Chapter 6 Multirate Signal Processing

6.2 Up-sampling and Down-sampling

Definition: Given an integer $L$, we define the up-sampling operator $S_L$ by

$$y[n] = S_L\{x[n]\} = \begin{cases} x[\frac{n}{L}], & \text{if } \frac{n}{L} \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

The operator $S_L$ increases the sampling frequency by a factor $L$, by adding more samples with zero value.

Definition: Given an integer $D$, we define the downsampling operator $S_D$ by

$$y[n] = S_D\{x[n]\} = x[nD]$$

The operator $S_D$ decreases the sampling frequency by a factor $D$, by keeping one
Sample out of D samples.

Symbolic Notations

Upsampling:

\[ x[n] \rightarrow \text{up} \rightarrow y[n] = S_2 \{ x[n] \} \]

Downsampling:

\[ x[n] \rightarrow \text{down} \rightarrow y[n] = S_{\frac{1}{2}} \{ x[n] \} \]

Example:
Let \( x[n] = \{ \ldots, 1, 2, 3, 4, 5, \ldots \} \)
\[ y[n] = S_2 \{ x[n] \} \]

then \( y[n] = \{ \ldots, 0, 1, 0, 2, 0, 3, 0, 4, \ldots \} \)

Example:
Let \( x[n] = \{ \ldots, 1, 2, 3, 4, 5, \ldots \} \)
\[ y[n] = S_{\frac{1}{2}} \{ x[n] \} \]

then \( y[n] = \{ \ldots, 1, 2, 3, 5, 2, \ldots \} \)
6.3 Analysis of Upsampling and Downsampling

A sampling sequence \( s_p[n] \) is defined as:

\[
s_p[n] = \begin{cases} 
1, & \text{if } n/D \text{ is an integer} \\
0, & \text{otherwise}
\end{cases}
\]

A subset of sequence \( x[n] \) inserted with zeros is defined as:

\[
u[n] = s_p[n] x[n].
\]

It has the z-transform:

\[
u(z) = \mathcal{Z}\{v[n]\}
\]

\[
= \sum_{n=-\infty}^{\infty} v[n] z^{-n} = \sum_{m=-\infty}^{\infty} x[mD] z^{-mD}
\]

Since \( s_p[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{j \frac{2\pi nk}{D}} \)

\[
s_p[n] = \begin{cases} 
1, & \text{if } n/D \text{ is an integer} \\
0, & \text{otherwise}
\end{cases}
\]
\[ V[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n] e^{-j\frac{2\pi kn}{D}} \]

\[ V(z) = \mathbb{Z} \{ V[n] \} = \frac{1}{D} \sum_{n=-\infty}^{\infty} x[n] \left( z e^{-j\frac{2\pi R}{D}} \right)^n \]

\[ = \frac{1}{D} \sum_{R=0}^{D-1} \mathbb{X} \left( e^{-j\frac{2\pi R}{D}} \right) \]

where \( \mathbb{X}(z) = \mathbb{Z} \{ x[n] \} \).

The DTFT \( \tilde{V}(w) = V(z) \big|_{z=e^{jw}} \) is given by

\[ \tilde{V}(w) = \frac{1}{D} \sum_{R=0}^{D-1} \mathbb{X}(w - \frac{2\pi R}{D}) \]

where \( \mathbb{X}(w) = \text{DTFT} \{ x[n] \} \).

Hence the sampling sequence \( S_D[n] \) is the operator to obtain the summation of all \( D \)-polyphase expansions of \( x[n] \).
Example:
Consider \( x[n] = 5 \cos \left( \frac{0.1}{\pi} \right) \). Determine the DTFT of \( x[n] \) and \( X(e^{j\omega}) \).

Solution:
\[
X(\omega) = \text{DTFT} \{ x[n] \} \\
= 5 \pi \delta(\omega - 0.1\pi) + 5 \pi \delta(\omega + 0.1\pi), 
\]
\(-\pi < \omega < \pi\)

\[
U(\omega) = \text{DTFT} \{ u[n] \} \\
= \frac{1}{2} \sum_{k=0}^{\infty} X(\omega - \frac{2\pi k}{3})
\]
\[ U(\omega) = \frac{\pi}{3} \delta(\omega - 0.1\pi) + \frac{\pi}{3} \delta(\omega + 0.1\pi) \]
\[ + \frac{\pi}{3} \delta(\omega - 0.1\pi - \frac{2\pi}{3}) + \frac{\pi}{3} \delta(\omega + 0.1\pi - \frac{2\pi}{3}) \]
\[ + \frac{\pi}{3} \delta(\omega - 0.1\pi + \frac{2\pi}{3}) + \frac{\pi}{3} \delta(\omega + 0.1\pi + \frac{2\pi}{3}) \]

We can apply the analysis for the downsampling operation.

Downsampling

\( y[n] \) is called the decimated sequence, such that

\[ y[n] = S_{1/6} \{ x[n] \} = x[nD] \]

The z-transform of \( y[n] \) can be written as
\[ Y(z) = z \{ y[n] \} = \sum_{n=-\infty}^{\infty} y[nD] z^{-n} \]

If \( y[n] = x[n] \delta[n] \), then

\[ Y(z) = z \{ x[n] \} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

\[ = \sum_{m=-\infty}^{\infty} x[mD] z^{-mD} \]

Thus,

\[ Y(z) = U(z)^{1/2} \]

\[ = \sqrt{D^{-1} \sum_{k=0}^{D-1} X \left( e^{-j\frac{2\pi k}{D} \theta} \right)} \]

where \( X(z) = z \{ x[n] \} \).

The DTFT of \( y[n] \) can be written as

\[ Y(\omega) = \text{DTFT} \{ y[n] \} \]

\[ = \left. Y(z) \right|_{z = e^{j\omega}} \]

\[ = \sqrt{D^{-1} \sum_{k=0}^{D-1} X \left( \frac{2\pi k}{D} - k\frac{2\pi}{D} \right)} \]
To construct the DTFT of $Y(w)$, we have to take the following steps:

1. Compute $U(w) = \frac{1}{D} \sum_{k=0}^{D-1} X(w - \frac{2\pi k}{D})$

2. Stretch the $U(w)$ by changing $w$ to $\frac{w}{D}$.

Illustration:

$X(n) = X(nT_s)$

$Y(n) = X(nD) = X(nDT_s)$, $D=2$
It is obvious from the illustration that the decimation (downsampling) will cause the aliasing if

\[ DF > \frac{\pi}{D}, \]

where \( F_x \) is the bandwidth of \( x(w) \) and \( D \) is the decimation rate.

**Example:** Consider a signal

\[ x[n] = 2 \cos(0.2\pi n - 0.3\pi) + \cos(0.6\pi n + 0.2\pi) \]

Determine the frequency spectrum of \( y[n] = x[2n] \) and what is the maximum decimation rate \( D \) for aliasing-free.

**Solution:**

\[
x(w) = DTF \{ x[n] \} = 2\pi e^{-j0.2\pi} \delta(w-0.2\pi) + 2\pi e^{j0.2\pi} \delta(w+0.2\pi) + 4\pi e^{j0.2\pi} \delta(w-0.6\pi) + 4\pi e^{-j0.2\pi} \delta(w+0.6\pi), \quad -\pi < w < \pi.
\]
\[ \mathcal{Y}(\omega) = \text{DTFT} \left\{ y[n] \right\} = \text{DTFT} \left\{ x[2n] \right\} \]

\[ = \frac{1}{2} X \left( \frac{\omega}{2} \right) + \frac{1}{2} X \left( \frac{\omega}{2} - \pi \right) \]

\[ = \pi e^{-j0.1\pi} \delta \left( \frac{\omega}{2} - 0.2\pi \right) + \pi e^{j0.3\pi} \delta \left( \frac{\omega}{2} + 0.1\pi \right) + \pi e^{j0.1\pi} \delta \left( \frac{\omega}{2} - 0.6\pi \right) + 2\pi e^{-j0.2\pi} \delta \left( \frac{\omega}{2} + 0.6\pi \right) \]

\[ + \pi e^{-j0.1\pi} \delta \left( \frac{\omega}{2} + 0.8\pi \right) + \pi e^{j0.3\pi} \delta \left( \frac{\omega}{2} - 0.8\pi \right) \]

\[ + 2\pi e^{j0.1\pi} \delta \left( \frac{\omega}{2} - 0.4\pi \right) + 2\pi e^{j0.3\pi} \delta \left( \frac{\omega}{2} - 0.4\pi \right) \]

\[-\pi < \frac{\omega}{2} \leq \pi.\]
The 2-transform of $y[n]$ is written as

$$\mathcal{Z}^{-1}\{y[n]\} = x[k]$$

where $x[k] = E\{x(n)\}$ for $n$ is an integer.

In general, $y[n]$, $x[n]$, and $x[k]$ are related as follows:

$$D \geq 2 \rightarrow \frac{1}{D} \rightarrow \frac{1}{D^2}$$

Upsampling

The maximum downsampling: free decimation

$$F_x = 0.67r$$

$D \times 0.67r > \frac{1}{D}$$
The DTFT of $y[n]$ can be obtained as

$$\bar{Y}(\omega) = \frac{Y(\omega)}{2\pi} \mid_{z=e^{j\omega}}$$

where $X(\omega)$ is the DTFT of $x[n]$.

Illustration:

$$X(\omega) = \text{DTFT} \{ x[n] \}$$

![Graph showing up-sampling and DTFT relations]
Example:

Consider the signal

\[ X[n] = 2 \cos(2\pi n - 0.8\pi) + 4 \cos(0.6\pi n + 0.2\pi) \]

Call \( y[n] = X\left[\frac{n}{2}\right] \delta[n] \) the signal after upsampling by \( L = 2 \). Determine the DTFT of \( y[n] \).

Solution:

\[ \tilde{X}(\omega) = 2\pi \sum_{\omega_0 = -\pi}^{\pi} \delta(\omega - \omega_0) + 2\pi e^{j0.3\pi} \delta(\omega + 0.3\pi) + 4\pi e^{j0.3\pi} \delta(\omega - 0.3\pi) + 4\pi e^{-j0.3\pi} \delta(\omega + 0.6\pi), \]

\[ -\pi < \omega < \pi \]

\[ \tilde{Y}(\omega) = \tilde{X}(2\omega) \]

\[ = 2\pi \sum_{\omega_0 = -\pi}^{\pi} \delta(2\omega - \omega_0) + 2\pi e^{j0.3\pi} \delta(2\omega + 0.3\pi) + 2\pi e^{j0.3\pi} \delta(2\omega - 0.3\pi + 2\pi) + 2\pi e^{-j0.3\pi} \delta(2\omega + 0.6\pi - 2\pi) + 4\pi e^{j0.3\pi} \delta(2\omega - 0.6\pi) + 4\pi e^{-j0.3\pi} \delta(2\omega + 0.6\pi) + 4\pi e^{j0.3\pi} \delta(2\omega - 0.6\pi + 2\pi) + 4\pi e^{-j0.3\pi} \delta(2\omega + 0.6\pi - 2\pi) \]
6.4 Fractional Sampling

The fractional sampling is the technique for resampling a digital signal $x[n]$ from the original sampling rate $F_s$ Hz into a new sampling rate $(1/D) F_s$ Hz, where both $L$ & $D$ are integers.

We can establish such a fractional sampler through downsampling and upsampling in series.

1. Interpolation by an integer factor $L$:
   The signal can be first upsampled by a factor $L$. The upsampling alone introduces artifacts in the frequency domain as image frequencies. All image frequencies are outside the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and can be eliminated by a lowpass filtering.
Upsampling by $L$, $L = 2$

$I(\omega L)$

Image frequencies

Lowpass filtering to eliminate image frequencies

Lowpass filtered
Decimation by an integer factor \( D \)

The signal will be downsampled after the interpolation and lowpass-filtering. As mentioned before, the downsampling stretches the spectrum. It can follow a lowpass filter to avoid the aliasing.

\[
\tilde{X}(w)
\]

\[
\downarrow \text{Downsampling by } D, \ D = 2 \]

\[
\tilde{Y}(w)
\]
- Fractional Sampling by a rational factor \( \frac{L}{D} \)

A general case of sampling-rate conversion by a rational factor \( \frac{L}{D} \) is combining an interpolation operation by \( L \) with a decimation operation by \( D \) right after.

\[ X[n] \xrightarrow{\uparrow L} \mathrm{LPF} \xrightarrow{\downarrow D} Y[n] \]
Example:
Consider a signal $x[n]$ sampled at $F_s = 10k$ Hz. Let's resample it as follows:

a. Resample it at $22k$ Hz. (aliasing-free; $22k$ Hz $> 10k$ Hz)

\[
\frac{L}{D} = \frac{22k}{10k} = \frac{11}{5}
\]

Thus, $L = 11$ and $D = 5$ are the minimum factors in the fractional sampling scheme. $C = \max(L, D) = 11$

Hence, the fractional sampler can be implemented as:

\[
x[n] \rightarrow 9 \frac{11}{15} \rightarrow LPF \rightarrow \downarrow s \rightarrow y[n]
\]

Assume $F_s$ is the Nyquist rate (twice of the signal bandwidth).
We can illustrate the spectra as follows:

\[
\hat{x}(ω) = \text{DTFT} \{x[n]\}
\]

\[
\downarrow \quad \text{upsampling by 11}
\]

\[
\hat{x}(11ω)
\]

\[
\downarrow \quad \text{lowpass filtered}
\]

\[
\hat{x}(11ω) \cdot H_{lp}(ω)
\]

\[
\downarrow \quad \text{downsampling by 5}
\]

\[
\hat{x}(ω) \quad \text{aliasing-free}
\]
b. Resample it at 8 kHz: (aliasing, $8 \text{ kHz} < 10 \text{ kHz}$)

$$\frac{L}{D} = \frac{8k}{10k} = \frac{4}{5}$$

Thus, $L = 4, D = 5$ are the minimum factors in the fractional sampling scheme. $C = \max(L, D) = 5$.

Hence, the fractional sampler can be implemented as:

```
\[ X[n] \xrightarrow{\uparrow 4} \Box \quad \frac{1}{2} \quad \Box \quad \downarrow 5 \quad Y[n] \]
```

Assume $F_s$ is the Nyquist rate.

We can illustrate the spectra as follows:
\[ X(\omega) \]

\[ \downarrow \text{upsampling by 4} \]

\[ X(4\omega) \]

\[ \downarrow \text{low pass filtered} \]

\[ X(4\omega) \text{ } \text{H}_{\text{lp}}(\omega) \]

\[ \downarrow \text{downsampling by 5} \]

\[ X(\omega) \]
6.5 Multistage Implementation of Digital Filters

Read by yourselves!

6.6 Efficient Implementation of Multirate Systems

The up sampler - filter - down sampler implementation is inefficient since only partial data is reserved or nontrivial in these schemes.

We can apply the noble identities and the polyphase decomposition to make no unnecessary computations.

In general, a linear filter does not compute with either an up sampler or a
Example:

\[ X[n] \rightarrow \downarrow D \rightarrow Y_1[n] \]
\[ X[n] \rightarrow \downarrow D \rightarrow \downarrow C \rightarrow Y_2[n] \]

where \( h[n] \) is defined as:
\[ h[n] = \begin{cases} 
 2^M, & 0 \leq n \leq M-1 \\
 0, & \text{otherwise}
\end{cases} \]

Check if \( Y_1[n] = Y_2[n] \).

Solution:

\[ X[n] \otimes h[n] = \sum_{k=0}^{M} X[n-k] h[k] \]
\[ Y_1[n] = \sum_{k=0}^{M} X[n] \otimes h[k] \]
\[ Y_2[n] = h[n] \otimes X[nD] = \sum_{k=0}^{M} X[nD-kD] h[k] \]

\( Y_2[n] \) is not necessarily equivalent to \( Y_1[n] \).
Example:
\[ x[n] \xrightarrow{\text{ } \uparrow L \text{ } } h[n] \xrightarrow{\text{ } \uparrow L \text{ } } y_1[n] \]
\[ x[n] \xrightarrow{\text{ } h[n] \text{ } } y_2[n] \]

where \( h[n] \neq 0 \), \( 0 \leq n \leq M \)

\[ X[n] \ast h[n] = \sum_{n=0}^{M} X[n-k] h[k] \]

\[ y_1[n] = \frac{1}{L} \sum_{k=0}^{L-1} \sum_{\ell=0}^{M} x[\ell] h[k] e^{-j \frac{2 \pi \ell n}{L}} \]

\[ y_2[n] = \sum_{k=0}^{M} \sum_{\ell=0}^{L-1} x[\ell] h[k] e^{-j \frac{2 \pi \ell n}{L}} \]

\( y_1[n] \) is not necessarily equivalent to \( y_2[n] \).

Noble Identities

However, if the filter has a particular structure, an up-sampler or a down-sampler can commute with the filter. This leads to the Noble identities. The transfer function
complying with the Noble identities has to be of the form:

\[ G(z^M) = \sum_{n=-\infty}^{\infty} g[n] z^{-nM} \]

\[ = + \cdots + 0 + g[0] z^0 + \cdots + 0 + g[1] z^{-M} + 0 + \cdots \]

In other words, its impulse response has to be zero at times that are not multiples of some integer \( M \), which is either the upsampling or the downsampling rate shown in Figure 6.32. The Noble identities can be shown in the following figure:
The Noble identities can help us to design a more efficient filtering scheme. From the figure above, the leftmost implementations involve a lot of zero samples in the filters $G(z^p)$ and $G(z^l)$, but the rightmost counterparts don’t have any unnecessary zero values in the filter $G(z)$. It can easily be visualized as follows:
Polyphase Decomposition of LTI Filters

Given an LTI filter with the transfer function \( H(z) = \sum_n h(n) z^{-n} \), we want to decompose it into the superposition of the filters of the form \( G(z^M) \). Therefore, we can use the Noble identities thereafter.

The transfer function can be easily decomposed as

\[
H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M)
\]

or

\[
H(z) = \sum_{k=0}^{M-1} z^k H_k^{-1}(z^M)
\]

where \( H_k(z^M) = \sum_n h[nM+k] z^{-nM} \)

Proof:

\[
H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{M-1} h[(mM+k)] z^{-mM-k}
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{M-1} h[(mM+k)] z^{-mM-k} = \sum_{k=0}^{M-1} z^{-k} \sum_{m=-\infty}^{\infty} h[(mM+k)] z^{-mM}
\]
\[ H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M) \]

\[ H(z) = \sum_{m=0}^{\infty} h_{m} \text{conj.} z^{-m} \]

\[ H(z) = \sum_{m=0}^{\infty} \sum_{k=0}^{M-1} h_{[mM-k]} z^{-mM-k} \]

Example:

Consider the transfer function of an FIR filter:

\[ H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5} \]

Determine the polyphase decomposition for \( M = 3 \).

Solution:

\[ H(z) = (1 + 4z^{-1} + 2z^{-2} + 5z^{-3} + z^{-4}) \]

\[ H_0(z^{3}) = 1 + 4z^{-1}, \quad H_1(z^{3}) = 2 + 5z^{-1} \]

\[ H_2(z^{3}) = 3 + 6z^{-1} \]

\[ H_0(z^{3}) = 1 + 4z^{-1}, \quad H_1(z^{3}) = 2 + 5z^{-1} \]

\[ H_2(z^{3}) = 3 + 6z^{-1} \]
The significance of this polyphase decomposition can be easily seen when it is incorporated with the Noble identities.

Efficient implementation
In general, the decimation and polyphase decomposition can be efficiently implemented as follows:

\[ \begin{align*}
X[n] & \rightarrow \downarrow D \rightarrow H_0[n] \rightarrow + \rightarrow y[n] \\
& \downarrow D \rightarrow H_1[n] \rightarrow + \rightarrow \\
& \vdots \\
& \downarrow D \rightarrow H_m[n] \rightarrow \end{align*} \]

6.7. Application of Multirate DSP:
Read by yourselves!