Chapter 4 Digital Filters

4.2 Ideal and Nonideal Filters

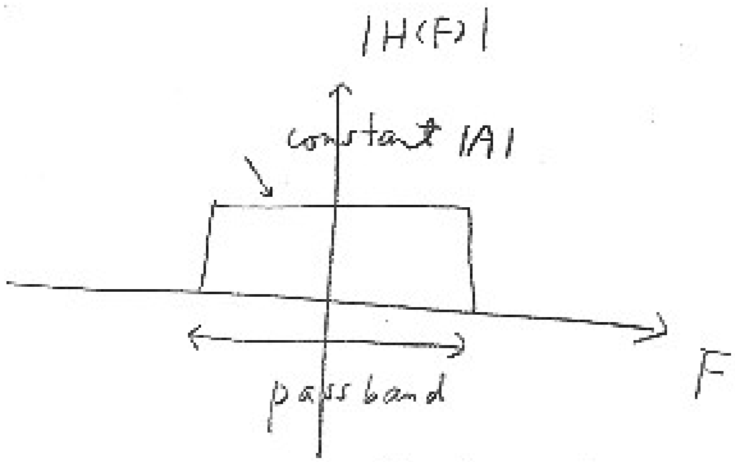
$$x(t) = \delta(t) \longrightarrow \underset{h(t)}{\boxed{\text{Ideal Filter}}} \longrightarrow y(t) = h(t) \circledast x(t)$$

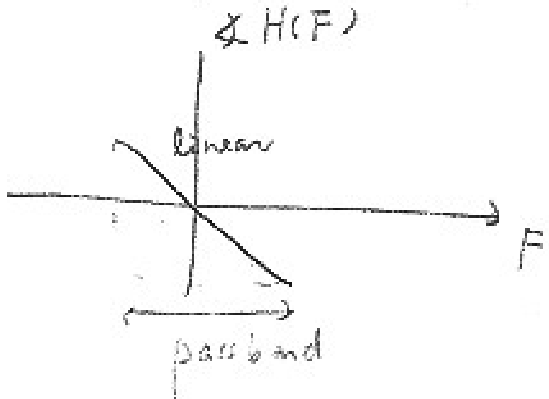
$$= A\delta(t-T)$$

$$H(F) = \underbrace{FT}_{\text{Fourier Transform}} \{h(t)\}$$

$$= \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi F t} \, dt$$

$$= \begin{cases} A e^{-j2\pi F T}, & \text{if } F \text{ is in the passband} \\ \emptyset, & \text{otherwise} \end{cases}$$

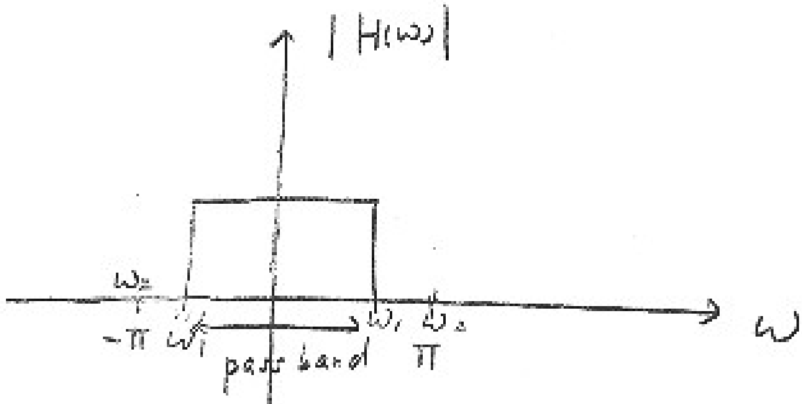


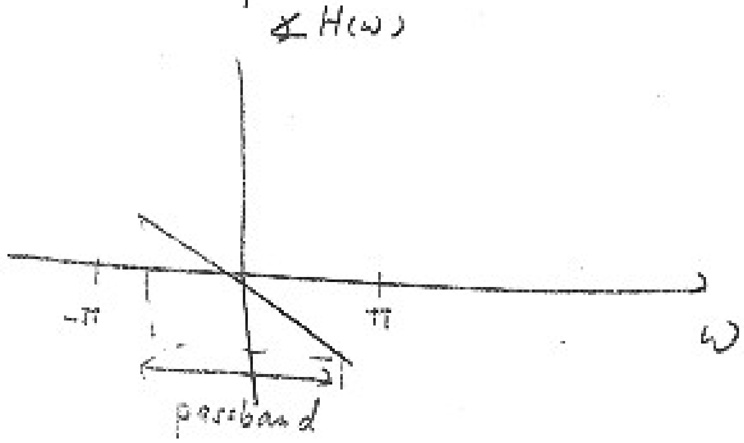


For digital signals, we have the ideal filter as

$$x[n] \longrightarrow \boxed{\text{ideal filter } h[n]} \longrightarrow y[n] = x[n] \circledast h[n]$$

$$H(\omega) = \text{DTFT}\{h[n]\}$$





Now the question is, can we build an ideal filter for real-time DSP applications?

The answer is NO! If we want the filter to be implementable in real time, or to be causal in other words, it can never be ideal. Let's address the following theorem to verify it.

Paley-Wiener Theorem: If $h[n]$ is causal, or $h[n] = 0$ for $n < 0$, and it's energy is finite, then

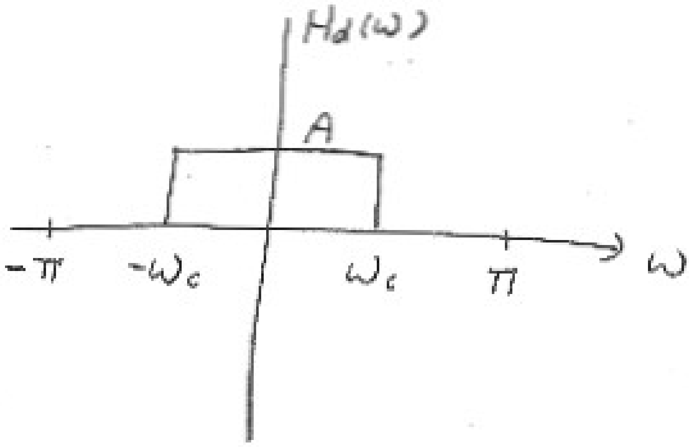
$$\int_{-\pi}^{\pi} \left| \log |H(\omega)| \right| d\omega < \infty$$

where $H(\omega) = \mathrm{DTFT}\{h[n]\}$.

The immediate consequence of this result is that any ideal filter cannot be causal, because it is zero within an interval of frequencies ω_1, ω_2, as shown in Figure 4.6, and therefore $\int_{-\pi}^{\pi} |\log |H(\omega)|| d\omega$ would be infinite.

If we really want to build the impulse response $h[n]$ from an ideal filter spectrum, we will have an infinitely long $h[n]$.

Example: What is the impulse response with respect to the ideal low-pass spectrum depicted as below?



Solution:

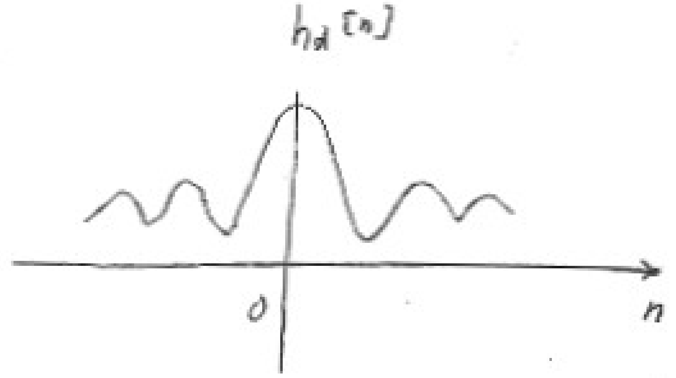
$$h_d[n] = \text{IDTFT}\{H_d(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{A}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{A \sin(\omega_c n)}{\pi n} = \frac{A\omega_c}{\pi} \text{sinc}(\omega_c n)$$

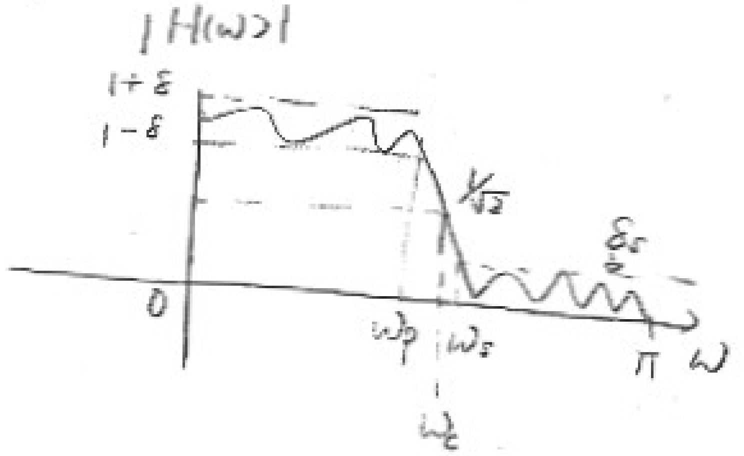
where $\text{sinc}(x) = \frac{\sin(x)}{x}$



$h_d[n]$ is noncausal.

Characteritics of Nonideal Filters

A typical magnitude of a nonideal lowpass filter is shown in the following figure



Three regions in the frequency domain are defined:

1. Passband ($0 \rightarrow \omega_p$): the nonideal filter has fluctuations within $1 \pm \delta$ for some constant δ

rather than 1 as the ideal filter. The parameter $\delta_p = \frac{1}{1-\delta}$ is defined as the passband ripple in dB.

2. Stopband ($\omega_s \to \pi$): In the ideal filter, the magnitude is zero; the nonideal filter has a small nonzero magnitude bounded by the stopband ripple, or the attenuation δ_s in dB.

3. Transition band ($\omega_p \to \omega_s$): the ideal filter has a sharp transition from passband to stopband while the nonideal filter has a smooth transition, where ω_c is called the cut-off or the half-power frequency in the midway transition such that

$$|H(\omega_c)|^2 = \frac{1}{2}|H(0)|^2$$

To design a nonideal lowpass filter, we have to specify the passband and stopband frequencies ω_p and ω_s with the respective ripples δ_p and δ_s

Example:

A lowpass filter is speified as follows:

Passband frequency : 4 kHz with 1 dB ripple

stopband frequency : 4.5 kHz with 50 dB attenuation

Sampling frequency : 22 kHz

In digital frequencies,

Passband : $0 \to \omega_p = 2\pi \frac{4}{22} = 1.14$ radians

stopband : $\omega_s = 2\pi \frac{4.5}{22} = 1.28 \to \pi$ radians

$$20 \log_{10} \left(\frac{1}{1-\delta}\right) = 1 \text{ dB}$$

$$\Rightarrow \delta = 0.11$$

$$20 \log_{10} \delta_s = -50 \text{ dB}$$

$$\Rightarrow \delta_s = 3.2 \times 10^{-3}$$

The magnitude spectrum $|H(\omega)|$ can be specified as

$$0.89 \le |H(\omega)| \le 1.11 \quad , \quad 0 \le \omega \le 1.12 \text{ rad}$$

$$0 \le |H(\omega)| \le 3.2 \times 10^{-3} \quad , \quad 1.28 \le \omega \le \pi \text{ rad}$$

Recursive and Nonrecursive Filters

The general form of the linear filters can be addressed by a linear difference equation

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \dots + b_N x[n-N] ,$$

where $x[n]$ and $y[n]$ are input and output sequences, respectively. Therefore, the transfer function is determined as

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_N}{z^N + a_1 z^{N-1} + \dots + a_N}$$

To distinguish two cases, we define:

1. Finite impulse response (FIR) filters, when $a_1 = a_2 = \cdots = a_N = 0$. In this case the difference equation is nonrecursive, of the form

$$y[n] = h[0]\,x[n] + h[1]\,x[n-1] + \cdots + h[N]\,x[n-N],$$

where $h[n]$ is the impulse response.

FIR filters are very important because they not only guarantee the stability always but can be sophisticately designed with a perfectly linear phase.

2. Infinite Impulse response (IIR) filters, when at least one of the coefficients a_i is nonzero.

This leads to filters that are recursive in the sense that at any time n, the response $y[n]$ depends on past values $y[n-1]$, $y[n-2]$, The stability can never be guaranteed for an IIR filter. Nor is the linear phase

Example: $y[n] = -0.2y[n-1] + x[n] + x[n-1]$

is a recursive IIR filter.

The transfer function can be described as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+0.2z^{-1}}$$

pole $z = -0.2$ is within the unit circle. Hence $H(z)$ is stable.

Example:

$$y[n] = x[n] - 0.8\,x[n-1] + 0.6\,x[n-2] - 0.4\,x[n-3] + 0.2\,x[n-4]$$

is a nonrecursive FIR filter.

The impulse response $h[n]$ driven by the input $x[n] = \delta[n]$, is

$$h[n] = \delta[n] - 0.8\,\delta[n-1] + 0.6\,\delta[n-2] - 0.4\,\delta[n-3] + 0.2\,\delta[n-4]$$

4.3 FIR Filter Design

Window-based Filters

We can always apply a window function $w[n]$ to restrict any impulse response, which characterizes a filter, to be finite-support, such that

$$h_w[n] = h[n]\,w[n],$$

which is zero outside the interval $-L \le n \le L$. Such a window can lead to the causal filter by simply shifting $h_w[n]$ to form $h_w[n-L]$.

However, the frequency response would be altered through this windowing. The windowing effects to the resultant spectra can be stated as:

a. Windowing can broaden the spectrum through the convolution in the frequency domain, such that

$$H_w(\omega) = \text{DTFT}\{h[n]\,w[n]\}$$
$$= \frac{1}{2\pi} H(\omega) \circledast \overline{W}(\omega),$$

where $H(\omega) = \text{DTFT}\{h[n]\}$

$\overline{W}(\omega) = \text{DTFT}\{w[n]\}$

b. Time shift by L causes a linear phase offset in the frequency domain:

$$\text{DTFT}\{h_w[n-L]\} = e^{-j\omega L} H_w(\omega)$$

where $\measuredangle H(\omega) = \measuredangle H_w(\omega) \underbrace{- \omega L}_{\text{linear phase offset}}$

Example:

We want to design a digital lowpass filter with the following characteristics:

Passband: 4 kHz

stopband 5 kHz, with at least 40 dB attenuation

Sampling frequency: 20 kHz.

Solution:

$$\omega_p = 2\pi \frac{4k}{20k} = \frac{2\pi}{5} \text{ rad/sec}$$

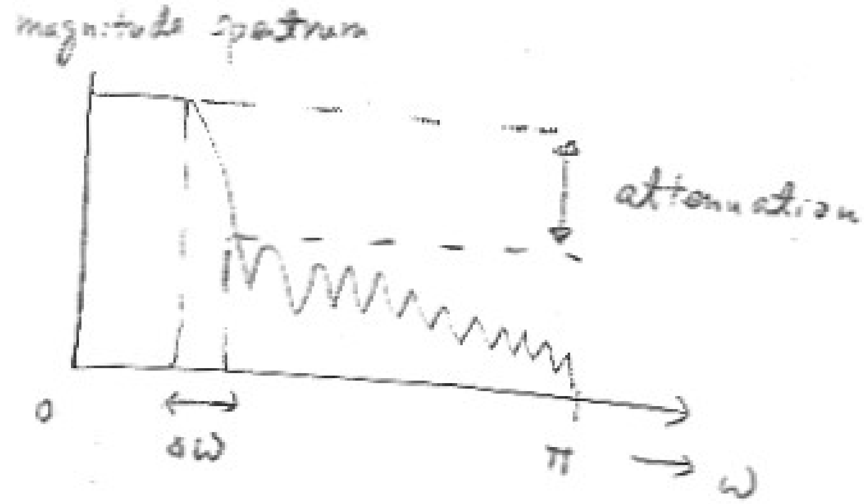
$$\omega_s = 2\pi \frac{5k}{20k} = \frac{\pi}{2} \text{ rad/sec}$$

Therefore, the transition band,

$$\Delta\omega = \omega_s - \omega_p = \frac{\pi}{2} - \frac{2\pi}{5} = \frac{\pi}{10} \text{ rad/sec}.$$

According to Figure 4.14, 4.15

	$\Delta\omega$	Attenuation
Rectangular	$4\pi/N$	-13 dB
Bartlet	$8\pi/N$	-27 dB
Hanning	$8\pi/N$	-32 dB
Hamming	$8\pi/N$	-43 dB
Blackman	$12\pi/N$	-58 dB



If a Hamming window is apply for the truncated ideal lowpass filter impulse response,

$$\Delta\omega = \frac{\pi}{10} \geq \frac{8\pi}{N} \Rightarrow N \geq 80$$

N is the total number of points of the windowing function, or $N = 2L+1$.

$$L = \frac{N-1}{2} = \frac{81-1}{2} = 40.$$

The hamming window can be described as

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right), & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$$

Hence, $h[n] = h_d[n]\, w[n]$

$$= \frac{\omega_c}{\pi} \operatorname{sinc}(\omega_c n)\; w[n]$$

where $\omega_c = \frac{1}{2}(\omega_p + \omega_s) = \frac{15\pi}{20} = \frac{3\pi}{4}$

The causal implementation of $h[n]$ is

$$h[n-L] = h[n-40]$$

$$= \frac{3}{4} \operatorname{sinc}\left(\frac{3\pi}{4}(n-40)\right)$$

$$\begin{cases} \times \left[0.54 - 0.46\cos\left(\frac{2\pi n}{81}\right)\right], & 0 \le n \le 80 \\ 0, & \text{otherwise} \end{cases}$$

Symmetry and Linear Phase

If an impulse response is an even sequence, or symmetric in time, then its DTFT has linear phase.

Proof: $h[n] = h[-n]$, $\forall n$

Then $H(\omega) = \text{DTFT}\{h[n]\}$

$$= \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} h[n]\, e^{-j\omega n} + h[0]$$

$$+ \sum_{n=1}^{\infty} h[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} h[-n]\, e^{-j\omega n} + \sum_{n=1}^{\infty} h[n]\, e^{-j\omega n}$$

$$+ h[0]$$

$$= h[0] + 2 \sum_{n=1}^{\infty} h[n] \cos(\omega n)$$

$\measuredangle H(\omega) = \pm\pi$

Therefore $h[n-L] \overset{DTFT}{\Longrightarrow} H(\omega) e^{-j\omega L}$.

$\measuredangle \{ H(\omega) e^{-j\omega L} \} = \pm\pi - \omega L$ is the

causal implementation of a filter, with the linear phase. Hence the linear phase can be guaranteed by the symmetric impulse response shifted by an L in time.

Design by the Kaiser Window Method

If we want to specify both attenuation in the stopband and width of the transition region, we can use a Kaiser window.

Consider a lowpass filter with the same ripple δ within the passband and the stopband. In other words, let

$$1-\delta \le |H(\omega)| \le 1+\delta, \quad 0 \le \omega \le \omega_p$$
$$0 \le |H(\omega)| \le \delta, \quad \omega_s \le \omega \le \pi$$

The Kaiser window, $w[n]$, $n = 0, 1, \ldots, N$, is defined on the basis of the Bessel function, $I_0(x)$ of order zero:

$$w[n] = K(\beta, N) = I_0\left(\beta\left(1 - \left[2\left(n - \frac{N}{2}\right)/N\right]^2\right)^{\frac{1}{2}}\right)$$

The parameter β and the window length N control both the ripple, $A = -20 \log_{10} \delta$, and the width of the transition region, $\Delta\omega = \omega_s - \omega_p$.

Empirically, the parameter β can be selected to control the ripples as follows:

$$\beta = \begin{cases} 0.1102\,(A - 8.7), & \text{if } A > 50\,\text{dB} \\ 0.5842\,(A - 21)^{0.4} + 0.07886\,(A - 21), & \text{if } 21 \le A \le 50 \\ 0.0 & \text{if } A < 21\,\text{dB} \end{cases}$$

and the window length N can be selected to control the transition region as

$$N = \frac{A - 8}{2.285\,\Delta\omega}$$

Example :

Consider a lowpass filter with the specs as follows :

Passband : 4 kHz

Stopband : 5 kHz with at least 40 dB attenuation

Sampling frequency : 20 kHz

$$\omega_p = \frac{2\pi \times 4k}{20k} = \frac{2\pi}{5} \text{ rad}, \quad \omega_s = \frac{2\pi \times 5k}{20k} = \frac{\pi}{2} \text{ rad}$$

Hence, $A = 40$

$$\Delta\omega = \omega_s - \omega_p = \frac{\pi}{10} \text{ rad}$$

$$\therefore N = \frac{A-8}{2.285\Delta\omega} = 45$$

$$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21) = 3.3953$$

Using Matlab,

```
w = kaiser(45, 3.3953);
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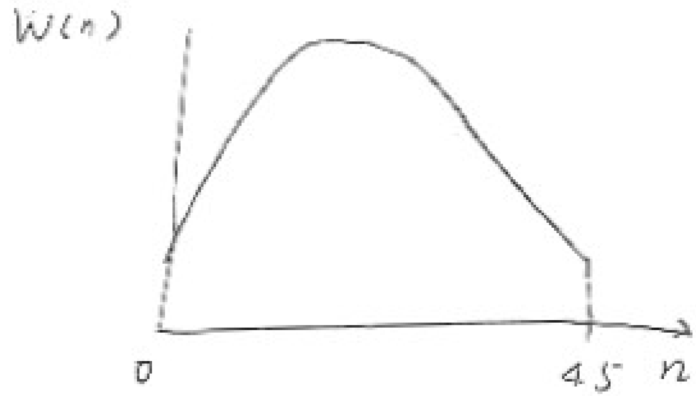


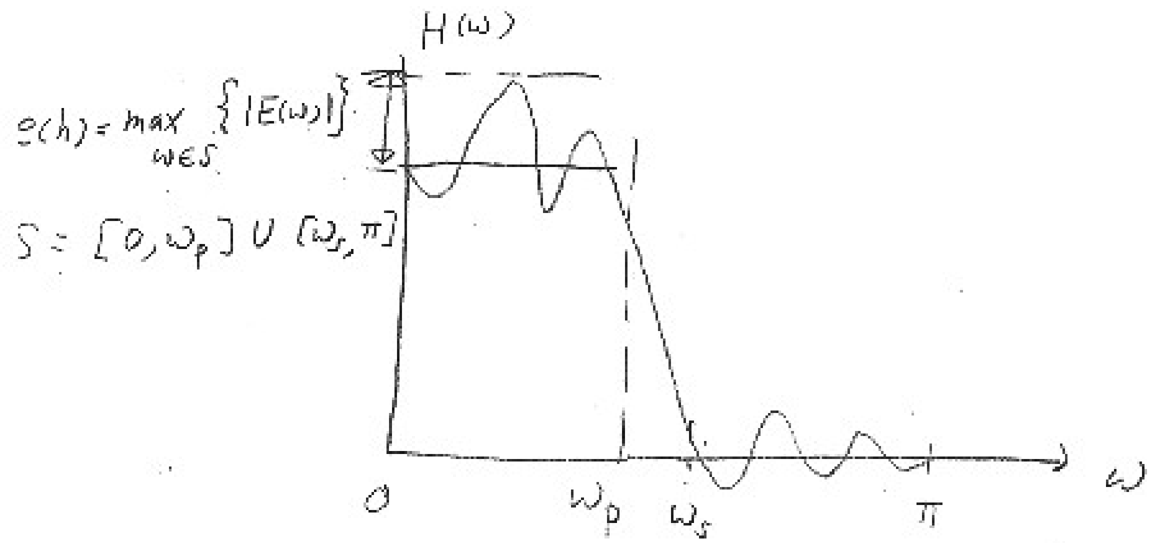
Kaiser window with $N = 45$, $\beta = 3.3953$

Equiripple Filters

The limitation can be found when we design the FIR filters: the filters' attributes such as the attenuation in the stopband do not improve with the order of the filters. Therefore, we will seek an optimization, which minimizes the error $E(\omega) = H(\omega) - H_d(\omega)$. Given a desired ideal filter $H_d(\omega)$ and a nonideal approximation $H(\omega) = \sum_{n=-L}^{L} h[n] e^{-j\omega n}$, the error can be represented by its maximum over the frequency range S of interest:

$$e(h) = \max_{\omega \in S} |H(\omega) - H_d(\omega)|.$$

An example can be shown as below:



Using such a criterion, the optimal FIR filter can have an impulse response $\hat{h}$, determined from the minization of $e(\hat{h}) = \min_h e(h)$. This problem can lead to a solution of the equiripple filter as depicted in Fig 4.22.

To maintain the linear phase, we assume a symmetric impulse response, that is,

$$h[n] = \pm h[-n],$$

First take the example of a noncausal lowpass filter with $h[n] = h[-n]$, we have

$$H(\omega) = \sum_{n=-L}^{L} h[n] e^{-j\omega n} = h[0] + 2 \sum_{n=1}^{L} h[n] \cos(\omega n)$$

Recall the trigonometric identities.

$$\cos(2\omega) = 2\cos^2(\omega) - 1$$

$$\cos(3\omega) = 4\cos^3(\omega) - 3\cos(\omega)$$

$$\begin{aligned}\cos(4\omega) &= 2\cos^2(2\omega) - 1 \\ &= 2\left[2\cos^2(\omega) - 1\right]^2 - 1 \\ &= 8\cos^4(\omega) - 8\cos^2(\omega) + 1\end{aligned}$$

$\vdots$

We can verify that, the frequency response can be written in a polynomial form,

$$H(\omega) = \sum_{k=0}^{L} a_k \cos^k(\omega) = A[\cos(\omega)],$$

where $A[x] = \sum_{k=0}^{L} a_k x^k$ is a polynomial of degree L.

The goal is to approximate $H_d(\omega)$ by

$H(\omega) = A[\omega s(\omega)]$ in the minimax sense

(minimize the maximum approximation error $e(h)$).

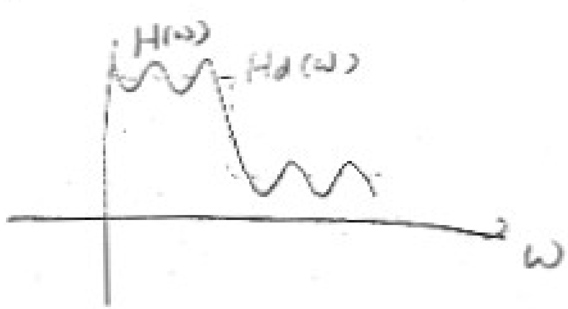
The optimal filter solution will have the sign alternation property, that is there exist at least $L+2$ frequencies $0 \le \omega_1 < \omega_2 < \dots < \omega_{L+2} \le \pi$ where

$$E(\omega_i) = -E(\omega_{i+1}), \quad \text{for } i = 1, 2, \dots, L+1$$

$$|E(\omega_i)| = \max_{\omega \in S} |E(\omega)|, \quad \text{for } i = 1, 2, \dots, L+2$$

and $E(\omega) = H(\omega) - H_d(\omega)$

In other words, as shown in Fig 4.23, the error has to be spreaded evenly over the local maxima and minima on the set of frequency S.



The remaining problem is how to apply this sign alternation property to achieve the solution numerically. We have to follow the iterative procedure below:

① Start:

Initialize an arbitrary set of distinct frequencies, $0 \le \omega_1 < \omega_2 < \cdots < \omega_{L+2} \le \pi$, where $\omega_i \in S$, $i = 1, 2, \cdots, L+2$.

② Compute:

Obtain the polynomial coefficients $a_0, a_1, \cdots, a_L$ and a constant δ, where $A(x_i) = \sum_{k=0}^{L} a_k x_i^k$

and $x_i = \cos(\omega_i)$, $i = 1, 2, \cdots, L+2$

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_L \\ \delta \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & 1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & -1 \\ \vdots & & & & & \vdots \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & (-1)^{L+2} \end{bmatrix}^{-1} \begin{bmatrix} H_d(\omega_1) \\ H_d(\omega_2) \\ \vdots \\ H_d(\omega_{L+2}) \end{bmatrix}$$

③ Update:

Define $P_E(\omega_i') \equiv \max_{\omega_i' \in (\omega_i, \omega_{i+1})} \{ |E(\omega_i')| \}$

If $P_E(\omega_i') > \delta$, then replace ω_i by ω_i'

such that $w_i = w_i'$. After inspect and substitute w_i's appropriately, go back to Step ②. If nothing is needed to be substituted for w_i, $i = 1, 2, \cdots, L+2$, then stop here.

The whole procedure can be found in Matlab as:

$$h = \text{remez}(N, \underset{\substack{\uparrow \\ [0\ \omega_p\ \omega_s\ \pi]}}{F}, \overset{\nwarrow\ [1\ 1\ 0\ 0]}{M})$$

Example:

We want to design an equiripple lowpass filter to meet the following specs:

Passband: 4 kHz

Stopband: 5 kHz with at least 40 dB attenuation

Sampling frequency: 20 kHz.

Solution:

$$\omega_p = 2\pi \frac{4}{20} = \frac{2\pi}{5} \text{ rads}, \quad \omega_s = 2\pi \frac{5}{20} = \frac{\pi}{2} \text{ rads}$$

Choose the filter length $N = 80$. The ideal filter has value $M = 1$ at frequency $f = \frac{\omega_p}{\pi} = 0.4$ and $M = 0$ at frequencies $f = \frac{\omega_s}{\pi} = 0.5$ and $f = \frac{\pi}{\pi} = 1$. Therefore, we call it as

h = remez (80, [0, 0.4, 0.5, 1], [1, 1, 0, 0])

The frequency response is depicted in Fig 4.29

4.4 Infinite Impulse Response (IIR) Filters

A recursive filter with the difference equation

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + \cdots + b_N x[n-N]$$

has an impulse response of infinite length when not all the coefficients a_i are zero.

The main advantage of these IIR filters is their simplicity, in the sense that we may design a very selective filter with just a few parameters. On the other hand, the IIR filters have the main disadvantages as that

(a) they are not linear phase in general,

(b) their stability is often the severe problem.

The general IIR filter design technique uses discretization of analog continuous time filters. Because analog filters can all have infinite impulse responses based on, the simple analyses in the s-domain, we can just build a discrete-time IIR filter by transforming an analog filter through s-domain to z-domain conversion.

Hence the crucial problem is to convert an analog filter with transfer function $H_A(s)$ to a discrete-time realization with the transfer function $H_D(z)$.

Ideally, this conversion should follows that:

a. It preserves stability. If the analog filter is stable, then the transformed digital filter should be stable too. In terms of transfer function, this means that if $H_A(s)$ has all poles on the left half s-plane, the discrete time transfer function $H_D(z)$ should have all poles inside the unit circle in the z-plane

b. It preserves the frequency response, in the sense that the frequency response of the analog continuous-time filter and the discrete-time implementation should have the same shape.

In terms of transfer functions, this means that the $j\Omega$ axis in the s-plane should be mapped into the unit circle in the z-plane.

There are two ways of analog-to-digital conversion for IIR filter design: ① Euler approximation, $S = \frac{1 - z^{-1}}{T_s}$, ② Bilinear transformation, $S = \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}$, where T_s is the sampling interval

Example:

Consider an analog filter with transfer function

$$H_A(s) = \frac{2}{s+1}$$

Find the discrete-time transfer function obtained by the Euler approximation where $F_s = 5$ Hz.

Solution:

$$H_D(z) = \left. \frac{2}{s+1} \right|_{s = F_s(1 - z^{-1})}$$

$$= \frac{2}{5(1-z^{-1})+1} = \frac{2z}{6z-5} = \frac{2}{6-5z^{-1}}$$

$$\Rightarrow \quad \underset{\uparrow \atop \text{output}}{Y(z)} = \underset{\uparrow \atop \text{IIR filter}}{H_D(z)} \; \underset{\uparrow \atop \text{input}}{X(z)}$$

$$\Rightarrow \quad 6y[n] = 5y[n-1] + 2x[n]$$

$$\Rightarrow \quad y[n] = \frac{5}{6} y[n-1] + \frac{1}{3} x[n]$$

Example: Use the previous example to obtain an IIR filter by the bilinear transformation.

Solution:

$$H_D(z) = \left. \frac{2}{s+1} \right|_{s = 2F_s \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$$

$$= \frac{2}{10 \, \frac{1-z^{-1}}{1+z^{-1}} + 1}$$

$$= \frac{2+2z^{-1}}{11-9z^{-1}}$$

$\Rightarrow \; Y(z) = H_D(z)\, X(z)$

$\Rightarrow \; 11\, y[n] = 9\, y[n-1] + 2x[n] + 2x[n-1]$

$\Rightarrow \; y[n] = \frac{9}{11}\, y[n-1] + \frac{2}{11}\, x[n] + \frac{2}{11}\, x[n-1]$

Properties of two Analog-to-digital Filter Conversion Methods

a. Both Euler approximation and bilinear transformation methods preserve the stability, i.e; if the transfer function $H_A(s)$ has poles with a negative real part, then the transformed transfer function $H_D(z)$ has poles inside the unit circle.

b. The frequency response is not preserved or the $j\Omega$ axis in the s-plane is not mapped into the unit circle in the z-plane by the Euler applroximation ($|z| = \frac{1}{|1 - j\Omega T_s|}$ is not necessarily equal to one); whereas the frequency response is preserved

$$\left(|z| = \left| -\frac{j\Omega + 2/T_s}{j\Omega - 2/T_s} \right| = 1 \right)$$ by the bilinear

transformation.

Example

Suppose we want to design a digital lowpass filter with a bandwidth $F_c = 8$ kHz and a sampling frequency $F_s = 24$ kHz using the bilinear transformation. We proceed through the following steps:

Step 1: Determine the specifications in the digital frequency domain. The bandwidth of the digital lowpass filter is $w_c = 2\pi \times \frac{8}{24} = \frac{2\pi}{3}$ rad

Step 2. Determine the specifications of the analog filter $H_A(s)$ to be transformed

We need an analog lowpass filter with bandwidth

$$\Omega_c = 2F_s \tan\left(\frac{\omega_c}{2}\right) = 2 \times (24 \times 10^3) \times \tan\left(\frac{\pi}{3}\right)$$

$$= 83.13 \times 10^3 \text{ rads/sec}$$

Step 3: Design an analog lowpass filter with bandwidth $\Omega_c = 83.13 \times 10^3$ rads/sec. Its transfer function is $H_A(s)$.

Step 4: Apply the bilinear transformation

$$s = \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}}$$ to obtain the discrete time transfer function $H_D(z)$.

From the complete procedure to design a digital filter in the previous example, the only problem unsolved is the step 3.

The appropriate analog fiters which can be used are Butterworth and Chebyshev filters.

Design of Analog Filters

Butterworth Filters:

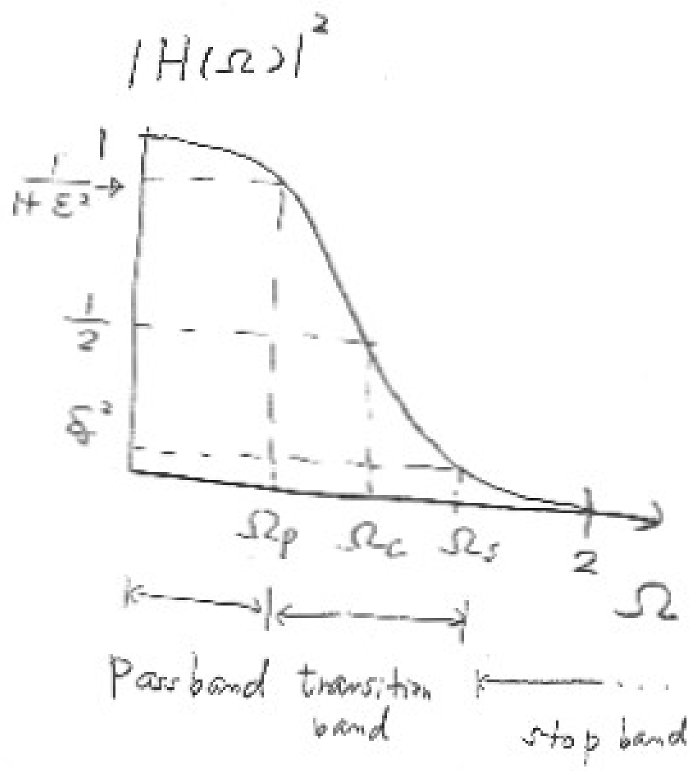
A Butterworth filter is characterized by the frequency response

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$= \frac{1}{1 + \left(\varepsilon^2 \frac{\Omega}{\Omega_p}\right)^{2N}}$$

where Ω_p is the passband frequency and Ω_c is the cutoff frequency and $\Omega_c = \frac{\Omega_p}{\varepsilon^2}$ both in rads/sec.

The magnitude response of $|H(\Omega)|^2$ can be depicted in the following figure for the Butterworth filter of order N:



The passband ripple δ_p is related to ε as

$$\delta_p^2 = 1+\varepsilon^2$$

The stopband ripple is given by δ_s^2, such that

$$\delta_s^2 = \frac{1}{1+\left(\varepsilon^2 \frac{\Omega_s}{\Omega_p}\right)^{2N}}$$

Thus, the order N can be determined as

$$N = \frac{\ln\left[\frac{1-\delta_s^2}{\varepsilon^2 \delta_s^2}\right]}{2 \ln\left(\frac{\Omega_s}{\Omega_p}\right)}$$

How do we determine $H(s)$?

Since $|H(\Omega)|^2 = H(s)\,H(-s)\big|_{s=j\Omega}$,

$$H(s)\,H(-s) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}\Bigg|_{\Omega = s/j}$$

$$= \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}$$

There are no zeros and poles are

$$\frac{-s^2}{\Omega_c^2} = \sqrt[N]{(-1)} \Rightarrow s = \Omega_c\, e^{j\frac{\pi}{2}} e^{j\frac{(2k+1)\pi}{2N}},$$

for $k = 0, 1, 2, \ldots, 2N-1$

In order to have a stable $H(s)$, we assign N poles on the left-half s-plane to $H(s)$ and the remaining N poles on the right-half s-plane are automatically assigned to $H(-s)$.

Example: Design a digital lowpass filter based on the analog Butterworth filter with a 3dB passband freq. at 100 Hz and 40 dB stopband frequency at 10000 Hz, $F_s = 10^4$ Hz

Solution:

$$\Omega_c = 2\pi \times 100 = 200\pi$$

$$\Omega_s = 2\pi \times 10{,}000 = 20{,}000\pi$$

$$|H_A(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{200\pi}\right)^{2N}}$$

$$-40 = 20 \log_{10} \delta_s$$

$$\delta_s = 10^{-2} = 0.01$$

$$\Rightarrow \quad |H_A(\Omega_s)|^2 = \frac{1}{1+(100)^{2N}} = \delta_s^2 \leq 10^{-4}$$

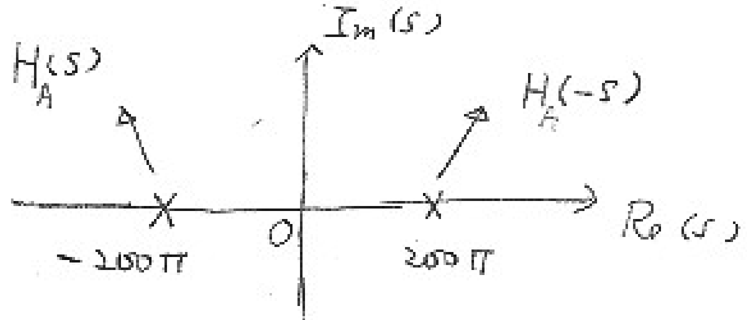
$$\Rightarrow \quad 10^{-4} + 10^{-4} \times 10^{4N} \geq 1$$

$$\Rightarrow \quad 10^{4N} \geq 10^{4} - 1 \quad \Rightarrow N \geq 1$$

We only need to choose an order-1 Butterworth filter

$$H_A(s)\, H_A(-s) = \frac{1}{1 + \left(\frac{-s^2}{(200\pi)^2}\right)^1} = \frac{1}{1 - \frac{s^2}{(200\pi)^2}}$$

$$\Rightarrow \quad \text{poles:} \quad s = \pm\, 200\pi$$



$$\therefore \quad H_A(s) = \frac{1}{1 + \frac{s}{200\pi}} = \frac{200\pi}{200\pi + s}$$

There are two ways to do the final s-domain to z-domain conversion.

① Euler approximation:

$$H_D(z) = H_A(s)\Big|_{s = \frac{1-z^{-1}}{T_s}} = (1-z^{-1}) \times 10^4$$

$$= \frac{200\pi}{200\pi + 10^4(1-z^{-1})}$$

$$= \frac{.628\, z}{10628\, z - 10000}$$

$$= 0.059 \frac{z}{z - 0.94}$$

We can drop the gain 0.059 or normalize it to the unity

$\Rightarrow$ $y[n] = 0.94\, y[n-1] + x[n]$

is the desired IIR - lowpass filter

3 Bilinear transformation:

$$H_D(z) = H_A(s)\Big|_{s = \frac{2}{T_s}\frac{z-1}{z+1} = 2\times10^4 \frac{z-1}{z+1}}$$

$$= \frac{200\pi}{200\pi + 2\times10^4\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{628\,(z+1)}{20628\,z - 19372}$$

$$= 0.03\ \frac{z+1}{z-0.94}$$

normalize the gain 0.03 to the unity.

$\Rightarrow$ $y[n] = 0.94\, y[n-1] + x[n] + x[n-1]$

is the desired IIR - lowpass filter.

X. Chebyshev Filters:

We may also choose the analog Chebyshev filters to design $H_A(s)$ at the beginning.

The frequency response of an order-N Chebyshev filter is of the form.

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

where $T_N(x)$ is the Chebyshev polynomial of degree N and it is such that

$$-1 \le T_N(x) \le 1 \qquad \text{for } -1 \le x \le 1.$$

Thus, $\frac{1}{1+\varepsilon^2} \le |H(\Omega)|^2 \le 1$, for $0 \le \Omega \le \Omega_p$

The Chebyshev polynomials are defined as:

$$T_N(x) = \cos(Nt)\Big|_{t=\cos^{-1}(x)}$$

for $-1 \le x \le 1$

$T_0(x) = \cos(0) = 1$

$T_1(x) = x$

$T_2(x) = 2\cos^2(t) - 1 \Big|_{t=\cos^{-1}(x)} = 2x^2 - 1$

$\vdots$

for $N > 2$,

$$\cos[(N+1)t] = 2\cos(t)\cos(Nt) - \cos[(N-1)t]$$

$$\Rightarrow \quad T_{N+1}(x) = 2x\,T_N(x) - T_{N-1}(x)$$

.... the recursive formula to obtain an degree-N Chebyshev polynomial

Since $|T_N(x)| \le 1$ for $|x| \le 1$,

$$\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right) \le \varepsilon^2 \text{ for } -\Omega_p \le \Omega \le \Omega_p.$$

and $$N = \frac{\ln\left[\frac{1}{\varepsilon\delta_s}\left(\sqrt{(1-\delta_s^2)} + \sqrt{1-\delta_s^2(1+\varepsilon^2)}\right)\right]}{\ln\left[\frac{\Omega_s}{\Omega_p} + \sqrt{\left(\frac{\Omega_s}{\Omega_p}\right)^2 - 1}\right]}$$

Example: Design a digital filter based on the analog Chebyshev filter with the following characteristics:

Passband frequency at 500 Hz with 1 dB ripple.

Stopband frequency at 5 kHz with 40 dB attenuation.

Sampling frequency at 10 kHz.

Solution:

$$\Omega_p = 1000\pi \text{ rads/sec}$$

$$\Omega_s = 10{,}000\pi \text{ rads/sec}$$

$$F_s = 10{,}000$$

$$20\log_{10}\left(\sqrt{1+\varepsilon^2}\right) = 1\text{ dB}$$

$$\Rightarrow \varepsilon = \sqrt{10^{\frac{1}{10}} - 1} = 0.5088$$

$$-40 = 20\log_{10}\delta_s$$

$\Rightarrow \quad \delta_s = 0.01$

Thus,
$$N \geq \frac{\ln\left[\frac{1}{0.01 \times 0.5088}\left(\sqrt{1-0.01^2} + \sqrt{1-0.01^2(1+0.5088^2)}\right)\right]}{\ln(10+\sqrt{99})}$$

$$= 2$$

Hence, $H_A(s)H_A(-s) = |H_A(\Omega)|^2 \Big|_{\Omega = \frac{s}{j}}$

$$= \frac{1}{1+\varepsilon^2\left[4\frac{\Omega^4}{\Omega_p^4} - 4\frac{\Omega^2}{\Omega_p^2} + 1\right]}\Bigg|_{\Omega = \frac{s}{j}}$$

$$= \frac{1}{1+0.51^2\left[\frac{4}{\Omega_p^4}s^4 + \frac{4}{\Omega_p^2}s^2 + 1\right]}$$

$$= \frac{1}{\frac{4\times 0.51^2}{\Omega_p^4}s^4 + \frac{4\times 0.51^2}{\Omega_p^2}s^2 + 1.2589}$$

$$= \frac{1}{1.063\times 10^{-14} \times (s \pm 1724.43 \pm 2812.20j)}$$

$$\Rightarrow \quad H_A(s) = \frac{1}{s+1704.43+2812.20j} \times \frac{1}{s+1724.43-2812.21j}$$

$$= \frac{1}{s^2 + 3448.87s + 1.088\times 10^7}$$

Using the Euler approximation :

$$H_D(z) = H_A(s)\Big|_{s = (1-z^{-1})\times 10^4}$$

$$= \frac{1}{10^8 \times (z^{-2} - 2z^{-1} + 1) + 3.449\times 10^7 (1 - z^{-1}) + 1.088\times 10^7}$$

$$= \frac{z^2}{10^8 (1.4536 z^2 - 2.3449 z + 1)}$$

Drop the gain $\frac{1}{1.4536\times 10^8}$

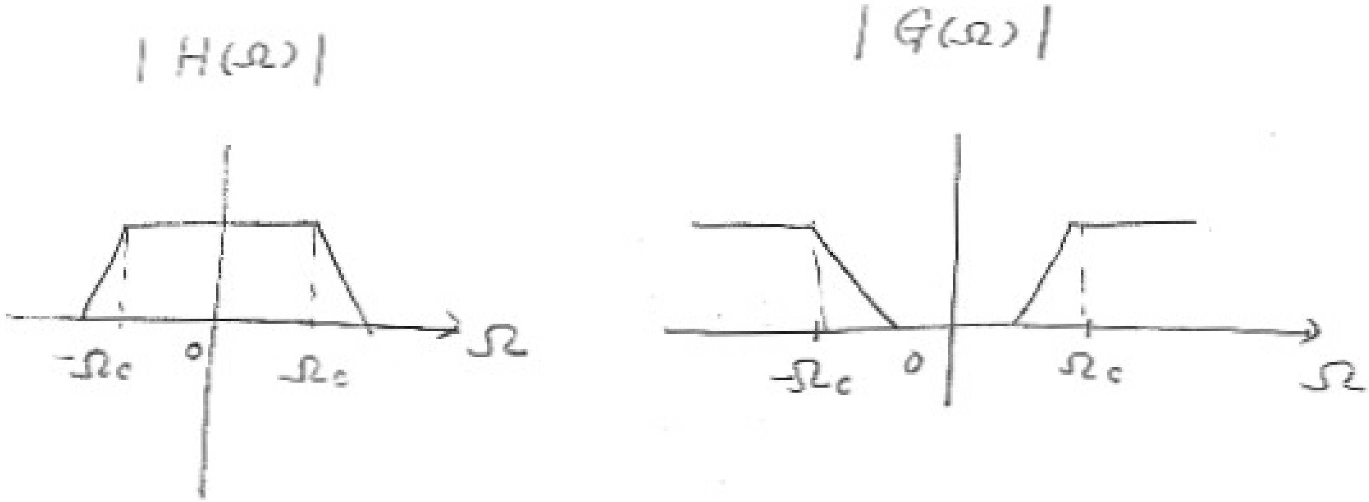
$$H_D(z) = \frac{1}{1 - 1.6123 z^{-1} + 0.6879 z^{-2}}$$

$$\Rightarrow \quad y[n] = 1.612\, y[n-1] - 0.688\, y[n-2] + x[n]$$

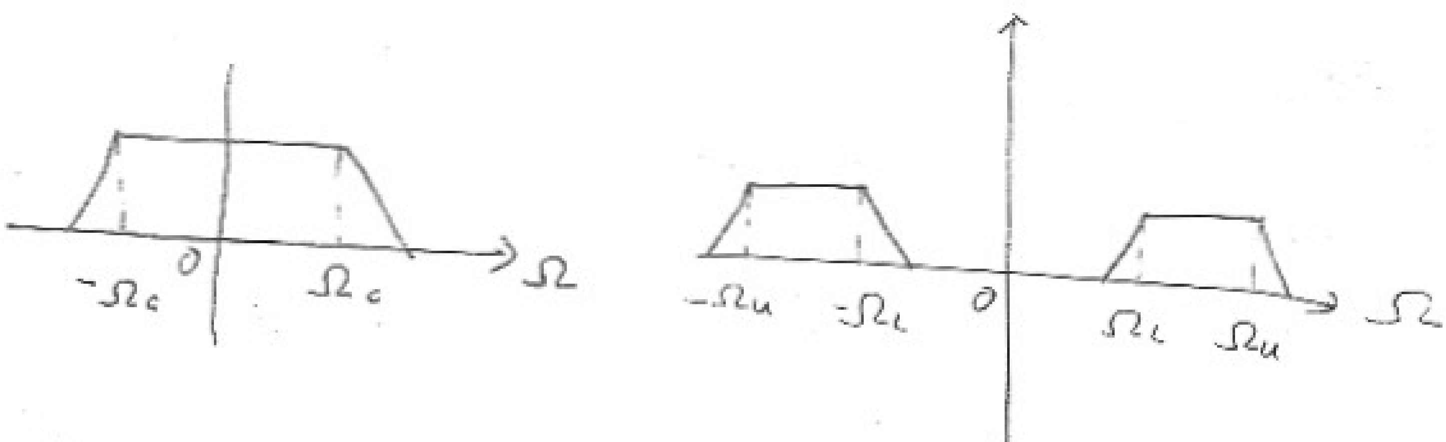
is the desired IIR - lowpass filter

The frequency spectra can be depicted as below.

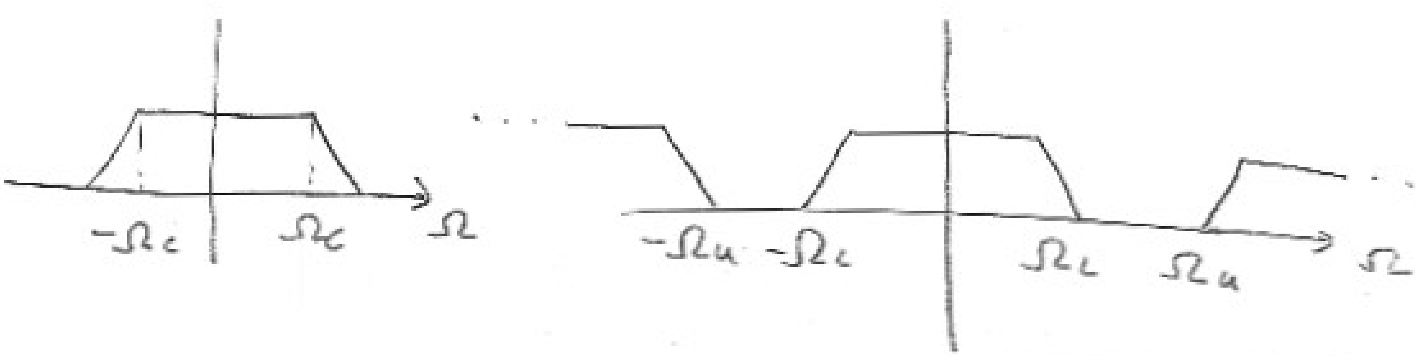
Lowpass to Highpass



Lowpass to Bandpass



Lowpass to Bandstop



Frequency Transformations:

We want to design a analog lowpass filter first and then transform it to the different classes of filters, such as highpass, bandpass and bandstop filters.

The transformation can be generalized as follows: we map a lowpass filter $H(s)$ into a different class of filter $G(s)$ such that

$$G(s) = H(s')\Big|_{s' = g(s)}$$

where

Lowpass to Highpass: $g(s) = \dfrac{\Omega_c^2}{s}$

Lowpass to Bandpass: $g(s) = \Omega_c^2 \dfrac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)}$

Lowpass to Bandstop: $g(s) = \Omega_c \dfrac{s(\Omega_u - \Omega_L)}{s^2 + \Omega_u \Omega_L}$