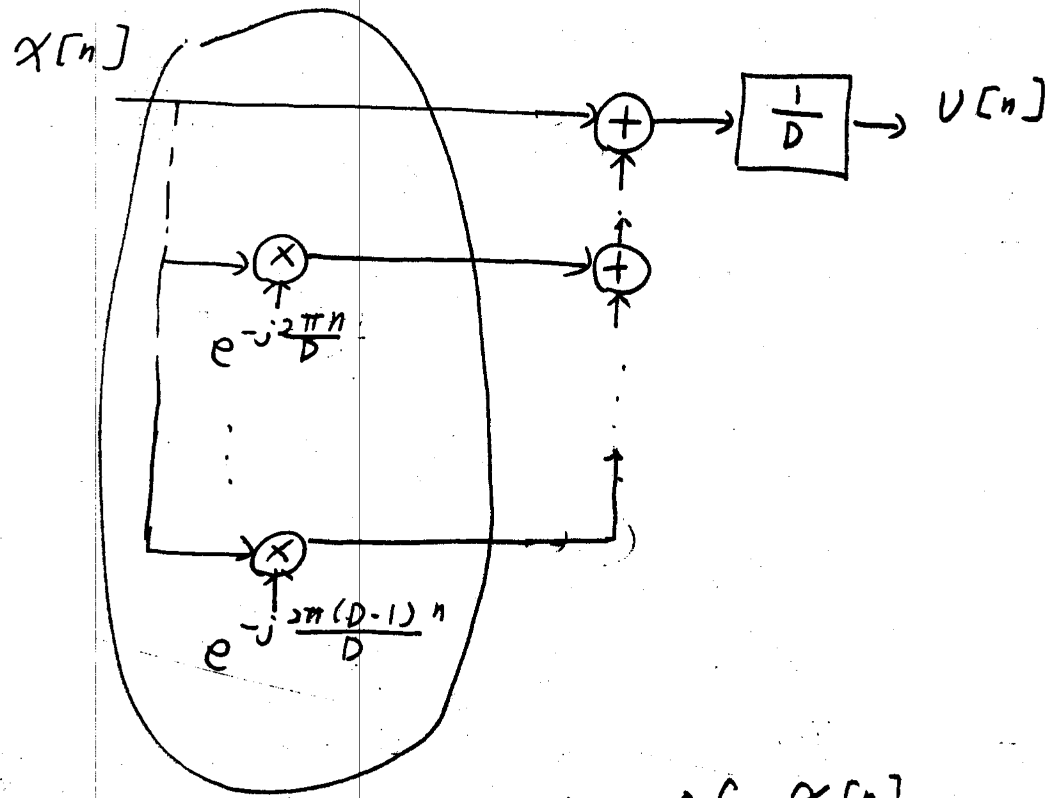
The DTFT $V(\omega) = V(z)\Big|_{z=e^{j\omega}}$

is given by

$$V(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\omega - \frac{2\pi k}{D}\right)$$

where $X(\omega) = \mathrm{DTFT}\{x[n]\}$.

Hence the sampling sequence $\delta_D[n]$ is the operator to obtain the summation of all D-polyphase expansions of $x[n]$.



polyphase expansion of $x[n]$

Example:

Consider $x[n] = 5\cos[0.1\pi n]$. Determine the DTFT of $v[n] = \delta_3[n]\, x[n]$.

Solution:

$$X(\omega) = \text{DTFT}\{x[n]\}$$

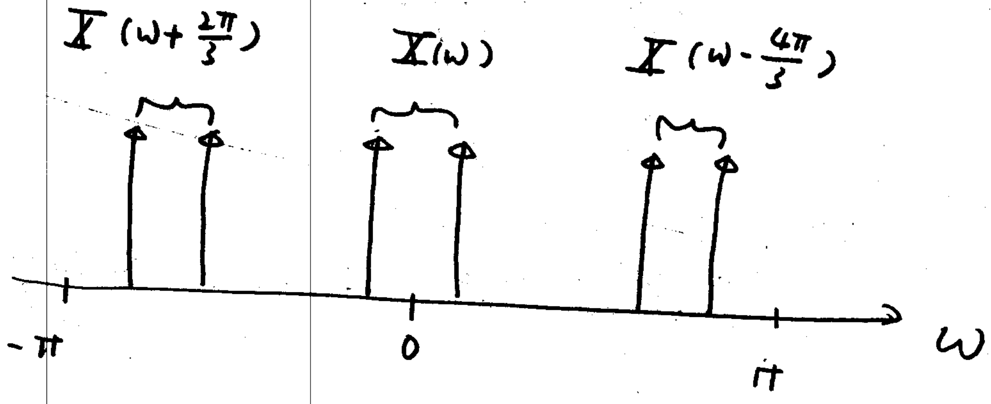
$$= 5\pi\, \delta(\omega - 0.1\pi) + 5\pi\, \delta(\omega + 0.1\pi), \quad -\pi < \omega \le \pi$$

$$V(\omega) = \text{DTFT}\{v[n]\}$$

$$= \frac{1}{3} \sum_{k=0}^{2} X\left(\omega - \frac{2\pi k}{3}\right)$$

$$
\begin{aligned}
V(\omega) = {} & \frac{5}{3}\pi\,\delta(\omega - 0.1\pi) + \frac{5}{3}\pi\,\delta(\omega + 0.1\pi) \\
& + \frac{5}{3}\pi\,\delta\left(\omega - 0.1\pi - \frac{2\pi}{3}\right) + \frac{5}{3}\pi\,\delta\left(\omega + 0.1\pi - \frac{2\pi}{3}\right) \\
& + \frac{5}{3}\pi\,\delta\left(\omega - 0.1\pi + \frac{2\pi}{3}\right) + \frac{5}{3}\pi\,\delta\left(\omega + 0.1\pi + \frac{2\pi}{3}\right)
\end{aligned}
$$

$V(\omega)$



We can apply the analysis for the downsampling operation.

Downsampling

$y[n]$ is called the decimated sequence, such that

$$y[n] = S_{1/D}\{x[n]\} = x[nD].$$

The z-transform of $y[n]$ can be written as

$$Y(z) = \mathcal{Z}\{y[n]\}$$

$$= \sum_{n=-\infty}^{\infty} x[nD]\, z^{-n}$$

If $v[n] = x[n]\,\delta_D[n]$, then

$$V(z) = \mathcal{Z}\{v[n]\} = \sum_{n=-\infty}^{\infty} v[n]\, z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} v[mD]\, z^{-mD}$$

Thus, $Y(z) = V(z^{1/D})$

$$= \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D} z^{1/D}\right)$$

where $X(z) = \mathcal{Z}\{x[n]\}$.

The DTFT of $y[n]$ can be written as

$$Y(\omega) = \text{DTFT}\{y[n]\}$$

$$= Y(z)\Big|_{z=e^{j\omega}}$$

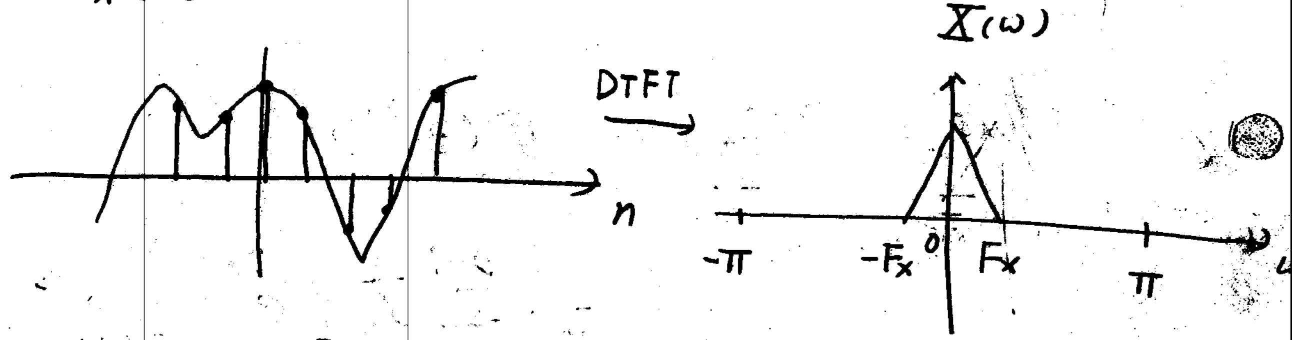
$$= \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega}{D} - k\frac{2\pi}{D}\right)$$

To construct the DTFT of $Y(\omega)$, we have to take the following steps:

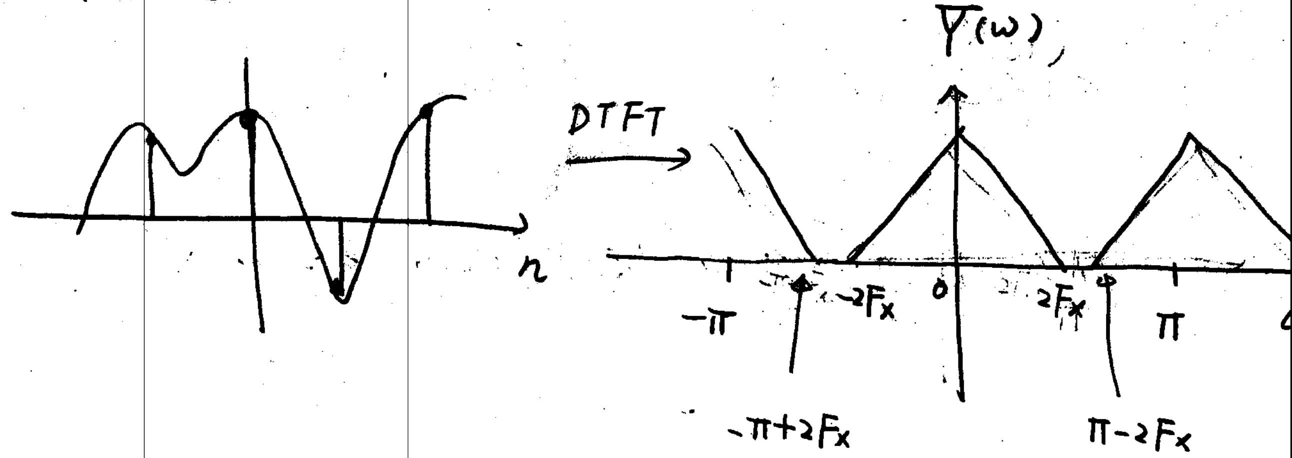
1. Compute $V(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X(\omega - \frac{2\pi k}{D})$
2. Stretch the $V(\omega)$ by changing ω to ω/D.

Illustration:

$x[n] = x(nT_s)$



$y[n] = x[nD] = x(nDT_s)$, $D=2$



It is obvious from the illustration that the decimation (downsampling) will cause the aliasing if

$$D F_x > \frac{\pi}{D},$$

where F_x is the bandwidth of $X(\omega)$ and D is the decimation rate.

Example: Consider a signal

$x[n] = 2\cos(0.2\pi n - 0.3\pi) + 4\cos(0.6\pi n + 0.2$

Determine the frequency spectrum of $y[n] = x[2n]$ and what is the maximum decimation rate D for aliasing-fee.

Solution:

$$X(\omega) = \text{DTFT}\{x[n]\} = 2\pi e^{-j0.3\pi}\delta(\omega - 0.2\pi) + 2\pi e^{j0.3\pi}\delta(\omega + 0.2\pi) + 4\pi e^{j0.2\pi}\delta(\omega - 0.6\pi) + 4\pi e^{-j0.2\pi}\delta(\omega + 0.6\pi), \quad -\pi < \omega \le \pi$$

$$Y(\omega) = DTFT\{y[n]\} = DTFT\{x[2n]\}$$

$$= \frac{1}{2} X\left(\frac{\omega}{2}\right) + \frac{1}{2} X\left(\frac{\omega}{2} - \pi\right)$$

$$= \pi e^{-j0.3\pi} \delta\left(\frac{\omega}{2} - 0.2\pi\right) + \pi e^{j0.3\pi} \delta\left(\frac{\omega}{2} + 0.2\right.$$

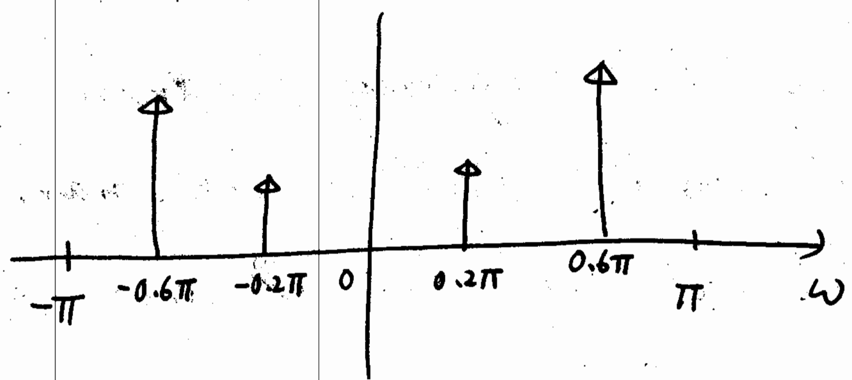
$$+ 2\pi e^{j0.2\pi} \delta\left(\frac{\omega}{2} - 0.6\pi\right) + 2\pi e^{-j0.2\pi} \delta\left(\frac{\omega}{2} + 0.6\pi\right)$$

$$+ \pi e^{-j0.3\pi} \delta\left(\frac{\omega}{2} + 0.8\pi\right) + \pi e^{j0.3\pi} \delta\left(\frac{\omega}{2} - 0.8\pi\right)$$

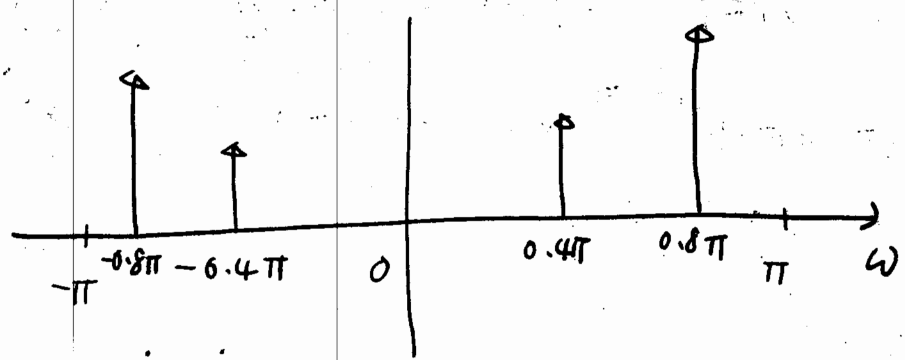
$$+ 2\pi e^{j0.2\pi} \delta\left(\frac{\omega}{2} + 0.4\pi\right) + 2\pi e^{-j0.2\pi} \delta\left(\frac{\omega}{2} - 0.4\pi\right.$$

$$-\pi < \frac{\omega}{2} \le \pi .$$

$|X(\omega)|$



$|Y(\omega)|$



$F_x = 0.6\pi$, $\quad D \times 0.6\pi > \frac{\pi}{D}$

$\Rightarrow \quad D^2 > \frac{1}{0.6} = \frac{5}{3}$

$\Rightarrow \quad D \geq 2$ will cause aliasing.

$\Rightarrow$ The maximum aliasing-free decimation rate is $D = 1$.

<u>Up sampling</u>

In general, $y[n] = S_L\{x[n]\}$.

The Z-transform of $y[n]$ is written as

$$Y(z) = \mathcal{Z}\{y[n]\}$$

$$= \sum_{n=-\infty}^{\infty} x[\tfrac{n}{L}]\, z^{-n}, \quad \text{for } \tfrac{n}{L} \text{ is an integ}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, z^{-mL}, \quad \text{where } n = Lm$$

$$= X(z^L), \quad \text{where } X(z) = \mathcal{Z}\{x[n]\}$$

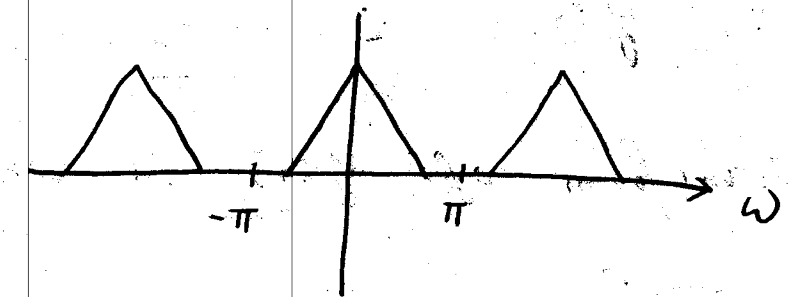
The DTFT of $y[n]$ can be obtained as

$$Y(\omega) = Y(z)\big|_{z=e^{j\omega}}$$
$$= X(\omega L)$$

where $X(\omega)$ is the DTFT of $x[n]$.

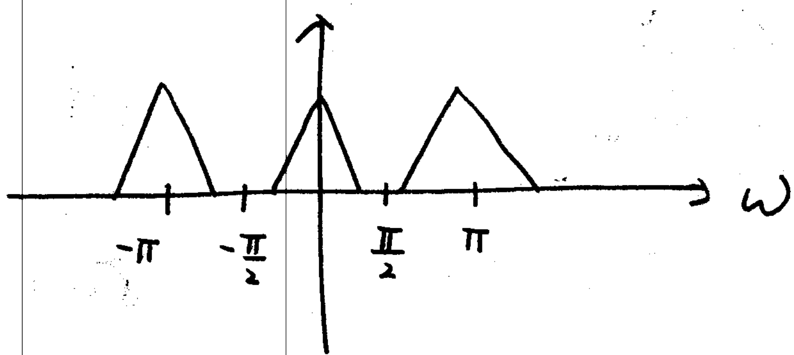
Illustration:

$X(\omega) = \text{DTFT}\{x[n]\}$



$\downarrow$ up sampling

$Y(\omega) = \text{DTFT}\{y[n]\} = X(\omega L), \; L=2$



Example:

Consider the signal

$$x[n] = 2\cos(2\pi n - 0.3\pi) + 4\cos(0.6\pi n + 0.2\pi)$$

Call $y[n] = x[\frac{n}{2}]\,\delta_2[n]$ the signal after upsampling by $L = 2$. Determine the DTFT of $y[n]$.

Solution:

$$X(\omega) = 2\pi e^{-j0.3\pi}\,\delta(\omega - 0.2\pi) + 2\pi e^{j0.3\pi}\,\delta(\omega + 0.2\pi)$$
$$+ 4\pi e^{j0.2\pi}\,\delta(\omega - 0.6\pi) + 4\pi e^{-j0.2\pi}\,\delta(\omega + 0.6\pi),$$
$$-\pi < \omega \le \pi$$

$$Y(\omega) = X(2\omega)$$
$$= 2\pi e^{-j0.3\pi}\,\delta(2\omega - 0.2\pi) + 2\pi e^{j0.3\pi}\,\delta(2\omega + 0.2\pi)$$
$$+ 2\pi e^{-j0.3\pi}\,\delta(2\omega - 0.2\pi + 2\pi) + 2\pi e^{j0.3\pi}\,\delta(2\omega + 0.2\pi - 2\pi$$
$$+ 4\pi e^{j0.2\pi}\,\delta(2\omega - 0.6\pi) + 4\pi e^{-j0.2\pi}\,\delta(2\omega + 0.6\pi)$$
$$+ 4\pi e^{j0.2\pi}\,\delta(2\omega - 0.6\pi + 2\pi) + 4\pi e^{-j0.2\pi}\,\delta(2\omega + 0.6\pi -$$

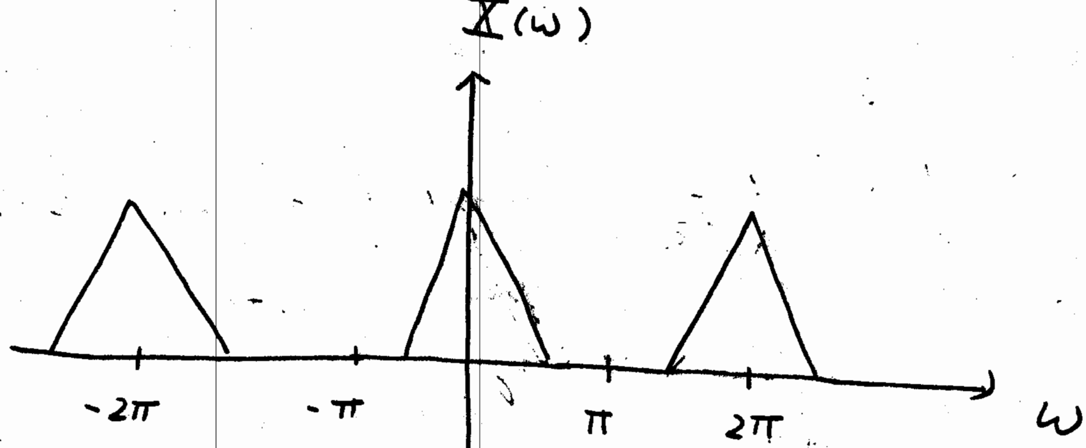
6.4 Fractional Sampling

The fractional sampling is the technique for resampling a digital signal $x[n]$ from the original sampling rate F_s Hz into a new sampling rate $(L/D) F_s$ Hz, where both L & D are integers.

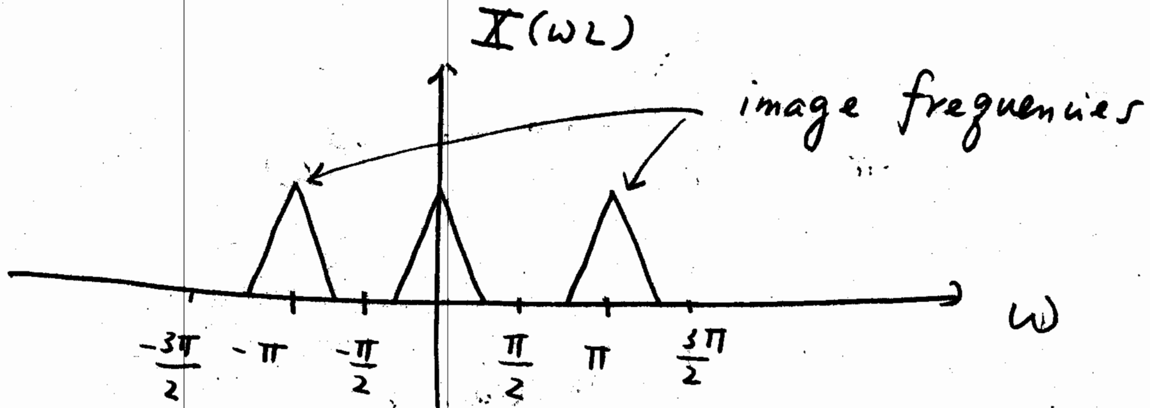
We can establish such a fractional sampler through downsampling and upsampling in series.

※ Interpolation by an integer factor L:

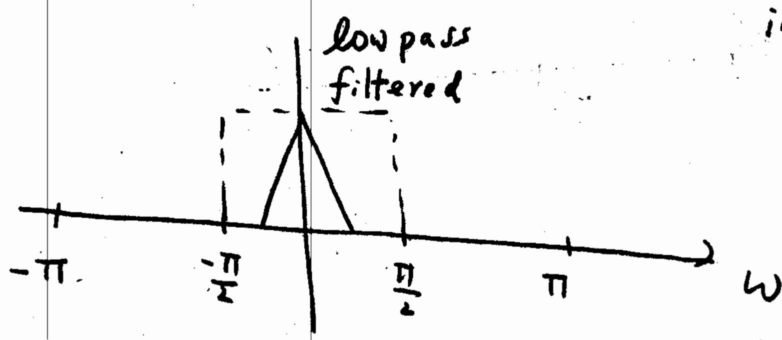
The signal can be first upsampled by a factor L. The upsampling alone introduces artifacts in the frequency domain as image frequencies. All image frequencies are outside the interval $[-\frac{\pi}{L}, \frac{\pi}{L}]$ and can be eliminated by a lowpass filtering.



upsampling by L, L = 2

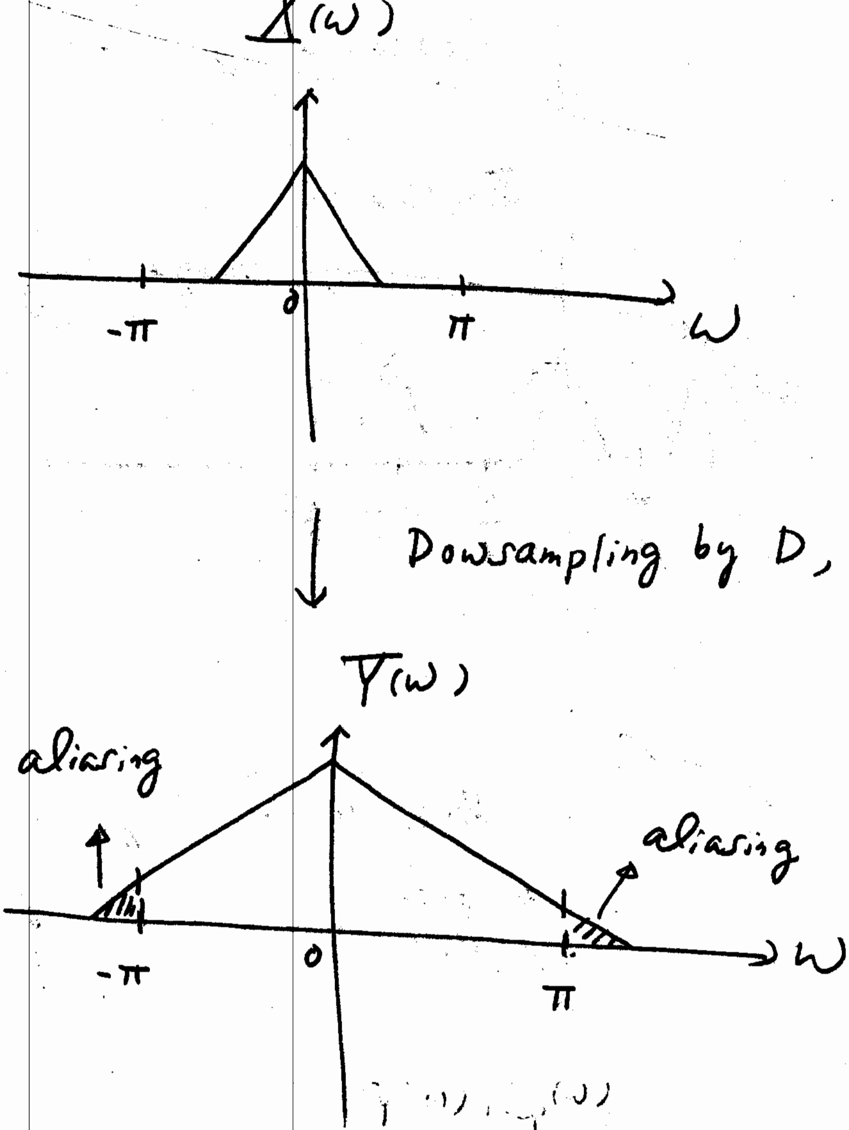


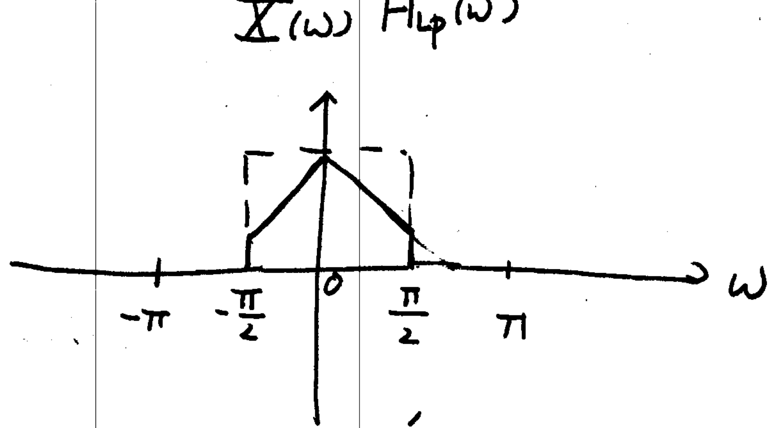
lowpass filtering to eliminate image frequencies

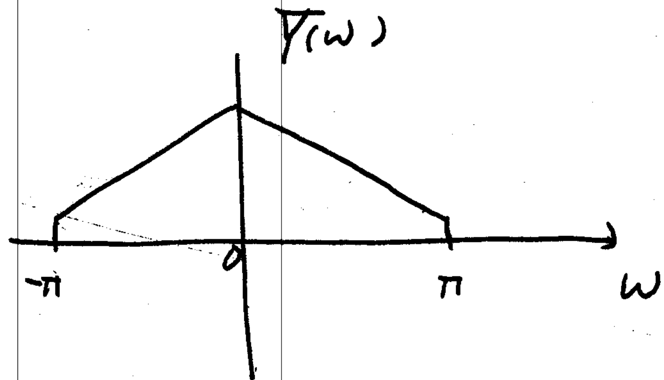


※ Decimation by an integer factor D

The signal will be downsampled after the interpolation and lowpass-filtering. As mentioned before, the downsampling stretches the spectrum. It can follow a lowpass filter to avoid the aliasing.



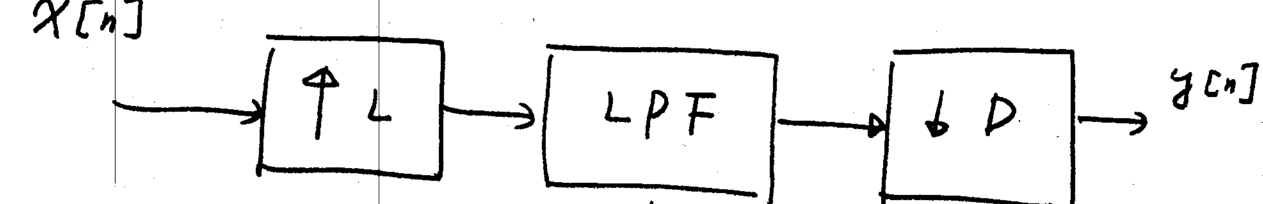


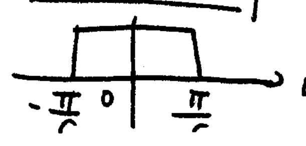


aliasing-free

X. Fractional Sampling by a rational factor L/D

A general case of sampling-rate conversion by a rational factor L/D is combining an interpolation operation by L with a decimation operation by D right after.





, $C = \max(D, L)$

Example:

Consider a signal $x[n]$ sampled at $F_s = 10\,kHz$.

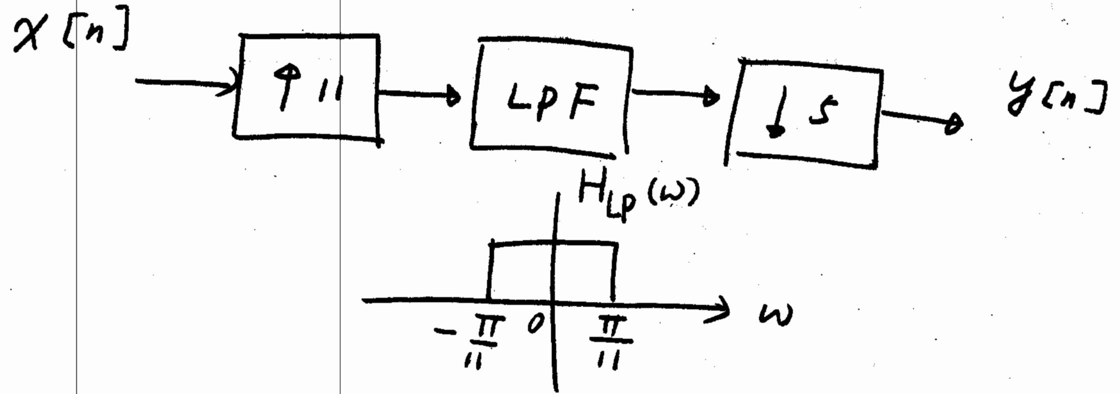
Let's resample it as follows:

a. Resample it at $22\,kHz$. (aliasing-free, $22\,kHz > 10\,kHz$)

$$\frac{L}{D} = \frac{22k}{10k} = \frac{11}{5}$$

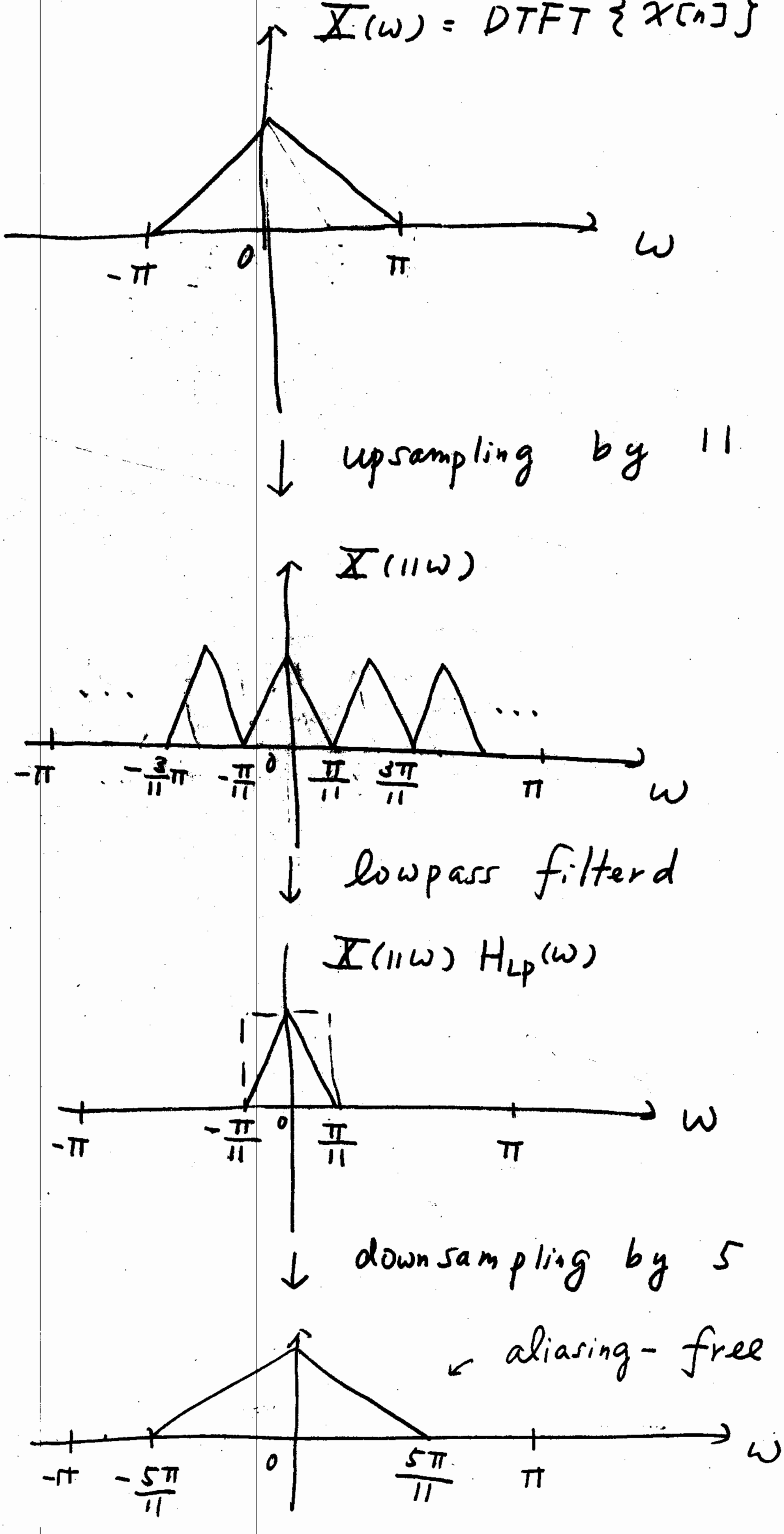
Thus, $L = 11$ and $D = 5$ are the minimum factors in the fractional sampling scheme. $C = \max(L, D) = 11$

Hence, the fractional sampler can be implemented as:



Assume F_s is the Nyquist rate (twice of the signal bandwidth).

We can illustrate the spectra as follows:



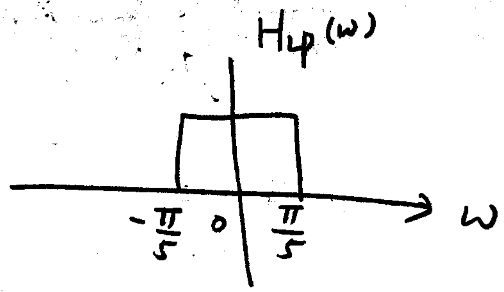
b. Resample it at 8 kHz. (aliasing. 8 kHz < 10 kHz)

$$\frac{L}{D} = \frac{8k}{10k} = \frac{4}{5}$$

Thus, $L = 4$, $D = 5$ are the minimum factors in the fractional sampling scheme. $C = \max(L, D) = 5$.

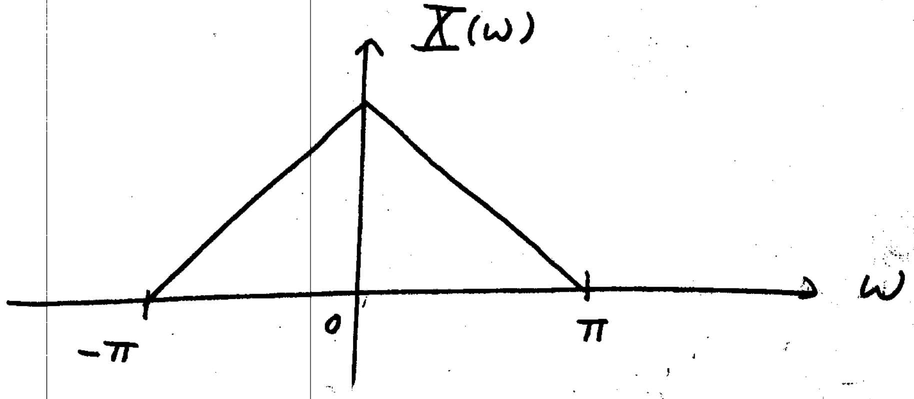
Hence, the fractional sampler can be implemented as:

$$x[n] \longrightarrow \boxed{\uparrow 4} \rightarrow \boxed{LPF} \longrightarrow \boxed{\downarrow 5} \longrightarrow y[n]$$

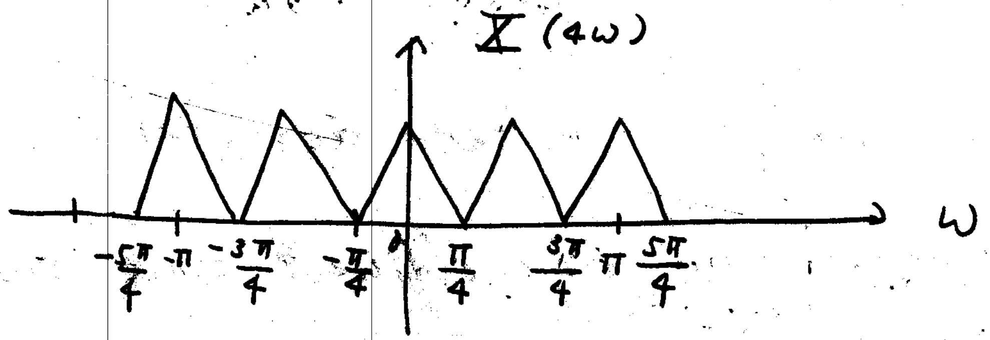


Assume Fs is the Nyquiste rate.

We can illustrate the spectra as follows:

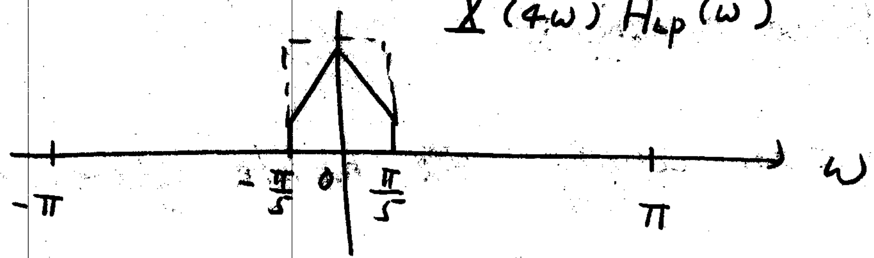


↓ upsampling by 4

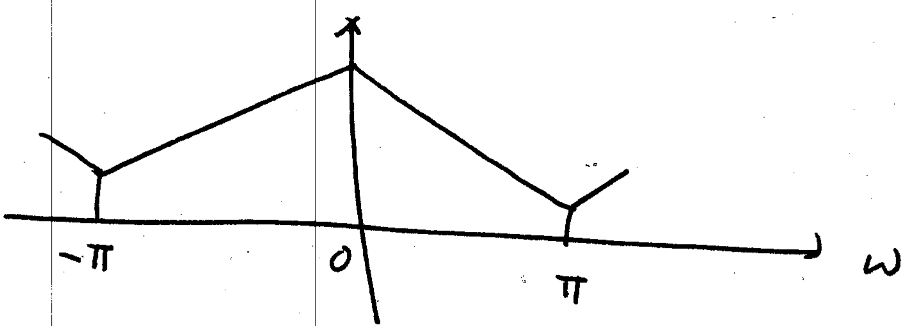


↓ lowpass filtered

$X(4\omega)\, H_{lp}(\omega)$



↓ downsampling by 5



6.5 Multistage Implementation of Digital Filters

Read by yourselves!

6.6 Efficient Implementation of Multirate Systems

The upsampler-filter-downsampler implementation is ineffiecent since only partial data is reserved or nontrivial in these schemes.

We can apply the noble identies and the polyphase decomposition to make no unnecessary computations.

In general, a linear filter does not commute with either an upsampler or a downsampler.

Example :

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \boxed{\downarrow D} \rightarrow y_1[n]$$

$$x[n] \rightarrow \boxed{\downarrow D} \rightarrow \boxed{h[n]} \rightarrow y_2[n]$$

where $h[n] \begin{cases} \neq 0, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$

Check if $y_1[n] = y_2[n]$.

Solution :

$$x[n] \circledast h[n] = \sum_{k=0}^{M} x[n-k]\, h[k]$$

$$y_1[n] = \sum_{k=0}^{M} x[nD-k]\, h[k]$$

$$y_2[n] = h[n] \circledast x[nD]$$

$$= \sum_{k=0}^{M} x[nD-kD]\, h[k]$$

$y_1[n]$ is not necessarily equivalent to $y_2[n]$.

Example:

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \boxed{\uparrow L} \rightarrow y_1[n]$$

$$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{h[n]} \rightarrow y_2[n]$$

where $h[n] \begin{cases} \neq 0, & 0 \leq n \leq M \\ = 0, & \text{otherwise} \end{cases}$

$$x[n] \circledast h[n] = \sum_{n=0}^{M} x[n-k]\, h[k]$$

$$y_1[n] = \frac{1}{L} \sum_{l=0}^{L-1} \sum_{k=0}^{M} x\left[\frac{n}{L} - k\right] h[k]\, e^{-j\frac{2\pi l n}{L}}$$

$$y_2[n] = \sum_{k=0}^{M} \sum_{l=0}^{L-1} x\left[\frac{n-k}{L}\right] e^{-j\frac{2\pi l n}{L}}\, h[k]$$

$y_1[n]$ is not necessarily equivalent to $y_2[n]$.

<u>Noble Identities</u>

However, if the filter has a particular structure, an upsampler or a downsampler can commute with the filter. This leads to the Noble identities. The transfer function complying with the Nòble identities has to be of the form:

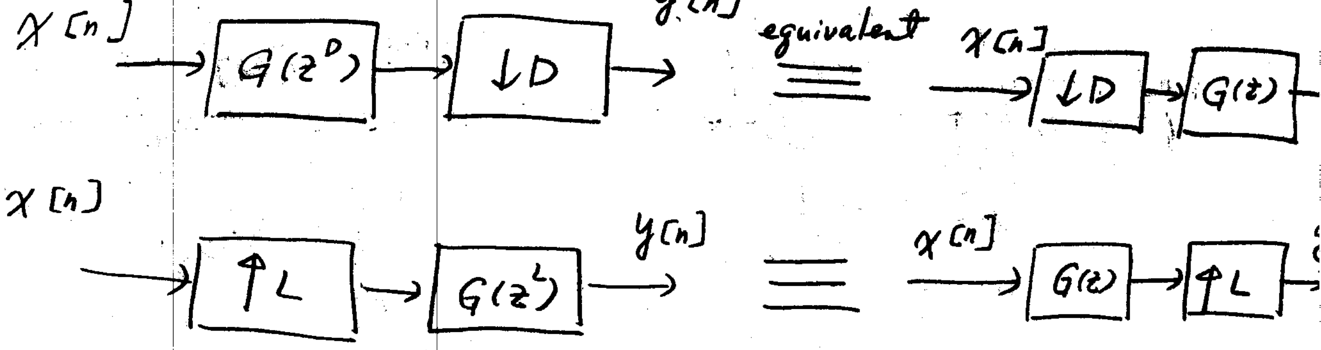
$$G(z^M) = \sum_{n=-\infty}^{\infty} g[n]\, z^{-nM}$$

$$= + \cdots + 0 + g[0] + 0 + \cdots$$

$$+ 0 + g[1]\, z^{-M} + 0 + \cdots + 0 +$$

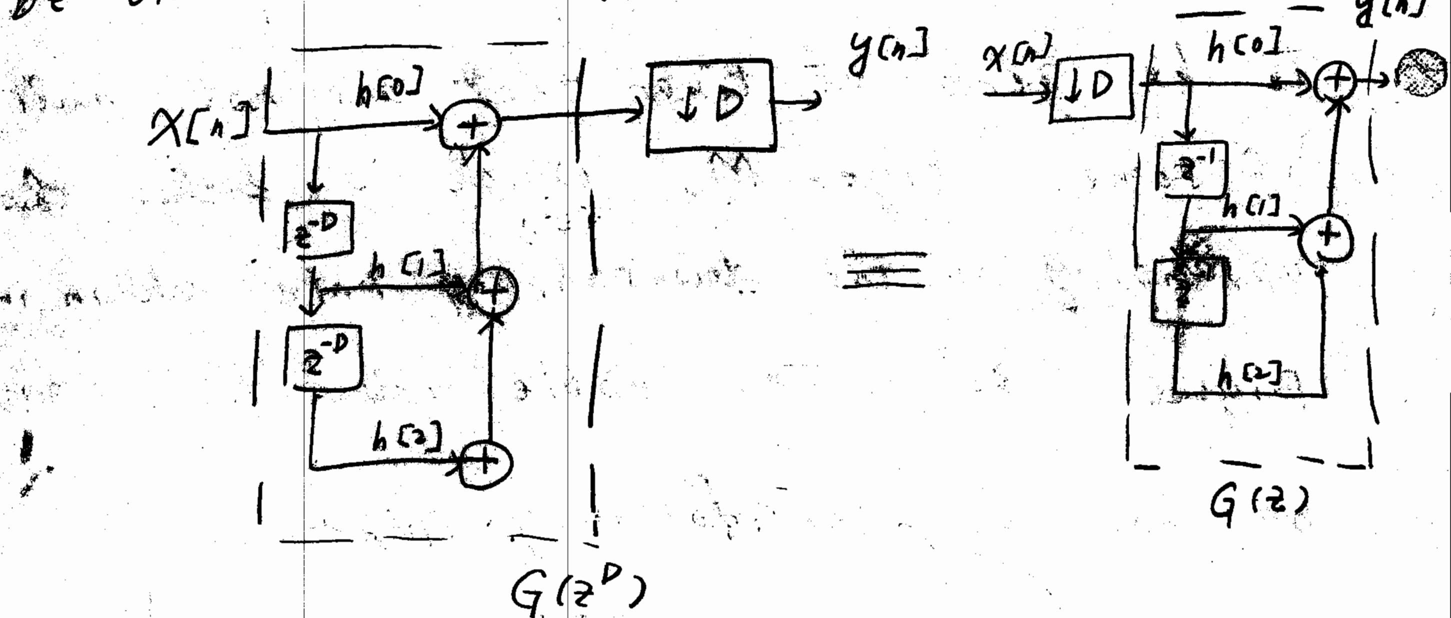
$$g[2]\, z^{-2M} + 0 + \cdots$$

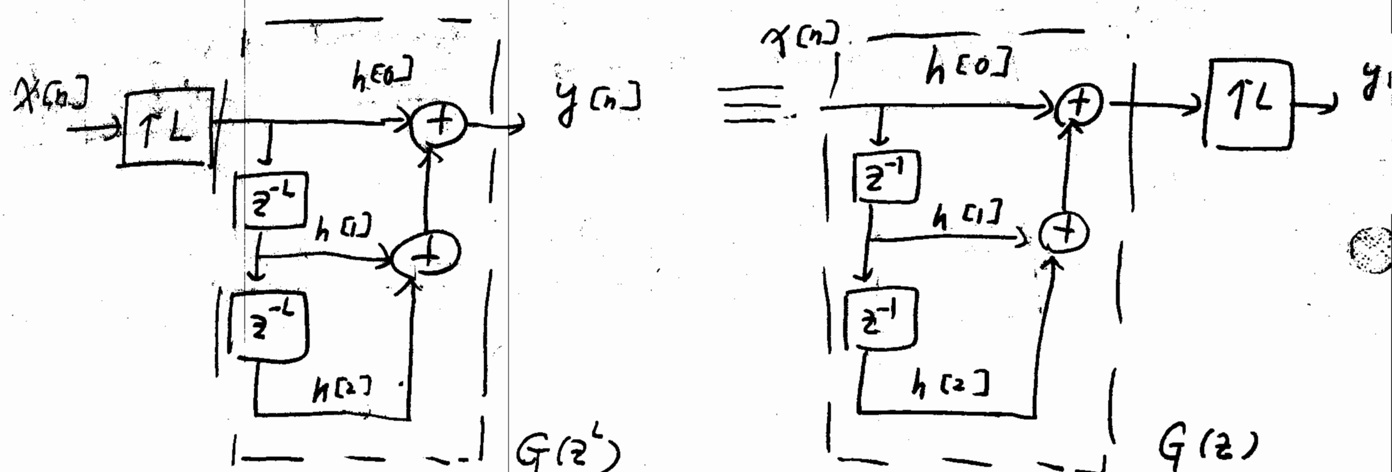
In orther words, its impulse response has to be zero at timer that are not multipl of some integer M, which is either the upsampling or the downsampling rate shown in Figue 6.32. The Nòble identities can be shown in the following figue:



The Noble identities can help us to design a more efficient filtering scheme.

From the figure above, the leftmost implementation involve a lot of zero samples in the filter $G(z^D)$ and $G(z^L)$, but the right most counterparts don't have any unnecessary zero values in the filter $G(z)$. It can easily be visualized as follows:





Polyphase Decomposition of

Given an LTI filter with the transfer function $H(z) = \sum_n h[n] z^{-n}$, we want to decompose it into the superposition of the filters of the form $G(z^M)$. Therefore we can use the Noble identities thereafter.

The transfer function can be easily decomposed as

$$H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M)$$

$$\text{or} \quad H(z) = \sum_{k=0}^{M-1} z^{k} H_{-k}(z^M)$$

where $H_{\pm k}(z^M) = \sum_n h[nM \pm k] z^{-nM}$

Proof: $$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{M-1} h[mM+k] z^{-mM-k}$$

$$= \sum_{k=0}^{M-1} z^{-k} \sum_{m=-\infty}^{\infty} h[mM+k] z^{-mM}$$

$$= \sum_{k=0}^{M-1} z^{-k} H_k(z^M)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{M-1} h[mM-k] z^{-mM+k}$$

$$= \sum_{k=0}^{M-1} z^{k} \sum_{m=-\infty}^{\infty} h[mM-k] z^{-mM}$$

$$= \sum_{k=0}^{M-1} z^{k} H_{-k}(z^M)$$

Example: Consider the transfer function of an FIR filter, $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5}$. Determine the polyphase decomposition for $M = 3$.

Solution:

$$H(z) = (1+4z^{-3}) + z^{-1}(2+5z^{-3}) + z^{-2}(3+6z^{-3})$$

$$= (1+4z^{-3}) + z(2z^{-2}+5z^{-5}) + z^{2}(3z^{-4}+6z^{-7})$$

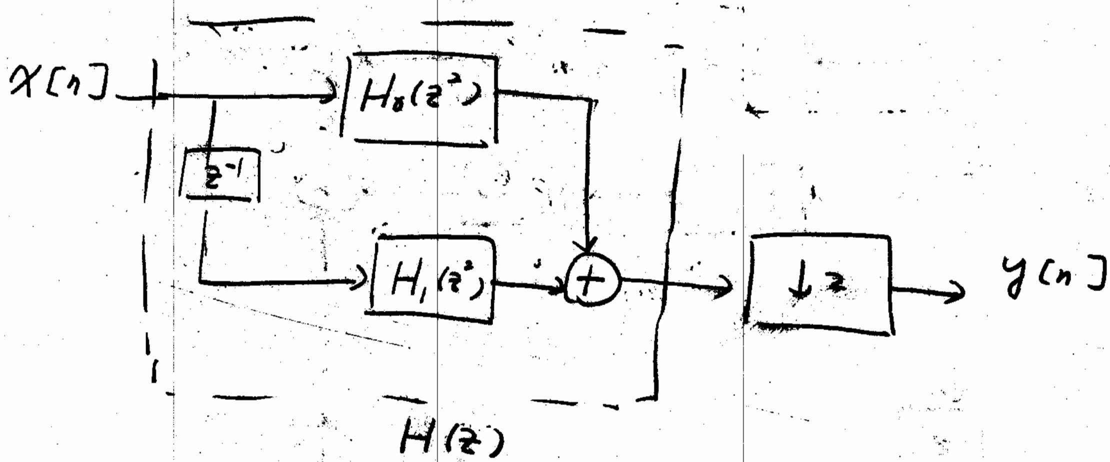
$\therefore$ $H_0(z^3) = 1+4z^{-3}$, $H_1(z^3) = 2+5z^{-3}$

$H_2(z^3) = 3+6z^{-3}$

$H_0(z^3) = 1+4z^{-3}$, $H_{-1}(z^3) = 2z^{-2}+5z^{-5}$

$H_{-2}(z^3) = 3z^{-4}+6z^{-7}$

The significance of this polyphase decomposition can be easily seen when it is incorporated with the Noble identities.



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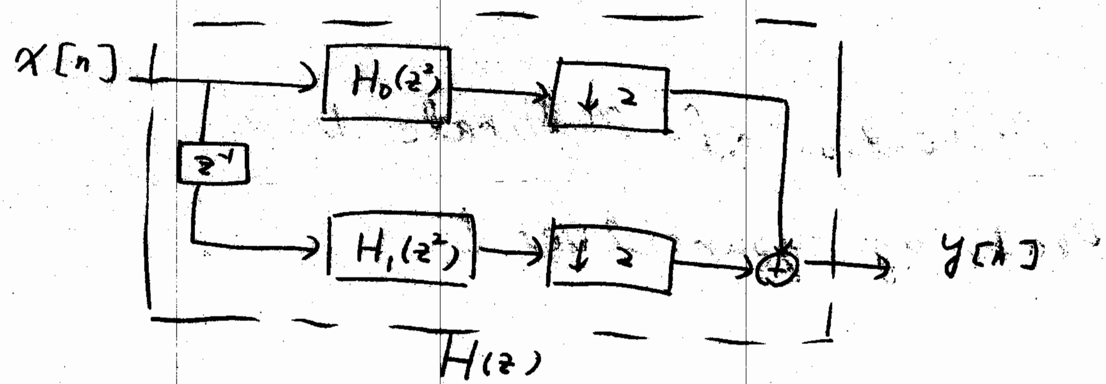

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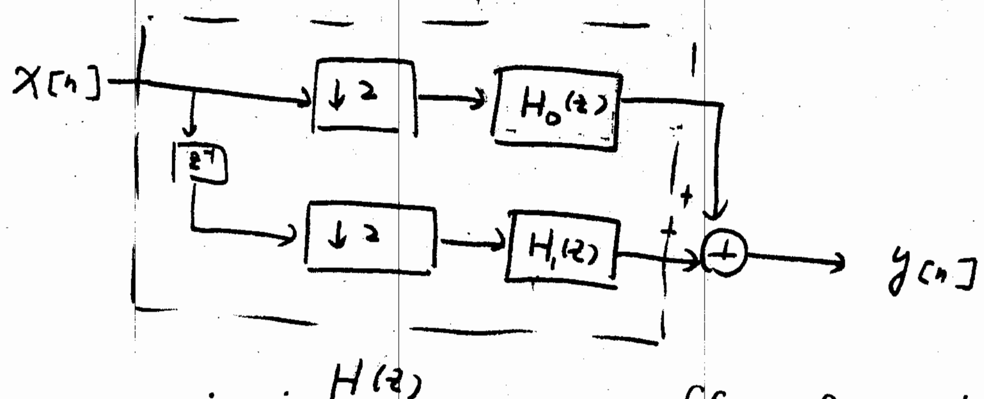
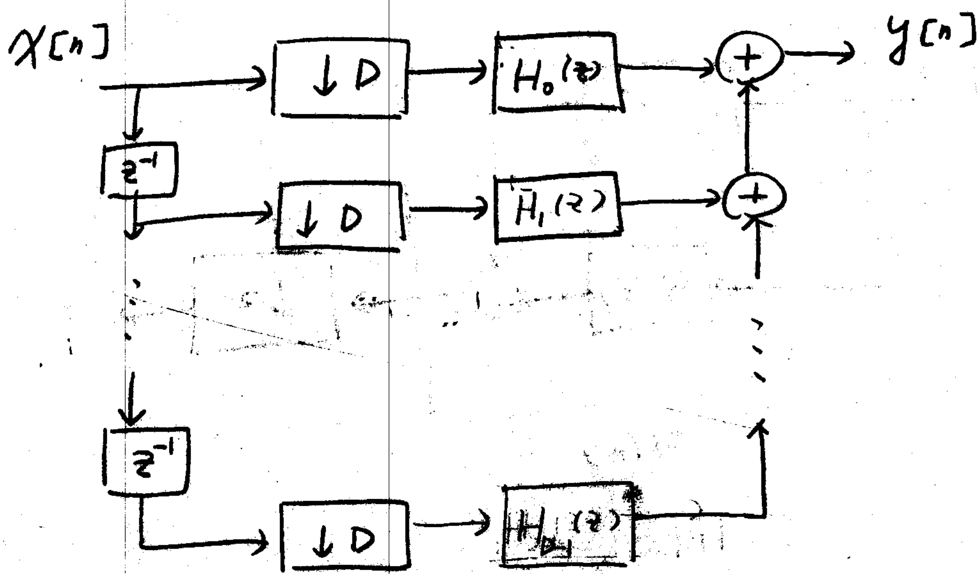


efficient implementation

In general, the decimation and polyphase decomposition can be efficiently implemented as follows:



6.7. Application of Multirate DSP.

Read by yourselves!