Chapter 5 Digital Filter Implementation

Read 5.1 and 5.2 by yourselves!!

5.3 State Space Realization

In general the transfer function of a digital filter can be written as

$$H(z) = \frac{B(z^{-1})}{A(z^{-1})}$$

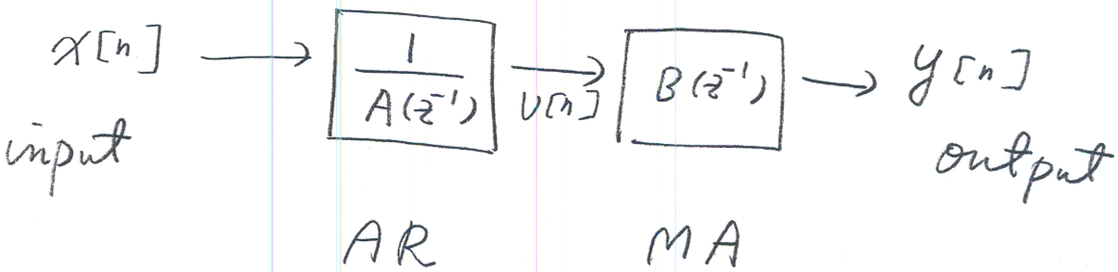
where

$$B(z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}$$

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

and $B(z^{-1})$ denotes the MA (moving average) filter while $A(z^{-1})$ denotes the AR (autoregressive) filter.

The input/output relationship can be illustrated in the following figure.



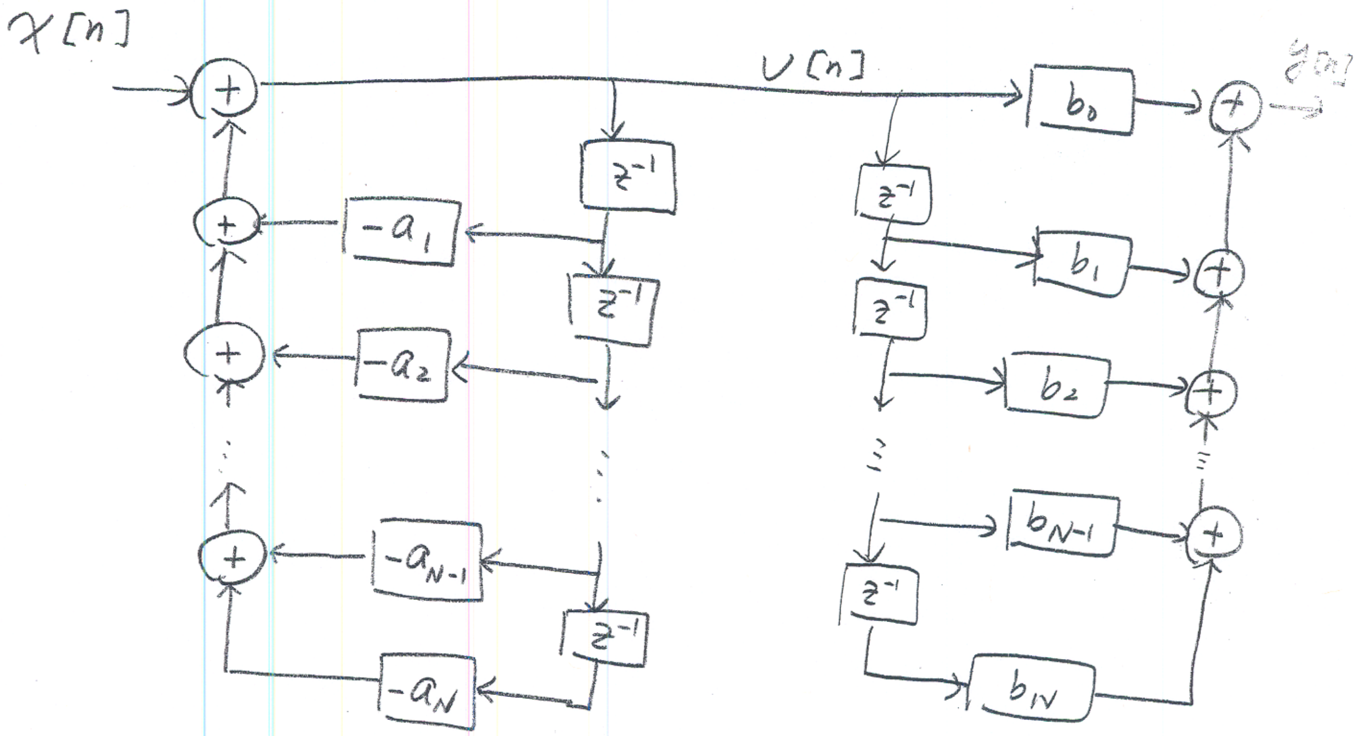
The two systems (AR, MA) can be implemented as the two difference equations:

$$v[n] = -a_1 v[n-1] - a_2 v[n-2] - \cdots - a_N v[n-N] + x[n]$$

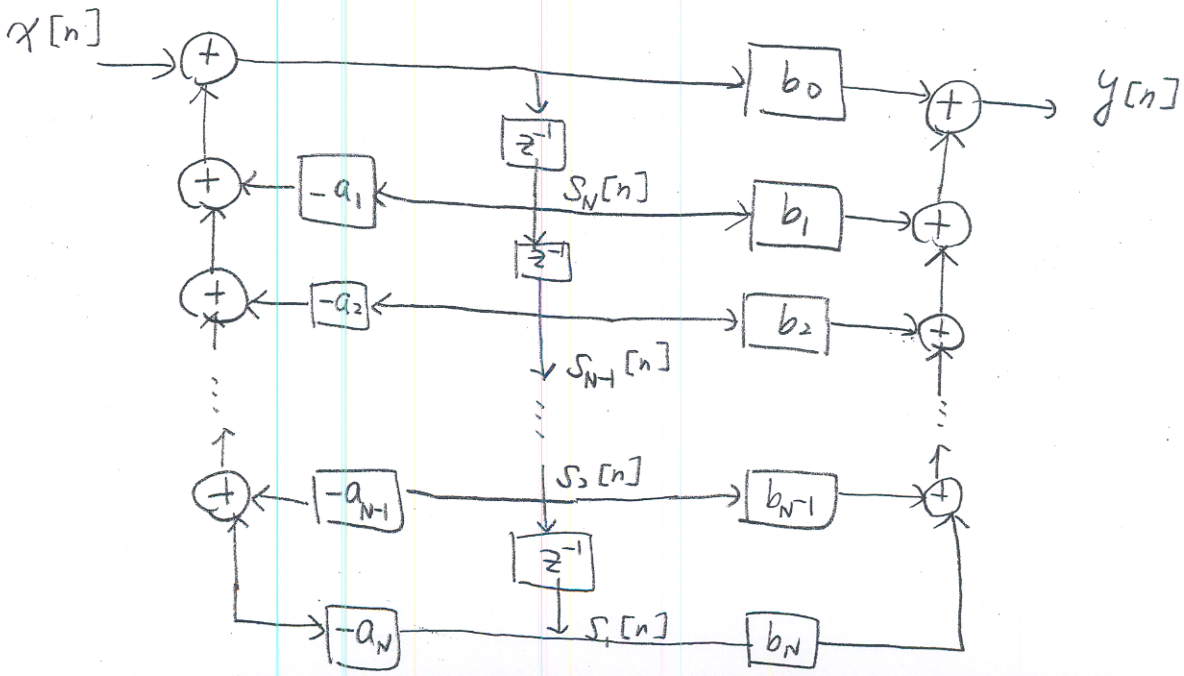
$$y[n] = b_0 v[n] + b_1 v[n-1] + b_2 v[n-2] + \cdots + b_N v[n-N]$$

There are two types of realization for this arbitrary transfer function $H(z)$. We depict as below.

Type I. realization



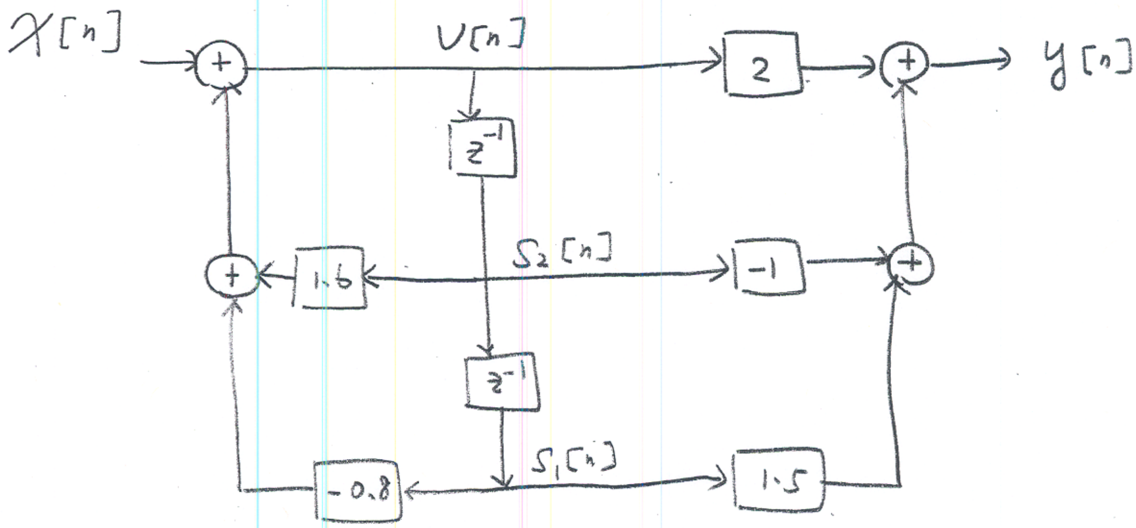
Type II realization



Example: Consider an IIR transfer function

$$H(z) = \frac{2z^2 - z + 1.5}{z^2 - 1.6z + 0.8}$$

Implement this transfer function using type II realization.



In terms of state space representation,

$$\vec{s}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.6 \end{bmatrix} \vec{s}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

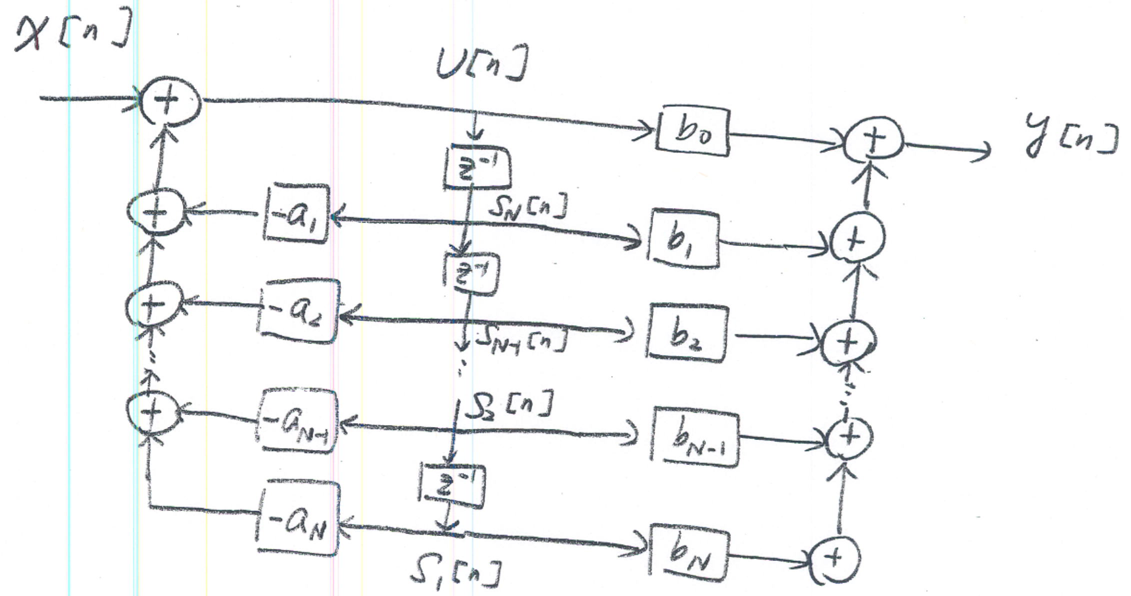
$$y[n] = \begin{bmatrix} -0.1 & 2.2 \end{bmatrix} \vec{s}[n] + 2x[n]$$

where $\vec{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$

In general, a transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

has a type II realization (Fig. 5.9) as below:



We can write the state-space realization as a set of first-order difference equations:

$$S_1[n+1] = S_2[n]$$

$$S_2[n+1] = S_3[n]$$

$$\vdots$$

$$S_N[n+1] = -a_N S_1[n] - \cdots - a_1 S_N[n] + x[n]$$

$$y[n] = (b_N - b_0 a_N) S_1[n] + \cdots + (b_1 - b_0) S_N[n] + b_0 x[n]$$

Thus, $$\vec{s}[n+1] = \tilde{A}\,\vec{s}[n] + \vec{B}\,x[n]$$
$$y[n] = \vec{C}^T \vec{s}[n] + D\,x[n],$$

where $\vec{s}[n] = [\vec{s}_1[n],\ s_2[n] \cdots s_N[n]]^T$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots \\ 0 & 0 & \vdots & \ddots & 0 \\ & & 0 & & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix}_{N\times N}$$

$$\vec{B} = [0, 0, \cdots 0, 1]^T_{N\times 1}$$

$$\vec{C} = [b_N - b_0 a_N,\ b_{N-1} - b_0 a_{N-1}, \cdots, b_1 - b_0 a_1]^T_{N\times 1}$$

$$D = b_0$$

By taking Z-transform, we will obtain

$$z\vec{S}(z) = \tilde{A}\,\vec{S}(z) + \vec{B}\,X(z)$$

where $\vec{S}(z) = \mathcal{Z}\{\vec{s}[n]\}$

$$= \begin{bmatrix} \mathcal{Z}\{s_1[n]\} \\ \mathcal{Z}\{s_2[n]\} \\ \vdots \\ \mathcal{Z}\{s_N[n]\} \end{bmatrix}_{N\times 1}$$

$$\mathbb{X}(z) = \mathcal{Z}\{x[n]\} \quad \cdots \text{ scalar}$$

Therefore, $\vec{S}(z) = (z\tilde{I}_{N\times N} - \tilde{A})^{-1}\vec{B}\,\mathbb{X}(z)$

$$\mathbb{Y}(z) = \vec{C}^T\vec{S}(z) + D\,\mathbb{X}(z)$$

where $\mathbb{Y}(z) = \mathcal{Z}\{y[n]\} \quad \cdots$ scalar

$$= \vec{C}^T(z\tilde{I} - \tilde{A})^{-1}\vec{B}\,\mathbb{X}(z) + D\,\mathbb{X}(z)$$

$$= [\vec{C}^T(z\tilde{I} - \tilde{A})^{-1}\vec{B} + D]\,\mathbb{X}(z)$$

Consequently, the transfer function can be formulated as

$$H(z) = \frac{\mathbb{Y}(z)}{\mathbb{X}(z)} = \vec{C}^T(z\tilde{I} - \tilde{A})^{-1}\vec{B} + D$$

$$= \vec{C}^T\,\frac{\text{adj}\,(z\tilde{I} - \tilde{A})}{\det\,(z\tilde{I} - \tilde{A})}\,\vec{B} + D$$

It is noted that the poles of the transfer function are simply the roots of

$$\det\,(z\tilde{I} - \tilde{A}) = 0,$$

or simply the eigenvalues of the matrix $\tilde{A}$.

Example: The state space realization for an IIR system can be given by

$$\vec{s}[n+1] = \underset{2\times 2}{\begin{bmatrix} 0.5 & 0.8 \\ -0.2 & 0.6 \end{bmatrix}} \vec{s}[n] + \underset{2\times 1}{\begin{bmatrix} 2 \\ -2 \end{bmatrix}} x[n]$$

$$y[n] = [1 \quad -5]\, \vec{s}[n] + x[n].$$

What are $\tilde{A}$, $\vec{B}$, $\vec{C}$ and D? Determine the transfa function $H(z)$.

Solution:

$$\tilde{A} = \begin{bmatrix} 0.5 & 0.8 \\ -0.2 & 0.6 \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad D = 1$$

$$H(z) = \vec{C}^T (z\tilde{I} - \tilde{A})^{-1} \vec{B} + D$$

$$= [1 \quad -5] \begin{bmatrix} z-0.5 & -0.8 \\ 0.2 & z-0.6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$+1$$

$$= [1 \quad -5] \frac{\begin{bmatrix} z-0.6 & 0.8 \\ -0.2 & z-0.5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}}{z^2 - 1.1z + 0.46} + 1$$

$$= \frac{\begin{bmatrix} z+0.4 & -5z+3.3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}}{z^2 - 1.1z + 0.46} + 1$$

$$= \frac{12z - 5.8}{z^2 - 1.1z + 0.46} + 1$$

$$= \frac{z^2 + 10.9z - 5.34}{z^2 - 1.1z + 0.46}$$

For a covenient realization, we would like to "decouple" the states $s_1[n]$, $s_2[n]$, .. $s_N[n]$ That means $s_i[n+1]$ depends on $s_i[n]$ only, $\forall i = 1, 2, \cdots, N$. However, in a general IIR system, it is impossible to directly write such a "decoupled" state space realization.

We need to apply a similarity transformation hae to decouple the states. A set of new states, which are decoupled, can be defined as $\vec{\bar{s}}[n] = \tilde{Q}^{-1} \vec{s}[n]$, where $\tilde{Q}$ is the so-called similarity transformation matrix. Hence we can rewrite the same system using the new state vector $\vec{\bar{s}}[n]$ as follows:

$$\vec{\bar{s}}[n+1] = \tilde{\bar{A}}\,\vec{\bar{s}}[n] + \vec{\bar{B}}\,x[n]$$

$$y[n] = \vec{\bar{C}}^{T}\vec{\bar{s}}[n] + D\,x[n]$$

where
$$\tilde{\bar{A}} = \tilde{Q}^{-1}\tilde{A}\,\tilde{Q}$$
$$\vec{\bar{B}} = \tilde{Q}^{-1}\vec{B}$$
$$\vec{\bar{C}} = \tilde{Q}^{T}\vec{C}.$$

The form $\tilde{\bar{A}} = \tilde{Q}^{-1}\tilde{A}\,\tilde{Q}$ reads as

$\tilde{\bar{A}}$ is the similarity transform of $\tilde{A}$ and

$\tilde{\bar{A}}$ is desired to be diagonal for the decoupled states $\vec{\bar{s}}[n]$.

The transfer function associated with the new states doesn't change, that is

$$\vec{C}^T (z\tilde{I} - \tilde{A})^{-1} \vec{B} = \vec{C}^T \tilde{Q} (z \tilde{Q}^{-1} \tilde{Q} - \tilde{Q}^{-1} \tilde{A} \tilde{Q})$$

$$\times \tilde{Q}^{-1} \vec{B}$$

$$= \vec{C}^T (z\tilde{I} - \tilde{A})^{-1} \vec{B}$$

Assume $\tilde{A}$ is diagonal, namely

$$\tilde{A} = \tilde{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & & & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}.$$

Then $\tilde{\Lambda} \vec{q}_i = \lambda_i \vec{q}_i, \quad i = 1, 2, \ldots, N$

where $\tilde{Q} = [\vec{q}_1, \vec{q}_2, \cdots \vec{q}_N]$ is the eigenvector matrix associated with $\tilde{A}$

Example: Consider the system with the transfer function

$$H(z) = \frac{2z^2 - z + 2}{z^2 - 0.81}$$

Apply the similarity transformation to decouple the states

$$\vec{s}[n+1] = \begin{bmatrix} 0 & 1 \\ 0.81 & 0 \end{bmatrix} \vec{s}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} 3.62 & -1 \end{bmatrix} \vec{s}[n] + 2x[n]$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 0.81 & 0 \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} 3.62 \\ -1 \end{bmatrix} \quad D = 2$$

Take the eigen decomposition, we have

$$\tilde{A}\tilde{Q} = \tilde{Q}\tilde{\Lambda},$$

where $\tilde{Q} = \begin{bmatrix} 0.7433 & -0.7433 \\ 0.6690 & 0.6690 \end{bmatrix}$

$$\tilde{\Lambda} = \begin{bmatrix} 0.9 & 0 \\ 0 & -0.9 \end{bmatrix}$$

$$\hat{\tilde{A}} = \tilde{\Lambda} = \tilde{Q}^{-1}\tilde{A}\tilde{Q} = \begin{bmatrix} 0.9 & 0 \\ 0 & -0.9 \end{bmatrix}$$

$$\bar{\vec{B}} = \tilde{Q}^{-1}\vec{B} = \begin{bmatrix} 0.7474 \\ 0.7474 \end{bmatrix}$$

$$\bar{\vec{C}} = \tilde{Q}^{T}\vec{C} = \begin{bmatrix} 2.0218 & -3.3597 \end{bmatrix}$$

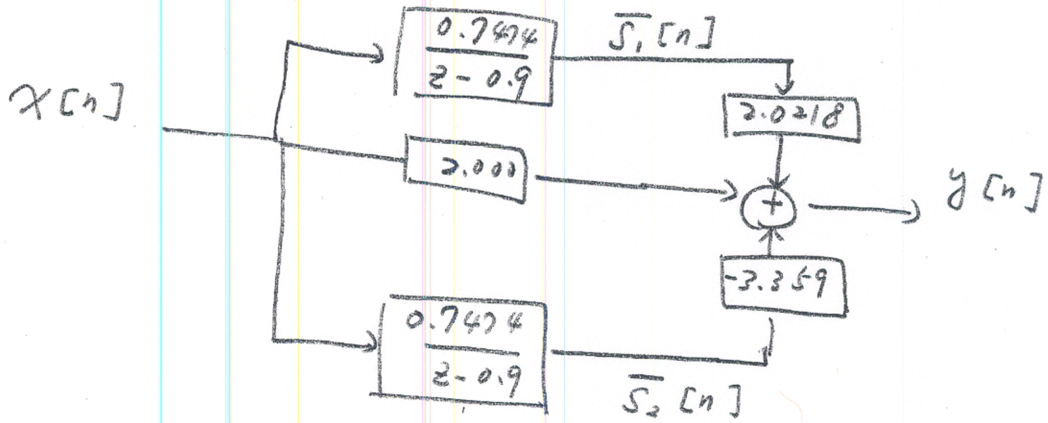
$$\bar{D} = D = 2$$

The decoupled difference equations can be written as:

$$\bar{s}_1[n+1] = 0.9\,\bar{s}_1[n] + 0.7474\,x[n]$$

$$\bar{s}_2[n+1] = -0.9\,\bar{s}_2[n] + 0.7474\,x[n]$$

$$y[n] = 2.0218\,\bar{s}_1[n] - 3.3597\,\bar{s}_2[n] + 2\,x[n]$$



The decoupled system

Example: Consider a system with the following transfer function:

$$H(z) = \frac{2z^2 - z + 2}{z^2 + z + 0.9}.$$

Using the type II realization, we have the state-space realization as follows:

$$\vec{s}[n+1] = \begin{bmatrix} 0 & 1 \\ -0.9 & -1 \end{bmatrix} \vec{s}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} 0.2 & -3 \end{bmatrix} \vec{s}[n] + 2x[n]$$

The eigen vector matrix of $\tilde{A} = \begin{bmatrix} 0 & 1 \\ -0.9 & -1 \end{bmatrix}$

is $\tilde{Q} = \begin{bmatrix} 0.6165 & -0.3824 \\ 0 & 0.6882 \end{bmatrix}$

$$\tilde{\hat{A}} = \hat{Q}^{-1} \tilde{A} \hat{Q} = 0.9487 \begin{bmatrix} -0.5270 & 0.8498 \\ -0.8498 & -0.5270 \end{bmatrix}$$

$$\vec{\bar{B}} = \hat{Q}^{-1} \vec{B} = \begin{bmatrix} 0.9013 \\ 1.4531 \end{bmatrix}$$

$$\vec{\bar{c}} = \tilde{Q}^{T} \vec{C} = \begin{bmatrix} 0.1233 & -2.1411 \end{bmatrix}$$

$$\bar{D} = D = 2$$